

Research Article

On the Dynamics of Nonautonomous Parabolic Systems Involving the Grushin Operators

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We study the long-time behavior of solutions to nonautonomous semilinear parabolic systems involving the Grushin operators in bounded domains. We prove the existence of a pullback \mathfrak{D} -attractor in $(L^2(\Omega))^m$ for the corresponding process in the general case. When the system has a special gradient structure, we prove that the obtained pullback \mathfrak{D} -attractor is more regular and has a finite fractal dimension. The obtained results, in particular, extend and improve some existing ones for the reaction-diffusion equations and the Grushin equations.

1. Introduction

Nonautonomous equations are of great importance and interest as they appear in many applications in the natural sciences. One way of studying the long-time behavior of solutions of such equations is using the theory of pullback attractors. This theory has been developed for both the nonautonomous and random dynamical systems and has shown to be very useful in the understanding of the dynamics of such dynamical systems (see [1] and references therein). In recent years, the existence of pullback attractors for reaction-diffusion equations has been studied widely (see, e.g., [2–6]). However, to the best of our knowledge, little seems to be known for the asymptotic behavior of solutions of nonautonomous degenerate equations.

One of the classes of degenerate equations that has been studied widely in recent years is the class of equations involving an operator of the Grushin type [7]

$$G_s u = \Delta_x u + |x|^{2s} \Delta_y u, \quad X = (x, y) \in \Omega \subset \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}, \quad s \geq 0. \quad (1.1)$$

The global existence and long-time behavior of solutions to semilinear parabolic equations involving the Grushin operator, in both autonomous and nonautonomous cases, have been studied in some recent works [8–10].

In this paper we consider the following nonautonomous semilinear parabolic system:

$$\begin{aligned} \frac{\partial u}{\partial t} - aG_s u + f(u) &= g(X, t), \quad X \in \Omega, \quad t > \tau, \\ u(X, t) &= 0, \quad X \in \partial\Omega, \quad t > \tau, \\ u(X, \tau) &= u_\tau(X), \quad X \in \Omega, \end{aligned} \quad (1.2)$$

where $X = (x, y) \in \Omega \subset \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$ ($N_1, N_2 \geq 1$), $u_\tau \in (L^2(\Omega))^m$ is given, $u = (u^1, \dots, u^m)$ is an unknown vector-function. Here $a \in \text{Mat}_m(\mathbb{R})$, $f(u) = (f^1(u^1, \dots, u^m), \dots, f^m(u^1, \dots, u^m))$, and $g(X, t) = (g^1(X, t), \dots, g^m(X, t))$ satisfy the following conditions:

(H1) $a \in \text{Mat}_m(\mathbb{R})$ has a positive symmetric part: $(1/2)(a + a^*) \geq \beta I_m, \beta > 0$;

(H2) $f : \mathbb{R}^m \rightarrow \mathbb{R}^m$ is a C^1 -vector function such that:

$$C_1|u|^p - C_0 \leq (f(u), u) = \sum_{j=1}^m f^j(u)u^j, \quad p \geq 2, \quad (1.3)$$

$$|f(u)| \leq C_2(|u|^{p-1} + 1), \quad (1.4)$$

$$-C_3|v|^2 \leq (f_u(u)v, v) = \sum_{i=1}^m \sum_{j=1}^m \frac{\partial f^i}{\partial u^j}(u)v^j v^i, \quad (1.5)$$

where C_0, C_1, C_2 , and C_3 are positive constants;

(H3) $g \in W_{\text{loc}}^{1,2}(\mathbb{R}; (L^2(\Omega))^m)$ such that

$$\begin{aligned} \int_{-\infty}^0 e^{\lambda_1 \beta s} \|g(s)\|_{(L^2(\Omega))^m}^2 ds < +\infty, \quad \int_{-\infty}^0 \int_{-\infty}^s e^{\lambda_1 \beta r} \|g(r)\|_{(L^2(\Omega))^m}^2 dr ds < +\infty, \\ \int_{-\infty}^0 e^{\lambda_1 \beta s} \|g'(s)\|_{(L^2(\Omega))^m}^2 ds < +\infty, \end{aligned} \quad (1.6)$$

where λ_1 is the first eigenvalue of the operator G_s in Ω with the homogeneous Dirichlet boundary condition.

In order to study problem (1.2), we will use the natural energy space $\mathbb{S}_0^1(\Omega) := (S_0^1(\Omega))^m$ defined as the complete of $(C_0^\infty(\Omega))^m$ in the following norm:

$$\|u\|_{\mathbb{S}_0^1(\Omega)} = \left(\int_{\Omega} (|\nabla_x u|^2 + |x|^{2s} |\nabla_y u|^2) dX \right)^{1/2}. \quad (1.7)$$

From the results in [11], we know that the embedding $\mathbb{S}_0^1(\Omega) \hookrightarrow \mathbb{L}^p(\Omega)$ is continuous if $1 \leq p \leq 2_s^* := (2N(s))/(N(s) - 2)$, where $N(s) := N_1 + (s + 1)N_2$; moreover, this embedding is compact if $1 \leq p < 2_s^*$.

Notations

Denote $\mathbb{L}^p(\Omega) := (L^p(\Omega))^m$, and $\mathbb{S}^{-1}(\Omega)$ the dual space of $\mathbb{S}_0^1(\Omega)$. For functions $u, v : \mathbb{R}^N \rightarrow \mathbb{R}^m$, we set

$$((\nabla u, \nabla v)) := \sum_{i=1}^m (\nabla u^i, \nabla v^i) = \sum_{i=1}^m \sum_{k=1}^N \frac{\partial u^i}{\partial X_k} \frac{\partial v^i}{\partial X_k}, \tag{1.8}$$

so if $a = (a_{ij})_{i,j=1}^m \in \text{Mat}_m(\mathbb{R})$, then

$$((a\nabla u, \nabla v)) = \sum_{i,j=1}^m a_{ij} (\nabla u^i, \nabla v^j) = \sum_{i,j=1}^m \sum_{k=1}^N a_{ij} \frac{\partial u^i}{\partial X_k} \frac{\partial v^j}{\partial X_k}, \tag{1.9}$$

where (\cdot, \cdot) denotes the inner product in \mathbb{R}^N .

Noting that by assumption (H1), we have

$$((a\nabla u, \nabla u)) = \frac{1}{2} \sum_{i,j=1}^m (a_{ij} + a_{ji}) (\nabla u^i, \nabla u^j) \geq \beta \sum_{j=1}^m (\nabla u^j, \nabla u^j) = \beta \sum_{j=1}^m |\nabla u^j|^2. \tag{1.10}$$

Hence

$$\int_{\Omega} [((a\nabla_x u, \nabla_x u)) + |x|^{2s} ((a\nabla_y u, \nabla_y u))] dX \geq \beta \|u\|_{\mathbb{S}_0^1(\Omega)}^2, \tag{1.11}$$

$$\int_{\Omega} (aG_s u, G_s u) dX \geq \beta \|G_s u\|_{\mathbb{L}^2(\Omega)}^2. \tag{1.12}$$

The aim of this paper is to study the long-time behavior of solutions to problem (1.2) by using the theory of pullback \mathfrak{D} -attractors. We first prove, under assumptions (H1)–(H3), the existence of a pullback \mathfrak{D} -attractor in $\mathbb{L}^2(\Omega)$ for the process $U(t, \tau)$ associated to problem (1.2). Then, with an additional condition that the system has a special gradient structure, namely, $a = \beta I_m$ and there exists a function $F : \mathbb{R}^m \rightarrow \mathbb{R}$ such that $f(u) = \text{grad}_u F(u)$, we prove the existence of a pullback \mathfrak{D} -attractor in the space $\mathbb{S}_0^1(\Omega) \cap \mathbb{L}^p(\Omega)$ for the process $U(t, \tau)$. Moreover, we prove the boundedness of the pullback \mathfrak{D} -attractor in $\mathbb{L}^{2p-2}(\Omega)$ and in $\mathbb{S}_0^2(\Omega)$, and give estimates of the fractal dimension of the pullback \mathfrak{D} -attractor. It is worth noticing that our results, in particular, extend and improve some recent results on the existence of pullback \mathfrak{D} -attractors for the reaction-diffusion equations [3–5] and for the Grushin equations [8].

Let us explain the methods used in the paper. We first prove the existence of a family of pullback \mathfrak{D} -absorbing sets in $\mathbb{S}_0^1(\Omega)$. Thanks to the compactness of the embedding $\mathbb{S}_0^1(\Omega) \hookrightarrow \mathbb{L}^2(\Omega)$, we immediately get the existence of a pullback \mathfrak{D} -attractor in $\mathbb{L}^2(\Omega)$. When the system has a special gradient structure, we are able to prove the existence of a pullback \mathfrak{D} -attractor in $\mathbb{S}_0^1(\Omega) \cap \mathbb{L}^p(\Omega)$. To do this, we follow the general lines of the approach used in [8], with some modifications so that we can improve conditions imposed on the external force g . In particular, we use the asymptotic *a priori* estimate method initiated in [12] to testify the pullback asymptotic compactness of the corresponding process. Moreover, in this case we also prove the regularity of the pullback \mathfrak{D} -attractor in the spaces $\mathbb{L}^{2p-2}(\Omega)$ and $\mathbb{S}_0^1(\Omega)$. Finally,

using the recent results in [13], we give an estimate of the fractal dimension of the pullback \mathfrak{D} -attractor. It is noticed that we do not impose the restriction on the exponent p in (H2) as in [13].

The rest of the paper is organized as follows. In Section 2, for the convenience of the reader, we recall some concepts and results on pullback \mathfrak{D} -attractors which we will use. In Section 3, we prove the existence of a pullback \mathfrak{D} -attractor in $\mathbb{L}^2(\Omega)$ in the general case. In Section 4, under the additional assumption that the system has a gradient structure, we prove the regularity and fractal dimension estimates of the pullback \mathfrak{D} -attractor.

2. Preliminaries

2.1. Pullback Attractors

For convenience of the reader, we recall in this section some concepts and results on the theory of pullback \mathfrak{D} -attractors, which will be used in the paper.

Let X be a metric space with metric d . Denote by $\mathcal{B}(X)$ the set of all bounded subsets of X . For $A, B \subset X$, the Hausdorff semidistance between A and B is defined by

$$\text{dist}(A, B) = \sup_{x \in A} \inf_{y \in B} d(x, y). \quad (2.1)$$

Let $\{U(t, \tau) : t \geq \tau, \tau \in \mathbb{R}\}$ be a process in X , that is, $U(t, \tau) : X \rightarrow X$ such that $U(\tau, \tau) = \text{Id}$ and $U(t, s)U(s, \tau) = U(t, \tau)$ for all $t \geq s \geq \tau, \tau \in \mathbb{R}$. The process $\{U(t, \tau)\}$ is said to be norm-to-weak continuous if $U(t, \tau)x_n \rightharpoonup U(t, \tau)x$, as $x_n \rightarrow x$ in X , for all $t \geq \tau, \tau \in \mathbb{R}$. The following result is useful for verifying the norm-to-weak continuity of a process.

Proposition 2.1 (see [14]). *Let X, Y be two Banach spaces, X^*, Y^* be, respectively, their dual spaces. Assume that X is dense in Y , the injection $i : X \rightarrow Y$ is continuous and its adjoint $i^* : Y^* \rightarrow X^*$ is dense, and $\{U(t, \tau)\}$ is a continuous or weak continuous process on Y . Then $\{U(t, \tau)\}$ is norm-to-weak continuous on X if and only if for $t \geq \tau, \tau \in \mathbb{R}$, $U(t, \tau)$ maps a compact set of X to be a bounded set of X .*

Suppose that \mathfrak{D} is a nonempty class of parameterized sets $\widehat{\mathfrak{D}} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{B}(X)$.

Definition 2.2. The process $\{U(t, \tau)\}$ is said to be pullback \mathfrak{D} -asymptotically compact if for any $t \in \mathbb{R}$, any $\widehat{\mathfrak{D}} \in \mathfrak{D}$, and any sequence $\{\tau_n\}_n$ with $\tau_n \leq t$ for all n , and $\tau_n \rightarrow -\infty$, any sequence $x_n \in D(\tau_n)$, the sequence $\{U(t, \tau_n)x_n\}$ is relatively compact in X .

Definition 2.3. A process $\{U(t, \tau)\}$ is called pullback ω - \mathfrak{D} -limit compact if for any $\varepsilon > 0$, any $t \in \mathbb{R}$, and $\widehat{\mathfrak{D}} \in \mathfrak{D}$, there exists a $\tau_0 = \tau_0(\widehat{\mathfrak{D}}, \varepsilon, t) \leq t$ such that

$$\alpha\left(\bigcup_{\tau \leq \tau_0} U(t, \tau)D(\tau)\right) \leq \varepsilon, \quad (2.2)$$

where α is the Kuratowski measure of noncompactness of $B \in \mathcal{B}(X)$,

$$\alpha(B) = \inf\{\delta > 0 \mid B \text{ has a finite open cover of sets of diameter } \leq \delta\}. \quad (2.3)$$

Lemma 2.4 (see [3]). *A process $\{U(t, \tau)\}$ is pullback \mathfrak{D} -asymptotically compact if and only if it is ω - \mathfrak{D} -limit compact.*

Definition 2.5. A family of bounded sets $\widehat{\mathcal{B}} \in \mathfrak{D}$ is called pullback \mathfrak{D} -absorbing for the process $\{U(t, \tau)\}$ if for any $t \in \mathbb{R}$ and any $\widehat{\mathcal{D}} \in \mathfrak{D}$, there exists $\tau_0 = \tau_0(\widehat{\mathcal{D}}, t) \leq t$ such that

$$\bigcup_{\tau \leq \tau_0} U(t, \tau)D(\tau) \subset B(t). \quad (2.4)$$

Definition 2.6. A family $\widehat{\mathcal{A}} = \{A(t) : t \in \mathbb{R}\} \subset \mathcal{B}(X)$ is said to be a pullback \mathfrak{D} -attractor for $\{U(t, \tau)\}$ if

- (1) $A(t)$ is compact for all $t \in \mathbb{R}$;
- (2) $\widehat{\mathcal{A}}$ is invariant, that is, $U(t, \tau)A(\tau) = A(t)$, for all $t \geq \tau$;
- (3) $\widehat{\mathcal{A}}$ is pullback \mathfrak{D} -attracting, that is,

$$\lim_{\tau \rightarrow -\infty} \text{dist}(U(t, \tau)D(\tau), A(t)) = 0, \quad (2.5)$$

for all $\widehat{\mathcal{D}} \in \mathfrak{D}$ and all $t \in \mathbb{R}$;

- (4) if $\{C(t) : t \in \mathbb{R}\}$ is another family of closed attracting sets, then $A(t) \subset C(t)$, for all $t \in \mathbb{R}$.

Theorem 2.7 (see [3]). *Let $\{U(t, \tau)\}$ be a norm-to-weak continuous process such that $\{U(t, \tau)\}$ is pullback \mathfrak{D} -asymptotically compact. If there exists a family of pullback \mathfrak{D} -absorbing sets $\widehat{\mathcal{B}} = \{B(t) : t \in \mathbb{R}\} \in \mathfrak{D}$, then $\{U(t, \tau)\}$ has a unique pullback \mathfrak{D} -attractor $\widehat{\mathcal{A}} = \{A(t) : t \in \mathbb{R}\}$ and*

$$A(t) = \bigcap_{s \leq t} \overline{\bigcup_{\tau \leq s} U(t, \tau)B(\tau)}. \quad (2.6)$$

2.2. Fractal Dimension of Pullback Attractors

Given a compact $K \subset X$ and $\varepsilon > 0$, we denote by $N(K, \varepsilon)$ the minimum number of open balls in X with radius ε which are necessary to cover K .

Definition 2.8. For any nonempty compact set $K \subset X$, the fractal dimension of K is the number

$$\dim_f(K) := \lim_{\varepsilon \rightarrow 0} \frac{\log N(K, \varepsilon)}{\log 1/\varepsilon}. \quad (2.7)$$

Definition 2.9. A bounded subset $B_0 \subset H$ is called a uniformly pullback absorbing set for process $U(t, \tau)$ if for every $B \subset H$ is bounded, there exists a $\tau_0 \geq 0$ such that

$$U(t, t - \tau_0)B \in B_0, \quad \forall \tau \geq \tau_0, \quad (2.8)$$

here, τ_0 does not depend on the choice of t .

Theorem 2.10 (see [13]). Let $U(t, \tau)$ be a process in a separable Hilbert space H , B be a uniformly pullback absorbing set in H , $\hat{A} = \{A(t) : t \in \mathbb{R}\}$ be a pullback attractor for $U(t, \tau)$, if there exists a finite dimensional projection P in the space H such that

$$\|P(U(t, t - T_0)u_1 - U(t, t - T_0)u_2)\|_H \leq l(T_0)\|u_1 - u_2\|_H \quad (2.9)$$

for all $u_1, u_2 \in B$ and some $T_0, l(T_0) > 0$ and

$$\|(I - P)(U(t, t - T_0)u_1 - U(t, t - T_0)u_2)\|_H \leq \delta\|u_1 - u_2\| \quad (2.10)$$

for all $u_1, u_2 \in B$, where $\delta < 1$, T_0 and $l(T_0)$ are independent of the choice of t . Then the family of pullback attractors $\hat{A} = \{A(t) : t \in \mathbb{R}\}$ possesses a finite fractal dimension especially

$$\dim_f(A(t)) \leq \dim P \log \left(1 + \frac{8l(T_0)}{1 - \delta} \right) \left[\log \frac{2}{1 + \delta} \right]^{-1}, \quad \forall t \in \mathbb{R}. \quad (2.11)$$

3. Existence of Pullback \mathfrak{D} -Attractors in $\mathbb{L}^2(\Omega)$

Denote

$$\begin{aligned} V &:= L^p(\tau, T; \mathbb{L}^p(\Omega)) \cap L^2(\tau, T; \mathbb{S}_0^1(\Omega)), \\ V^* &:= L^2(\tau, T; \mathbb{S}^{-1}(\Omega)) + L^{p'}(\tau, T; \mathbb{L}^{p'}(\Omega)), \end{aligned} \quad (3.1)$$

where p' is the conjugate of p (i.e., $1/p + 1/p' = 1$).

Definition 3.1. Let $T > 0$ and $u_\tau \in \mathbb{L}^2(\Omega)$ be given. A function u is called a weak solution of problem (1.2) on (τ, T) if

$$\begin{aligned} u &\in V, \quad \frac{\partial u}{\partial t} \in V^*, \\ u|_{t=\tau} &= u_\tau \quad \text{a.e. in } \Omega, \end{aligned} \quad (3.2)$$

$$\begin{aligned} &\int_\tau^T \int_\Omega \left[(u_t, \varphi) + ((a \nabla_x u, \nabla_x \varphi)) + |x|^{2s} ((a \nabla_y u, \nabla_y \varphi)) + (f(u), \varphi) \right] dX dt \\ &= \int_\tau^T \int_\Omega (g(t), \varphi) dX dt \end{aligned}$$

for all test functions $\varphi \in V$.

One can prove that if $u \in V$ and $\partial u / \partial t \in V^*$, then $u \in C([0, T]; \mathbb{L}^2(\Omega))$ (see [10]). This makes the initial condition in (1.2) meaningful.

Theorem 3.2. Under assumptions (H1)–(H3), for any $\tau \in \mathbb{R}$, $T > \tau$, $u_\tau \in \mathbb{L}^2(\Omega)$ given, problem (1.2) has a unique weak solution u on (τ, T) . Moreover, the solution u exists on the interval $(\tau, +\infty)$ and the following inequality holds:

$$\|u\|_{\mathbb{L}^2(\Omega)}^2 \leq e^{-\lambda_1\beta(t-\tau)} \|u_\tau\|_{\mathbb{L}^2(\Omega)}^2 + \frac{2C_0|\Omega|}{\lambda_1\beta} + e^{-\lambda_1\beta t} \int_{-\infty}^t e^{\lambda_1\beta s} \|g(s)\|_{\mathbb{L}^2(\Omega)}^2 ds, \quad \forall t \geq \tau. \quad (3.3)$$

Proof. The existence and uniqueness of a weak solution to problem (1.2) are proved similarly to the scalar case in [10], so it is omitted here.

We now prove inequality (3.3). Multiplying (1.2) by u , integrating over Ω , and using (1.11), we have

$$\frac{1}{2} \frac{d}{dt} \|u\|_{\mathbb{L}^2(\Omega)}^2 + \beta \|u\|_{\mathbb{S}_0^1(\Omega)}^2 + \int_{\Omega} (f(u), u) dX = \int_{\Omega} (g(t), u) dX. \quad (3.4)$$

Using condition (1.3) and the Cauchy inequality, we obtain

$$\frac{d}{dt} \|u\|_{\mathbb{L}^2(\Omega)}^2 + 2\beta \|u\|_{\mathbb{S}_0^1(\Omega)}^2 + 2C_1 \|u\|_{\mathbb{L}^p(\Omega)}^p - 2C_0|\Omega| \leq \frac{2}{\beta\lambda_1} \|g(t)\|_{\mathbb{L}^2(\Omega)}^2 + \frac{\beta\lambda_1}{2} \|u\|_{\mathbb{L}^2(\Omega)}^2. \quad (3.5)$$

Because $\|u\|_{\mathbb{S}_0^1(\Omega)}^2 \geq \lambda_1 \|u\|_{\mathbb{L}^2(\Omega)}^2$, so (3.5) becomes

$$\frac{d}{dt} \|u\|_{\mathbb{L}^2(\Omega)}^2 + \lambda_1\beta \|u\|_{\mathbb{L}^2(\Omega)}^2 \leq 2C_0|\Omega| + \frac{1}{\lambda_1\beta} \|g(t)\|_{\mathbb{L}^2(\Omega)}^2. \quad (3.6)$$

Applying the Gronwall inequality we get (3.3). \square

Now, we can define the family of two-parameter mappings

$$\begin{aligned} U(t, \tau) : \mathbb{L}^2(\Omega) &\longrightarrow \mathbb{S}_0^1(\Omega) \cap \mathbb{L}^p(\Omega), \\ u_\tau &\longmapsto U(t, \tau)u_\tau, \end{aligned} \quad (3.7)$$

where $U(t, \tau)u_\tau = u(t)$ is the unique weak solution of (1.2) with the initial datum u_τ at time τ . Then U defines a continuous process on $\mathbb{L}^2(\Omega)$.

Let \mathcal{R} be the set of all functions $r : \mathbb{R} \rightarrow (0, +\infty)$ such that $\lim_{t \rightarrow -\infty} e^{\lambda_1\beta t} r^2(t) = 0$ and denote by \mathfrak{D} the class of all families $\widehat{\mathfrak{D}} = \{D(t) : t \in \mathbb{R}\} \subset \mathcal{B}(\mathbb{S}_0^1(\Omega))$ such that $D(t) \subset \widehat{B}(r(t))$ for some $r(t) \in \mathcal{R}$, where $\widehat{B}(r(t))$ is the closed ball in $\mathbb{S}_0^1(\Omega)$ with radius $r(t)$.

Lemma 3.3. *Under assumptions (H1)–(H3), there exists a constant $C > 0$ such that the solution u of problem (1.2) satisfies the following inequality for all $t > \tau$:*

$$\begin{aligned} \|u\|_{\mathbb{S}_0^1(\Omega)}^2 &\leq C \left(\left(1 + (t - \tau) + \frac{1}{t - \tau}\right) e^{-\alpha(t-\tau)} \|u_\tau\|_{\mathbb{L}^2(\Omega)}^2 + \left(1 + \frac{1}{t - \tau}\right) \right. \\ &\quad \left. + \left(1 + \frac{1}{t - \tau}\right) e^{-\alpha t} \int_{-\infty}^t e^{\alpha s} \|g(s)\|_{\mathbb{L}^2(\Omega)}^2 ds \right. \\ &\quad \left. + \left(1 + \frac{1}{t - \tau}\right) e^{-\alpha t} \int_{-\infty}^t \int_{-\infty}^s e^{\alpha r} \|g(r)\|_{\mathbb{L}^2(\Omega)}^2 dr ds \right), \end{aligned} \quad (3.8)$$

where $\alpha = \beta\lambda_1$. This implies that there exists a family of pullback \mathfrak{D} -absorbing sets in $\mathbb{S}_0^1(\Omega)$ for the process $\{\mathcal{U}(t, \tau)\}$.

Proof. We multiply (1.2) by $-G_s u$ and integrate over Ω . After some standard transformations we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|u(t)\|_{\mathbb{S}_0^1(\Omega)}^2 + \beta \|G_s u(t)\|_{\mathbb{L}^2(\Omega)}^2 \\ \leq \int_{\Omega} \left[(f(u(t)), \Delta_x u(t)) + |x|^{2s} (f(u), \Delta_y u(t)) \right] dX + \int_{\Omega} (g, G_s u(t)) dX. \end{aligned} \quad (3.9)$$

Without loss of generality, we may assume that $f(0) = 0$. Otherwise we can replace $f(u)$ by $\tilde{f}(u) = f(u) - f(0)$. The function \tilde{f} satisfies the same conditions with modified constants C_i ($i = 0, 1, 2, 3$), because $|f(0)| \leq C_2$ (see (1.4)). Hence, since $f(u(t))|_{\partial\Omega} = 0$, we get

$$\begin{aligned} \int_{\Omega} (f(u), \Delta_x u(t)) dX &= \sum_{k=1}^{N_1} \sum_{i=1}^m \int_{\Omega} f^i(u) \frac{\partial^2 u^i}{\partial X_k^2} dX \\ &= - \sum_{k=1}^{N_1} \sum_{i=1}^m \sum_{j=1}^m \int_{\Omega} \frac{\partial f^i}{\partial u^j}(u) \frac{\partial u^j}{\partial X_k} \frac{\partial u^i}{\partial X_k} dX \\ &= - \sum_{k=1}^{N_1} \int_{\Omega} \left(f_u(u) \frac{\partial u}{\partial X_k}, \frac{\partial u}{\partial X_k} \right) dX \\ &\leq C_3 \sum_{k=1}^{N_1} \int_{\Omega} \left| \frac{\partial u}{\partial X_k} \right|^2 dX = C_3 \int_{\Omega} |\nabla_x u|^2 dX, \end{aligned} \quad (3.10)$$

where we have used condition (1.5). Similarly, we have

$$\int_{\Omega} |x|^{2s} (f(u), \Delta_y u(t)) dX \leq C_3 \int_{\Omega} |x|^{2s} |\nabla_y u|^2 dX. \quad (3.11)$$

Hence

$$\int_{\Omega} (f(u), \Delta_x u(t)) dX + \int_{\Omega} |x|^{2s} (f(u), \Delta_y u(t)) \leq C_3 \|u(t)\|_{\mathbb{S}_0^1(\Omega)}^2. \quad (3.12)$$

By the Cauchy inequality we have

$$\int_{\Omega} (g, G_s u(t)) dX \leq \frac{1}{2\beta} \|g\|_{\mathbb{L}^2(\Omega)}^2 + \frac{\beta}{2} \|G_s u(t)\|_{\mathbb{L}^2(\Omega)}^2. \quad (3.13)$$

From (3.9)–(3.13) we obtain

$$\frac{d}{dt} \|u(t)\|_{\mathbb{S}_0^1(\Omega)}^2 + \beta \|G_s u(t)\|_{\mathbb{L}^2(\Omega)}^2 \leq 2C_3 \|u(t)\|_{\mathbb{S}_0^1(\Omega)}^2 + \frac{1}{\beta} \|g\|_{\mathbb{L}^2(\Omega)}^2, \quad (3.14)$$

thus,

$$\frac{d}{dt} \|u(t)\|_{\mathbb{S}_0^1(\Omega)}^2 + \alpha \|u(t)\|_{\mathbb{S}_0^1(\Omega)}^2 \leq 2C_3 \|u(t)\|_{\mathbb{S}_0^1(\Omega)}^2 + \frac{1}{\beta} \|g\|_{\mathbb{L}^2(\Omega)}^2, \quad (3.15)$$

where $\alpha = \beta\lambda_1$. Multiplying (3.15) by $(t - \tau)e^{\alpha t}$ and integrating from τ to t , we obtain

$$(t - \tau)e^{\alpha t} \|u\|_{\mathbb{S}_0^1(\Omega)}^2 \leq (2C_3(t - \tau) + 1) \int_{\tau}^t e^{\alpha s} \|u(s)\|_{\mathbb{S}_0^1(\Omega)}^2 ds + \frac{t - \tau}{\beta} \int_{-\infty}^t e^{\alpha s} \|g(s)\|_{\mathbb{L}^2(\Omega)}^2 ds. \quad (3.16)$$

Multiplying (3.3) by $\alpha e^{\alpha t}$ and integrating from τ to t , we have

$$\alpha \int_{\tau}^t e^{\alpha s} \|u(s)\|_{\mathbb{L}^2(\Omega)}^2 ds \leq \alpha(t - \tau)e^{\alpha \tau} \|u_{\tau}\|_{\mathbb{L}^2(\Omega)}^2 + \frac{2C_0|\Omega|}{\alpha} e^{\alpha t} + \int_{-\infty}^t \int_{-\infty}^s e^{\alpha r} \|g(r)\|_{\mathbb{L}^2(\Omega)}^2 dr ds. \quad (3.17)$$

Now, from (3.5) we get

$$\frac{d}{dt} \|u\|_{\mathbb{L}^2(\Omega)}^2 + \beta \|u\|_{\mathbb{S}_0^1(\Omega)}^2 \leq \frac{1}{\alpha} \|g(t)\|_{\mathbb{L}^2(\Omega)}^2 + 2C_1|\Omega|. \quad (3.18)$$

Multiplying this equation by $e^{\alpha t}$ and integrating from τ to t , we deduce that

$$\begin{aligned} e^{\alpha t} \|u(t)\|_{\mathbb{L}^2(\Omega)}^2 + \beta \int_{\tau}^t e^{\alpha s} \|u(s)\|_{\mathbb{S}_0^1(\Omega)}^2 ds \\ \leq e^{\alpha \tau} \|u_{\tau}\|_{\mathbb{L}^2(\Omega)}^2 + \frac{2C_1|\Omega|}{\alpha} e^{\alpha t} + \frac{1}{\alpha} \int_{\tau}^t e^{\alpha s} \|g(s)\|_{\mathbb{L}^2(\Omega)}^2 ds + \alpha \int_{\tau}^t e^{\alpha s} \|u(s)\|_{\mathbb{L}^2(\Omega)}^2 ds. \end{aligned} \quad (3.19)$$

Using (3.17), (3.19) becomes

$$\begin{aligned}
 & e^{\alpha t} \|u(t)\|_{\mathbb{L}^2(\Omega)}^2 + \beta \int_{\tau}^t e^{\alpha s} \|u(s)\|_{\mathbb{S}_0^1(\Omega)}^2 ds \\
 & \leq e^{\alpha \tau} \|u_{\tau}\|_{\mathbb{L}^2(\Omega)}^2 + \frac{(C_0 + C_1)|\Omega|}{\alpha} e^{\alpha t} + \alpha(t - \tau) e^{\alpha \tau} \|u_{\tau}\|_{\mathbb{L}^2(\Omega)}^2 \\
 & \quad + \frac{1}{\alpha} \int_{-\infty}^t e^{\alpha \tau} \|g(s)\|_{\mathbb{L}^2(\Omega)}^2 ds + \int_{-\infty}^t \int_{-\infty}^s e^{\alpha r} \|g(r)\|_{\mathbb{L}^2(\Omega)}^2 dr ds.
 \end{aligned} \tag{3.20}$$

Substituting (3.20) into (3.16) we obtain

$$\begin{aligned}
 \|u(t)\|_{\mathbb{S}_0^1(\Omega)}^2 & \leq e^{-\alpha(t-\tau)} \left(2C_3 + \frac{1}{\lambda_1} + \frac{2C_3}{\lambda_1}(t - \tau) + \frac{1}{\beta(t - \tau)} \right) \|u_{\tau}\|_{\mathbb{L}^2(\Omega)}^2 \\
 & \quad + \left(2C_3 + \frac{1}{t - \tau} \right) \frac{C_0 + C_1}{\alpha\beta} |\Omega| + \frac{1}{\alpha\beta} \left(2C_3 + \alpha + \frac{1}{t - \tau} \right) e^{-\alpha t} \int_{-\infty}^t e^{\alpha s} \|g(s)\|_{\mathbb{L}^2(\Omega)}^2 ds \\
 & \quad + \frac{1}{\beta} \left(2C_3 + \frac{1}{t - \tau} \right) e^{-\alpha t} \int_{-\infty}^t \int_{-\infty}^s e^{\alpha r} \|g(r)\|_{\mathbb{L}^2(\Omega)}^2 dr ds.
 \end{aligned} \tag{3.21}$$

Hence we get (3.8) with $C = C(\beta, C_0, C_1, C_3, \lambda_1)$.

Let $r_0^2(t)$ be the right-hand side of (3.8), and let $\widehat{B}_0(r_0(t))$ be the closed ball in $\mathbb{S}_0^1(\Omega)$ centered at 0 with radius $r_0(t)$. Obviously for any $\widehat{\mathfrak{D}} \in \mathfrak{D}$ and any $t \in \mathbb{R}$, by (3.8) there exists $\tau_0 = \tau_0(\widehat{\mathfrak{D}}) \leq t$ such that the solution u with initial datum $u_{\tau} \in \mathfrak{D}(\tau)$ at time τ satisfies $\|u(t)\|_{\mathbb{S}_0^1(\Omega)} \leq r_0(t)$ for all $\tau \leq \tau_0$; that is, $\widehat{\mathfrak{B}} = \{\widehat{B}_0(r_0(t)) : t \in \mathbb{R}\}$ is a family of bounded pullback \mathfrak{D} -absorbing sets in $\mathbb{S}_0^1(\Omega)$. \square

From the above lemma we deduce that the process $\{U(t, \tau)\}$ maps a compact set of $\mathbb{S}_0^1(\Omega)$ to be a bounded set of $\mathbb{S}_0^1(\Omega)$, and thus by Proposition 2.1, the process $\{U(t, \tau)\}$ is norm-to-weak continuous in $\mathbb{S}_0^1(\Omega)$. Since $\{U(t, \tau)\}$ has a family of pullback \mathfrak{D} -absorbing sets in $\mathbb{S}_0^1(\Omega)$ and the embedding $\mathbb{S}_0^1(\Omega) \hookrightarrow \mathbb{L}^2(\Omega)$ is compact, we immediately get the following.

Theorem 3.4. *Under assumptions (H1)–(H3), the process $\{U(t, \tau)\}$ associated to problem (1.2) has a pullback \mathfrak{D} -attractor in $\mathbb{L}^2(\Omega)$.*

4. Some Further Results in the Gradient Case

In this section, instead of (H1)–(H3), we assume that

(H1bis) $a = \beta I_m$, where I_m is the unit matrix and $\beta > 0$;

(H2bis) f satisfies (H2) and $f(u) = \text{grad}_u F(u) = ((\partial F / \partial u^1)(u), \dots, (\partial F / \partial u^m)(u))$, where $F : \mathbb{R}^m \rightarrow \mathbb{R}$ is a potential function satisfying

$$\bar{C}_1 |u|^p - \bar{C}_0 \leq F(u) \leq \bar{C}_2 |u|^p + \bar{C}_0, \quad \forall u \in \mathbb{R}^m, \tag{4.1}$$

with $\bar{C}_1, \bar{C}_2, \bar{C}_0$ being positive constants

(H3bis) $g \in W_{\text{loc}}^{1,2}(\mathbb{R}, \mathbb{L}^2(\Omega))$ satisfies

$$\int_{-\infty}^0 e^{\alpha t} \left(\|g(t)\|_{\mathbb{L}^2(\Omega)}^2 + \|g'(t)\|_{\mathbb{L}^2(\Omega)}^2 \right) dt < +\infty, \tag{4.2}$$

where $\alpha = \beta \lambda_1$.

The aim of this section is to prove that the pullback \mathfrak{D} -attractor obtained in Section 3 is more regular and has a finite fractal dimension.

4.1. Existence of a Pullback \mathfrak{D} -Attractor in $\mathbb{S}_0^1(\Omega) \cap \mathbb{L}^p(\Omega)$

Denote by \mathcal{R} the set of all functions $r : \mathbb{R} \rightarrow (0, +\infty)$ such that $\lim_{t \rightarrow -\infty} e^{\lambda_1 \beta t} r^2(t) = 0$ and denote by \mathfrak{D} the class of all families $\hat{\mathfrak{D}} = \{D(t) : t \in \mathbb{R}\} \subset \mathfrak{B}(\mathbb{S}_0^1(\Omega) \cap \mathbb{L}^p(\Omega))$ such that $D(t) \subset \hat{B}(r(t))$ for some $r(t) \in \mathcal{R}$, where $\hat{B}(r(t))$ is the closed ball in $\mathbb{S}_0^1(\Omega) \cap \mathbb{L}^p(\Omega)$ with radius $r(t)$. Thanks to the above gradient structure, one can prove the existence of a pullback \mathfrak{D} -attractor, not only in $\mathbb{L}^2(\Omega)$, but also in the space $\mathbb{S}_0^1(\Omega) \cap \mathbb{L}^p(\Omega)$ for the process $\{U(t, \tau)\}$.

We first prove the following.

Lemma 4.1. *Under assumptions (H1bis)–(H3bis), the solution u of problem (1.2) satisfies the following inequality for all $t > \tau$:*

$$\|u\|_{\mathbb{S}_0^1(\Omega)}^2 + \|u\|_{\mathbb{L}^p(\Omega)}^p \leq C \left(e^{-\alpha(t-\tau)} \|u_\tau\|_{\mathbb{L}^2(\Omega)}^2 + 1 + e^{-\alpha t} \int_{-\infty}^t e^{\alpha r} \|g(r)\|_{\mathbb{L}^2(\Omega)}^2 dr \right), \tag{4.3}$$

where $C = C(C_0, C_1, \bar{C}_1, \bar{C}_0, \beta, \lambda_1)$. This implies that there exists a family of pullback \mathfrak{D} -absorbing sets in $\mathbb{S}_0^1(\Omega) \cap \mathbb{L}^p(\Omega)$ for the process $\{U(t, \tau)\}$.

Proof. Using (3.5) with $\alpha = \lambda_1 \beta$ and the fact that $\|u\|_{\mathbb{S}_0^1(\Omega)}^2 \geq \lambda_1 \|u\|_{\mathbb{L}^2(\Omega)}^2$, we have

$$\frac{d}{dt} \|u\|_{\mathbb{L}^2(\Omega)}^2 + \alpha \|u\|_{\mathbb{L}^2(\Omega)}^2 + \frac{\beta}{2} \|u\|_{\mathbb{S}_0^1(\Omega)}^2 + 2C_1 \|u\|_{\mathbb{L}^p(\Omega)}^p \leq 2C_0 |\Omega| + \frac{2}{\alpha} \|g(t)\|_{\mathbb{L}^2(\Omega)}^2, \tag{4.4}$$

thus

$$\frac{d}{dt} \left(e^{\alpha t} \|u\|_{\mathbb{L}^2(\Omega)}^2 \right) + C e^{\alpha t} \left(\beta \|u\|_{\mathbb{S}_0^1(\Omega)}^2 + 2 \|u\|_{\mathbb{L}^p(\Omega)}^p \right) \leq 2C_0 |\Omega| e^{\alpha t} + \frac{2}{\alpha} e^{\alpha t} \|g(t)\|_{\mathbb{L}^2(\Omega)}^2. \tag{4.5}$$

Integrating from τ to s , $\tau \leq s \leq t + 1$, and in particular, we have

$$e^{\alpha s} \|u(s)\|_{\mathbb{L}^2(\Omega)}^2 \leq e^{\alpha \tau} \|u_\tau\|_{\mathbb{L}^2(\Omega)}^2 + 2 \frac{C_0 |\Omega|}{\alpha} e^{\alpha s} + \frac{2}{\alpha} \int_\tau^s e^{\alpha r} \|g(r)\|_{\mathbb{L}^2(\Omega)}^2 dr. \quad (4.6)$$

Furthermore, multiplying (4.5) from s to $s + 1$ and using (4.6) we obtain

$$\begin{aligned} & \int_s^{s+1} e^{\alpha r} \left(\beta \|u(r)\|_{\mathbb{S}_0^1(\Omega)}^2 + 2 \|u(r)\|_{\mathbb{L}^p(\Omega)}^p \right) dr \\ & \leq e^{\alpha s} \|u(s)\|_{\mathbb{L}^2(\Omega)}^2 + 2 \frac{C_0 |\Omega|}{\alpha} e^{\alpha s} + \frac{2}{\alpha} \int_s^{s+1} e^{\alpha r} \|g(r)\|_{\mathbb{L}^2(\Omega)}^2 dr \\ & \leq C \left(e^{\alpha \tau} \|u_\tau\|_{\mathbb{L}^2(\Omega)}^2 + e^{\alpha s} + \int_s^{s+1} e^{\alpha r} \|u(r)\|_{\mathbb{L}^2(\Omega)}^2 + \int_\tau^s e^{\alpha r} \|u(r)\|_{\mathbb{L}^2(\Omega)}^2 \right) \\ & \leq C \left(e^{\alpha \tau} \|u_\tau\|_{\mathbb{L}^2(\Omega)}^2 + e^{\alpha t} + \int_\tau^t e^{\alpha r} \|u(r)\|_{\mathbb{L}^2(\Omega)}^2 \right). \end{aligned} \quad (4.7)$$

By assumption (H2bis), then (4.7) becomes

$$\int_s^{s+1} e^{\alpha r} \left(\beta \|u(r)\|_{\mathbb{S}_0^1(\Omega)}^2 + 2F(u(r)) \right) dr \leq C \left(e^{\alpha \tau} \|u_\tau\|_{\mathbb{L}^2(\Omega)}^2 + e^{\alpha t} + \int_\tau^t e^{\alpha r} \|u(r)\|_{\mathbb{L}^2(\Omega)}^2 \right). \quad (4.8)$$

Multiplying (1.2) by $\partial u / \partial t$ and integrating over Ω , we have

$$\|u_t\|_{\mathbb{L}^2(\Omega)}^2 + \frac{1}{2} \frac{d}{dt} \left(\beta \|u\|_{\mathbb{S}_0^1(\Omega)}^2 + \int_\Omega F(u) dX \right) = \int_\Omega (g(t), u_t) dX \leq \frac{1}{2} \|g(t)\|_{\mathbb{L}^2(\Omega)}^2 + \frac{1}{2} \|u_t\|_{\mathbb{L}^2(\Omega)}^2, \quad (4.9)$$

thus

$$\frac{d}{dt} e^{\alpha t} \left(\beta \|u(t)\|_{\mathbb{S}_0^1(\Omega)}^2 + 2 \int_\Omega F(u(t)) dX \right) \leq e^{\alpha t} \left(\beta \|u(t)\|_{\mathbb{S}_0^1(\Omega)}^2 + 2 \int_\Omega F(u(t)) dX \right) + e^{\alpha t} \|g(t)\|_{\mathbb{L}^2(\Omega)}^2. \quad (4.10)$$

Using (4.8), (4.10), and the uniform Gronwall inequality, we get

$$e^{\alpha t} \left(\beta \|u(t)\|_{\mathbb{S}_0^1(\Omega)}^2 + 2 \int_\Omega F(u(t)) dX \right) \leq C \left(e^{\alpha \tau} \|u_\tau\|_{\mathbb{L}^2(\Omega)}^2 + e^{\alpha t} + \int_\tau^t e^{\alpha r} \|g(r)\|_{\mathbb{L}^2(\Omega)}^2 dr \right). \quad (4.11)$$

Now, using (H2bis) once again we have from (4.11) that

$$e^{at} \left(\beta \|u(t)\|_{\mathbb{S}_0^1(\Omega)}^2 + 2\|u(t)\|_{\mathbb{L}^p(\Omega)}^p \right) \leq C \left(e^{a\tau} \|u_\tau\|_{\mathbb{L}^2(\Omega)}^2 + e^{at} + \int_\tau^t e^{ar} \|g(r)\|_{\mathbb{L}^2(\Omega)}^2 dr \right). \quad (4.12)$$

Thus we obtain (4.3) with a suitable positive constant

$$C = C(C_0, C_1, \bar{C}_1, \bar{C}_0, \beta, \lambda_1). \quad (4.13)$$

Hence, by the argument as in the end of the proof of Lemma 3.3, we obtain a family of bounded pullback \mathfrak{D} -absorbing sets in $\mathbb{S}_0^1(\Omega) \cap \mathbb{L}^p(\Omega)$. \square

To prove that the process $\{U(t, \tau)\}$ is pullback \mathfrak{D} -asymptotically compact in $\mathbb{L}^p(\Omega)$, we need the following lemma.

Lemma 4.2 (see [8, Lemma 3.6]). *Let $\{U(t, \tau)\}$ be a norm-to-weak continuous process in $\mathbb{L}^2(\Omega)$ and $\mathbb{L}^p(\Omega)$, and let $\{U(t, \tau)\}$ satisfy the following two conditions:*

- (i) $\{U(t, \tau)\}$ is pullback \mathfrak{D} -asymptotically compact in $\mathbb{L}^2(\Omega)$;
- (ii) for any $\varepsilon > 0$, $\widehat{B} \in \mathfrak{D}$, there exist constants $M(\varepsilon, \widehat{B})$ and $\tau_0(\varepsilon, \widehat{B}) \leq t$ such that

$$\left(\int_{\Omega(|U(t, \tau)u_\tau| \geq M)} |U(t, \tau)u_\tau|^p \right)^{1/p} < \varepsilon, \quad \text{for any } u_\tau \in B(\tau), \tau \leq \tau_0. \quad (4.14)$$

Then $\{U(t, \tau)\}$ is pullback \mathfrak{D} -asymptotically compact in $\mathbb{L}^p(\Omega)$.

Theorem 4.3. *Under assumptions (H1bis)–(H3bis), the process $\{U(t, \tau)\}$ associated to problem (1.2) has a pullback \mathfrak{D} -attractor in $\mathbb{L}^p(\Omega)$.*

Proof. It is sufficient to show that the process $\{U(t, \tau)\}$ satisfies the condition (ii) in Lemma 4.2. We will give some formal calculations, a rigorous proof is done by use of Galerkin approximations and Lemma 11.2 in [15].

Let M be a positive number, we will write $u \geq M$ (or $u \leq -M$) as any component of u is greater than or equal to M (or as any component of u is less than or equal to $-M$).

Using (1.3), (1.4), and for $u \geq M$ large enough, we have

$$\begin{aligned} (f(u), u - M) &\geq (f(u), u) - M\sqrt{m}|f(u)| \\ &\geq \tilde{C}_3|u|^p - \tilde{C}_2|u|^{p-1} \\ &\geq \frac{C_4}{2}|u - M|^p + \frac{\alpha}{p}|u - M|^2, \quad \text{with } 0 < \frac{C_4}{2} < \tilde{C}_3, \end{aligned} \quad (4.15)$$

because $\lim_{|u| \rightarrow +\infty} (\tilde{C}_3|u|^p - \tilde{C}_2|u|^{p-1}) / ((C_4/2)|u - M|^p + (\alpha/p)|u - M|^2) > 1$.

Multiplying (1.2) by $(u - M)_+ |u - M|^{p-2}$ and integrating over Ω we obtain

$$\begin{aligned} & \frac{1}{p} \frac{d}{dt} \int_{\Omega(u \geq M)} |u - M|^p dX + \int_{\Omega(u \geq M)} \left((a \nabla_x(u - M), \nabla_x((u - M)|u - M|^{p-2})) \right) dX \\ & + \int_{\Omega(u \geq M)} |x|^{2s} \left((a \nabla_y(u - M), \nabla_y((u - M)|u - M|^{p-2})) \right) dX \\ & + \int_{\Omega(u \geq M)} |u - M|^{p-2} (f(u), u - M) dX = \int_{\Omega} (g(t), u - M) dX, \end{aligned} \quad (4.16)$$

where

$$(u - M)_+ := \begin{cases} u - M & \text{if } u \geq M, \\ 0, & \text{in other cases.} \end{cases} \quad (4.17)$$

On the other hand, by the Cauchy inequality, we have

$$\left| (g(t), (u - M)) |u - M|^{p-2} \right| \leq |u - M|^{p-1} |g(t)| \leq \frac{C_4}{2} |u - M|^{2p-2} + \frac{1}{2C_4} |g|^2, \quad (4.18)$$

which implies that

$$(g(t), u - M) |u - M|^{p-2} \geq -\frac{C_4}{2} |u - M|^{2p-2} - \frac{1}{2C_4} |g|^2. \quad (4.19)$$

Hence, from (4.15) and (4.19), we have

$$|u - M|^{p-2} [(f(u), u - M) + (g(t), u - M)] \geq \frac{\alpha}{p} |u - M|^p - \frac{1}{2C_4} |g|^2. \quad (4.20)$$

From (4.16), using (4.20) and noting that

$$\begin{aligned} & \int_{\Omega(u \geq M)} \left((a \nabla_x(u - M), \nabla_x((u - M)|u - M|^{p-2})) \right) dX \\ & + \int_{\Omega(u \geq M)} |x|^{2s} \left((a \nabla_y(u - M), \nabla_y((u - M)|u - M|^{p-2})) \right) dX \geq 0, \quad a = \beta I_m, \end{aligned} \quad (4.21)$$

we have

$$\frac{d}{dt} \|u - M\|_{\mathbb{L}^p(\Omega(u \geq M))}^p + \alpha \|u - M\|_{\mathbb{L}^p(\Omega(u \geq M))}^p \leq \frac{p}{2C_4} \|g\|_{\mathbb{L}^2(\Omega)}^2. \quad (4.22)$$

Now, multiplying the above inequality by $(t - \tau)e^{\alpha t}$ and integrating from τ to t , we get

$$\begin{aligned} & (t - \tau)e^{\alpha t} \|u - M\|_{\mathbb{L}^p(\Omega(u \geq M))}^p \\ & \leq \int_{\tau}^t e^{\alpha s} \|u - M\|_{\mathbb{L}^p(\Omega(u \geq M))}^p ds + \frac{p}{2C_4} (t - \tau) \int_{\tau}^t e^{\alpha s} \|g(s)\|_{\mathbb{L}^2(\Omega)}^2 ds \\ & \leq \int_{\tau}^t e^{\alpha s} \|u\|_{\mathbb{L}^p(\Omega)}^p ds + \frac{p}{2C_4} (t - \tau) \int_{\tau}^t e^{\alpha s} \|g(s)\|_{\mathbb{L}^2(\Omega)}^2 ds. \end{aligned} \tag{4.23}$$

Then

$$\|(u - M)_+\|_{\mathbb{L}^p(\Omega)}^p \leq \frac{1}{t - \tau} e^{-\alpha t} \int_{\tau}^t e^{\alpha s} \|u\|_{\mathbb{L}^p(\Omega)}^p ds + \frac{pe^{-\alpha t}}{2C_4} \int_{-\infty}^t e^{\alpha s} \|g(s)\|_{\mathbb{L}^2(\Omega)}^2 ds. \tag{4.24}$$

On the other hand, integrating (4.5) from τ to t , we have

$$\int_{\tau}^t e^{\alpha s} \|u(s)\|_{\mathbb{L}^p(\Omega)}^p ds \leq e^{\alpha \tau} \|u_{\tau}\|_{\mathbb{L}^2(\Omega)}^2 + 2\frac{C_0}{\alpha} |\Omega| e^{\alpha t} + \frac{1}{\alpha} \int_{\tau}^t e^{\alpha s} \|g(s)\|_{\mathbb{L}^2(\Omega)}^2 ds. \tag{4.25}$$

Therefore, substituting (4.25) into (4.24), we obtain

$$\|(u - M)_+\|_{\mathbb{L}^p(\Omega)}^p \leq C \left(\frac{e^{-\alpha(t-\tau)}}{t - \tau} \|u_{\tau}\|_{\mathbb{L}^2(\Omega)}^2 + \frac{1}{t - \tau} + \frac{1}{t - \tau} e^{-\alpha t} \int_{-\infty}^t e^{\alpha s} \|g(s)\|_{\mathbb{L}^2(\Omega)}^2 ds \right). \tag{4.26}$$

Hence, for any $\varepsilon > 0$, there exists $M_1 > 0$ and $\tau_1 < t$ such that for any $\tau < \tau_1$ and any $M \geq M_1$, we have

$$\int_{\Omega(u(t) \geq M)} |u - M|^p dx \leq \varepsilon. \tag{4.27}$$

Repeating the same step above, just taking $(u + M)_-$ instead of $(u - M)_+$, we deduce that there exist $M_2 > 0$ and $\tau_2 < t$ such that for any $\tau < \tau_2$ and any $M \geq M_2$,

$$\int_{\Omega(u(t) \leq -M)} |u + M|^p dx \leq \varepsilon, \tag{4.28}$$

where

$$(u + M)_- = \begin{cases} u + M, & u \leq -M, \\ 0, & \text{in other cases.} \end{cases} \tag{4.29}$$

Let $M_0 = \max\{M_1, M_2\}$ and $\tau_0 = \min\{\tau_1, \tau_2\}$, we obtain

$$\int_{\Omega(|u| \geq M)} (|u| - M)^p dx \leq \varepsilon \quad \text{for } \tau \leq \tau_0, M \geq M_0. \quad (4.30)$$

So, we have

$$\begin{aligned} \int_{\Omega(|u| \geq 2M)} |u|^p dx &= \int_{\Omega(|u| \geq 2M)} ((|u| - M) + M)^p dx \\ &\leq 2^{p-1} \left(\int_{\Omega(|u| \geq 2M)} (|u| - M)^p dx + \int_{\Omega(|u| \geq 2M)} M^p dx \right) \\ &\leq 2^{p-1} \left(\int_{\Omega(|u| \geq M)} (|u| - M)^p dx + \int_{\Omega(|u| \geq M)} (|u| - M)^p dx \right) \\ &\leq 2^p \varepsilon. \end{aligned} \quad (4.31)$$

This completes the proof. \square

To prove the existence of a pullback \mathfrak{D} -attractor in $\mathbb{S}_0^1(\Omega) \cap \mathbb{L}^p(\Omega)$, we need the following lemma.

Lemma 4.4. *Under assumptions (H1bis)–(H3bis), for any $t \in \mathbb{R}$ and any bounded subset $B \subset \mathbb{L}^2(\Omega)$, there exists a positive constant $T = T(B, t) \leq t$ such that*

$$\|u_t(t)\|_{\mathbb{L}^2(\Omega)}^2 \leq C \left(1 + e^{-at} \int_{-\infty}^t e^{\alpha s} \left(\|g(s)\|_{\mathbb{L}^2(\Omega)}^2 + \|g'(s)\|_{\mathbb{L}^2(\Omega)}^2 \right) ds \right), \quad (4.32)$$

for all $\tau \leq T(B, t)$ and all $u_\tau \in B$, where $C > 0$ is independent of t and B .

Proof. We give some formal calculations, a rigorous proof is done by use of Galerkin approximations and Lemma 11.2 in [15].

Differentiating (1.2) in time and setting $v = u_t$, we get

$$v_t - aG_s v + f'(u)v = g'(r). \quad (4.33)$$

Multiplying this inequality by $e^{\alpha r} v$ and integrating over Ω and using (1.11), we get

$$\frac{1}{2} \frac{d}{dr} \left(e^{\alpha r} \|v\|_{\mathbb{L}^2(\Omega)}^2 \right) + \beta e^{\alpha r} \|v\|_{\mathbb{S}_0^1(\Omega)}^2 + e^{\alpha r} \int_{\Omega} (f'(u)v, v) dX \leq \frac{\alpha}{2} e^{\alpha r} \|v\|_{\mathbb{L}^2(\Omega)}^2 + e^{\alpha r} \int_{\Omega} (g'(r), v) dX. \quad (4.34)$$

By the Cauchy inequality and using (1.5), we obtain that

$$\frac{d}{dr} \left(e^{\alpha r} \|v\|_{\mathbb{L}^2(\Omega)}^2 \right) \leq (2C_3 + \alpha + 1) e^{\alpha r} \|v\|_{\mathbb{L}^2(\Omega)}^2 + e^{\alpha r} \|g'(r)\|_{\mathbb{L}^2(\Omega)}^2. \quad (4.35)$$

Let $\tau \leq s \leq t - 1$. Using (3.5), we have

$$\frac{d}{ds} \left(e^{\alpha s} \|u\|_{L^2(\Omega)}^2 \right) + \beta \|u\|_{S_0^1(\Omega)}^2 + 2C_1 e^{\alpha s} \|u\|_{L^p(\Omega)}^p \leq \frac{1}{\alpha} e^{\alpha s} \|g(s)\|_{L^2(\Omega)}^2 + C_0 e^{\alpha s} |\Omega|. \quad (4.36)$$

By (H2bis) we then infer from the above inequality that

$$\frac{d}{ds} \left(e^{\alpha s} \|u\|_{L^2(\Omega)}^2 \right) + C \left(\beta e^{\alpha s} \|u\|_{S_0^1(\Omega)}^2 + 2e^{\alpha s} \int_{\Omega} F(u) dX \right) \leq C \left(e^{\alpha s} \|g(s)\|_{L^2(\Omega)}^2 + e^{\alpha s} \right). \quad (4.37)$$

Integrating this inequality from r to $r + 1$, we obtain

$$\begin{aligned} & \int_r^{r+1} \left(\beta e^{\alpha s} \|u\|_{S_0^1(\Omega)}^2 + 2e^{\alpha s} \int_{\Omega} F(u) dX \right) ds \\ & \leq C \left(e^{\alpha r} \|u(r)\|_{L^2(\Omega)}^2 + \int_r^{r+1} \left(e^{\alpha s} \|g(s)\|_{L^2(\Omega)}^2 + e^{\alpha s} \right) ds \right). \end{aligned} \quad (4.38)$$

On the other hand, integrating (4.36) from τ to t , we obtain

$$e^{\alpha t} \|u\|_{L^2(\Omega)}^2 \leq e^{\alpha \tau} \|u_{\tau}\|_{L^2(\Omega)}^2 + \frac{1}{\alpha} \int_{-\infty}^t e^{\alpha s} \|g(s)\|_{L^2(\Omega)}^2 ds + \frac{C_0 |\Omega|}{\alpha} e^{\alpha t}. \quad (4.39)$$

So, substituting (4.39) into (4.38), we deduce

$$\begin{aligned} & \int_r^{r+1} \left(\beta e^{\alpha s} \|u\|_{S_0^1(\Omega)}^2 + 2e^{\alpha s} \int_{\Omega} F(u) dX \right) ds \\ & \leq C \left(e^{\alpha \tau} \|u_{\tau}\|_{L^2(\Omega)}^2 + \int_{-\infty}^t e^{\alpha s} \|g(s)\|_{L^2(\Omega)}^2 ds + e^{\alpha t} \right) < \infty, \quad \forall r \in [\tau, t - 1]. \end{aligned} \quad (4.40)$$

Now multiplying (1.2) by $e^{\alpha r} v$ and integrating over Ω , we have

$$\begin{aligned} & e^{\alpha r} \|v\|_{L^2(\Omega)}^2 + \frac{d}{dr} \left(\beta e^{\alpha r} \|u\|_{S_0^1(\Omega)}^2 + 2e^{\alpha r} \int_{\Omega} F(u) dX \right) \\ & \leq \alpha \left(\beta e^{\alpha r} \|u\|_{S_0^1(\Omega)}^2 + 2e^{\alpha r} \int_{\Omega} F(u) dX \right) + e^{\alpha r} \|g(r)\|_{L^2(\Omega)}^2. \end{aligned} \quad (4.41)$$

So applying the uniform Gronwall inequality, we get

$$\beta e^{\alpha r} \|u\|_{S_0^1(\Omega)}^2 + 2e^{\alpha r} \int_{\Omega} F(u) dX \leq C \left(e^{\alpha \tau} \|u_{\tau}\|_{L^2(\Omega)}^2 + e^{\alpha t} + \int_{-\infty}^t e^{\alpha s} \|g(s)\|_{L^2(\Omega)}^2 ds \right). \quad (4.42)$$

Integrating (4.41) from r to $r + 1$ and by (4.40)–(4.42), we have

$$\int_r^{r+1} e^{\alpha s} \|v\|_{\mathbb{L}^2(\Omega)}^2 ds \leq C \left(e^{\alpha r} \|u_r\|_{\mathbb{L}^2(\Omega)}^2 + e^{\alpha t} + \int_{-\infty}^t e^{\alpha s} \|g(s)\|_{\mathbb{L}^2(\Omega)}^2 ds \right). \quad (4.43)$$

Therefore, by (4.35), (4.43), using the uniform Gronwall inequality once again, we get

$$e^{\alpha t} \|v\|_{\mathbb{L}^2(\Omega)}^2 \leq C \left(e^{\alpha \tau} \|u_\tau\|_{\mathbb{L}^2(\Omega)}^2 + e^{\alpha t} + \int_{-\infty}^t e^{\alpha s} (\|g(s)\|_{\mathbb{L}^2(\Omega)}^2 + \|g'(s)\|_{\mathbb{L}^2(\Omega)}^2) ds \right). \quad (4.44)$$

Hence we get (4.32). \square

Theorem 4.5. *Under assumptions (H1bis)–(H3bis), the process $\{U(t, \tau)\}$ associated to problem (1.2) has a pullback \mathfrak{D} -attractor in $\mathbb{S}_0^1(\Omega) \cap \mathbb{L}^p(\Omega)$.*

Proof. By Lemma 4.1, $\{U(t, \tau)\}$ has a family of bounded pullback \mathfrak{D} -absorbing sets in $\mathbb{S}_0^1(\Omega) \cap \mathbb{L}^p(\Omega)$. It remains to show that $\{U(t, \tau)\}$ is pullback \mathfrak{D} -asymptotically compact in $\mathbb{S}_0^1(\Omega) \cap \mathbb{L}^p(\Omega)$, that is, for any $t \in \mathbb{R}$, any $\tilde{\mathfrak{B}} \in \mathfrak{D}$, and any sequence $\tau_n \rightarrow -\infty$, any sequence $u_{\tau_n} \in B(\tau_n)$, the sequence $\{U(t, \tau_n)u_{\tau_n}\}$ is precompact in $\mathbb{S}_0^1(\Omega) \cap \mathbb{L}^p(\Omega)$. Thanks to Theorem 4.3, we need only to show that the sequence $\{U(t, \tau_n)u_{\tau_n}\}$ is precompact in $\mathbb{S}_0^1(\Omega)$.

Let $u_n(t) = U(t, \tau_n)u_{\tau_n}$. By Theorem 3.4, we can assume that $\{u_n(t)\}$ is a Cauchy sequence in $\mathbb{L}^2(\Omega)$. We have

$$\begin{aligned} & \|u_n(t) - u_m(t)\|_{\mathbb{S}_0^1(\Omega)}^2 \\ &= -\langle G_s u_n(t) - G_s u_m(t), u_n(t) - u_m(t) \rangle \\ &= -\left\langle \frac{du_n}{dt}(t) - \frac{du_m}{dt}(t), u_n(t) - u_m(t) \right\rangle - \langle f(u_n(t)) - f(u_m(t)), u_n(t) - u_m(t) \rangle \quad (4.45) \\ &\leq \left\| \frac{d}{dt} u_n(t) - \frac{d}{dt} u_m(t) \right\|_{\mathbb{L}^2(\Omega)}^2 \|u_n(t) - u_m(t)\|_{\mathbb{L}^2(\Omega)}^2 + C_3 \|u_n(t) - u_m(t)\|_{\mathbb{L}^2(\Omega)}^2, \end{aligned}$$

where we have used condition (1.5). Because $\{u_n(t)\}$ is a Cauchy sequence in $\mathbb{L}^2(\Omega)$ and by Lemma 4.4, one gets

$$\|u_n(t) - u_m(t)\|_{\mathbb{S}_0^1(\Omega)} \rightarrow 0, \quad \text{as } m, n \rightarrow \infty. \quad (4.46)$$

The proof is complete. \square

4.2. $\mathbb{L}^{2p-2}(\Omega)$ and $\mathbb{S}_0^2(\Omega)$ -Boundedness of the Pullback \mathfrak{D} -Attractor

First, we prove the existence of a family of pullback \mathfrak{D} -absorbing sets for process $U(t, \tau)$ in $\mathbb{L}^{2p-2}(\Omega)$.

Proposition 4.6. *Under assumptions (H1bis)–(H3bis), then for any $t \in \mathbb{R}$ and any bounded subset $B \subset \mathbb{L}^2(\Omega)$, there exists a positive constant $\tau_0 = \tau_0(B, t) \leq t$ such that*

$$\|u\|_{\mathbb{L}^{2p-2}(\Omega)}^{2p-2} \leq C \left(1 + \|g(t)\|_{\mathbb{L}^2(\Omega)}^2 + e^{-at} \int_{-\infty}^t e^{as} (\|g(s)\|_{\mathbb{L}^2(\Omega)}^2 + \|g'(s)\|_{\mathbb{L}^2(\Omega)}^2) ds \right) \quad (4.47)$$

for all $\tau \leq \tau_0$ and all $u_\tau \in B$, where $C > 0$ is independent of t and B .

Proof. Multiplying (1.2) by $|u|^{p-2}u$ and integrating over Ω we obtain

$$\begin{aligned} & \int_{\Omega} \left[\left((a \nabla_x u, \nabla_x (|u|^{p-2}u)) \right) + |x|^{2s} \left((a \nabla_y u, \nabla_y (|u|^{p-2}u)) \right) \right] dX + \int_{\Omega} (f(u), u) |u|^{p-2} dX \\ & = - \int_{\Omega} (u_t, |u|^{p-2}u) dX + \int_{\Omega} (g(t), |u|^{p-2}u) dX. \end{aligned} \quad (4.48)$$

By the Cauchy inequality, (1.3) and note that

$$\int_{\Omega} \left[\left((a \nabla_x u, \nabla_x (|u|^{p-2}u)) \right) + |x|^{2s} \left((a \nabla_y u, \nabla_y (|u|^{p-2}u)) \right) \right] dX \geq 0; \quad \text{here } a = \beta I_m, \quad (4.49)$$

then we get

$$C_1 \|u\|_{\mathbb{L}^{2p-2}(\Omega)}^{2p-2} \leq \frac{1}{C_1} \|u_t\|_{\mathbb{L}^2(\Omega)}^2 + \frac{1}{C_1} \|g(t)\|_{\mathbb{L}^2}^2 + \frac{C_1}{2} \|u\|_{\mathbb{L}^{2p-2}}^{2p-2}. \quad (4.50)$$

Hence, by (4.32) we deduce from (4.16) that

$$\frac{C_1}{2} \|u\|_{\mathbb{L}^{2p-2}(\Omega)}^{2p-2} \leq \frac{1}{C_1} C \left(1 + e^{-at} \int_{-\infty}^t e^{as} (\|g(s)\|_{\mathbb{L}^2(\Omega)}^2 + \|g'(s)\|_{\mathbb{L}^2(\Omega)}^2) ds \right) + \frac{1}{C_1} \|g(t)\|_{\mathbb{L}^2(\Omega)}^2. \quad (4.51)$$

Therefore, we get (4.47) and the proof is complete. □

And now, we denote by $\mathbb{S}_0^2(\Omega)$ the closure of $(C_0^\infty(\Omega))^m$ in the norm

$$\|u\|_{\mathbb{S}_0^2(\Omega)} = \left(\int_{\Omega} (|\Delta_x u|^2 + |x|^{2s} |\Delta_y u|^2) dX \right)^{1/2}. \quad (4.52)$$

It is easy to see that $\mathbb{S}_0^2(\Omega)$ is a Banach space endowed with the above norm. We now prove the $\mathbb{S}_0^2(\Omega)$ -boundedness of the pullback \mathfrak{D} -attractor.

First, we recall a lemma (see [15]) which is necessary for our proof.

Lemma 4.7. Let X, Y be Banach spaces such that X is reflexive, and the inclusion $X \subset Y$ is continuous. Assume that $\{u_n\}$ is a bounded sequence in $L^\infty(\tau, T; X)$ such that $u_n \rightharpoonup u$ weakly in $L^q(\tau, T; X)$ for some $q \in [1, +\infty)$ and $u \in C([\tau, T]; Y)$. Then, $u(t) \in X$ for all $t \in [\tau, T]$ and $\|u(t)\|_X \leq \sup_{n \geq 1} \|u_n\|_{L^\infty(\tau, T; X)}$, for all $t \in [\tau, T]$.

Let $u_n(t)$ be the Galerkin approximations of the solution $u(t)$ of (1.2) then by Lemma 4.7 with noticing that $u_n = U(t, \tau)u_{n\tau} \rightharpoonup u = U(t, \tau)u_\tau$ in $L^2(\tau, T; \mathbb{S}_0^1(\Omega))$ and the inclusion $\mathbb{S}_0^2(\Omega) \subset \mathbb{S}_0^1(\Omega)$ is continuous, we only need the estimation on $u(t) = U(t, \tau)u_\tau$.

Theorem 4.8. Under assumptions (H1bis)–(H3bis), the pullback \mathfrak{D} -attractor $\widehat{\mathcal{A}} = \{A(t) : t \in \mathbb{R}\}$ in $\mathbb{S}_0^1(\Omega) \cap \mathbb{L}^p(\Omega)$ of the process $\{U(t, \tau)\}$ is bounded in $\mathbb{S}_0^2(\Omega)$. More precisely, for any $\tau < T_1 < T_2$, the set $\bigcup_{t \in [T_1, T_2]} A(t)$ is a bounded subset of $\mathbb{S}_0^2(\Omega)$.

Proof. Let us fix a bounded set $B \subset \mathbb{L}^2(\Omega)$, $\tau \in \mathbb{R}$, $\varepsilon > 0$, $t > \tau + \varepsilon$ and $u_\tau \in B$. Multiplying the first equation in (1.2) by $G_s u$ and integrating over Ω , we have

$$\begin{aligned} \int_{\Omega} (aG_s u(r), G_s u(r)) dX &= \int_{\Omega} (u'_r, G_s u(r)) dX + \int_{\Omega} (f(u(r)), G_s u(r)) dX \\ &\quad - \int_{\Omega} (g(r), G_s u(r)) dX. \end{aligned} \quad (4.53)$$

By the Cauchy inequality we have

$$\begin{aligned} - \int_{\Omega} (g(r), G_s u(r)) dX &\leq \frac{2}{\beta} \|g(r)\|_{\mathbb{L}^2(\Omega)}^2 + \frac{\beta}{8} \|G_s u(r)\|_{\mathbb{L}^2(\Omega)}^2, \\ \int_{\Omega} (u'_r, G_s u(r)) dX &\leq \frac{2}{\beta} \|u'_r\|_{\mathbb{S}_0^1(\Omega)}^2 + \frac{\beta}{8} \|G_s u(r)\|_{\mathbb{L}^2(\Omega)}^2. \end{aligned} \quad (4.54)$$

Using (3.12), (1.12), and (4.54), then from (4.53) we get

$$\begin{aligned} \beta \|G_s u(r)\|_{\mathbb{L}^2(\Omega)}^2 &\leq C_3 \|u(r)\|_{\mathbb{S}_0^1(\Omega)}^2 + \frac{2}{\beta} (\|u'_r\|_{\mathbb{L}^2(\Omega)}^2 + \|g(r)\|_{\mathbb{L}^2(\Omega)}^2) + \frac{\beta}{4} \|G_s u(r)\|_{\mathbb{L}^2(\Omega)}^2 \\ &= C_3 \int_{\Omega} (u(r), -G_s u(r)) dX + \frac{2}{\beta} (\|u'_r\|_{\mathbb{L}^2(\Omega)}^2 + \|g(r)\|_{\mathbb{L}^2(\Omega)}^2) + \frac{\beta}{4} \|G_s u(r)\|_{\mathbb{L}^2(\Omega)}^2 \\ &\leq \frac{2C_3^2}{\beta} \|u(r)\|_{\mathbb{L}^2(\Omega)}^2 + \frac{2}{\beta} \|u'_r\|_{\mathbb{L}^2(\Omega)}^2 + \frac{2}{\beta} \|g(r)\|_{\mathbb{L}^2(\Omega)}^2 + \frac{\beta}{2} \|G_s u(r)\|_{\mathbb{L}^2(\Omega)}^2. \end{aligned} \quad (4.55)$$

Hence,

$$\frac{\beta}{2} \|G_s u(r)\|_{\mathbb{L}^2(\Omega)}^2 \leq \frac{2C_3^2}{\beta} \|u(r)\|_{\mathbb{L}^2(\Omega)}^2 + \frac{2}{\beta} \|u'_r\|_{\mathbb{L}^2(\Omega)}^2 + \frac{2}{\beta} \|g(r)\|_{\mathbb{L}^2(\Omega)}^2. \quad (4.56)$$

Differentiating the first equation in (1.2) in time t and setting $v(r) = u'(r)$, then multiplying by $v(r)$ and using (1.11) we get

$$\begin{aligned} \frac{1}{2} \frac{d}{dr} \|v(r)\|_{\mathbb{L}^2(\Omega)}^2 + \beta \|v(r)\|_{\mathbb{S}_0^1(\Omega)}^2 &\leq - \int_{\Omega} (f'_u(u)v(r), v(r)) dX + \int_{\Omega} (g'(r), v(r)) dX \\ &\leq C_3 \|v(r)\|_{\mathbb{L}^2(\Omega)}^2 + \frac{1}{2} \|v(r)\|_{\mathbb{L}^2(\Omega)}^2 + \frac{1}{2} \|g'(r)\|_{\mathbb{L}^2(\Omega)}^2. \end{aligned} \quad (4.57)$$

Hence,

$$\frac{d}{dr} \|v(r)\|_{\mathbb{L}^2(\Omega)}^2 \leq (2C_3 + 1) \|v(r)\|_{\mathbb{L}^2(\Omega)}^2 + \|g'(r)\|_{\mathbb{L}^2(\Omega)}^2. \quad (4.58)$$

Integrating the above inequality, we have

$$\|v(r)\|_{\mathbb{L}^2(\Omega)}^2 \leq \|v(s)\|_{\mathbb{L}^2(\Omega)}^2 + (2C_3 + 1) \int_{\tau+\varepsilon/2}^t \|v(\theta)\|_{\mathbb{L}^2(\Omega)}^2 d\theta + \int_{\tau+\varepsilon/2}^t \|g'(\theta)\|_{\mathbb{L}^2(\Omega)}^2 d\theta, \quad (4.59)$$

for all $\tau + \varepsilon/2 \leq s \leq r \leq t$.

Now, integrating with respect to s between $\tau + \varepsilon/2$ and r , we get

$$\begin{aligned} &\left(r - \tau - \frac{\varepsilon}{2}\right) \|v(r)\|_{\mathbb{L}^2(\Omega)}^2 \\ &\leq \int_{\tau+\varepsilon/2}^r \|v(s)\|_{\mathbb{L}^2(\Omega)}^2 ds \\ &\quad + (2C_3 + 1) \left(r - \tau - \frac{\varepsilon}{2}\right) \int_{\tau+\varepsilon/2}^t \|v(\theta)\|_{\mathbb{L}^2(\Omega)}^2 d\theta + \left(r - \tau - \frac{\varepsilon}{2}\right) \int_{\tau+\varepsilon/2}^t \|g'(\theta)\|_{\mathbb{L}^2(\Omega)}^2 d\theta \\ &\leq \left((2C_3 + 1) \left(t - \tau - \frac{\varepsilon}{2}\right) + 1\right) \int_{\tau+\varepsilon/2}^t \|v(\theta)\|_{\mathbb{L}^2(\Omega)}^2 d\theta + \left(r - \tau - \frac{\varepsilon}{2}\right) \int_{\tau+\varepsilon/2}^t \|g'(\theta)\|_{\mathbb{L}^2(\Omega)}^2 d\theta, \end{aligned} \quad (4.60)$$

for all $\tau + \varepsilon/2 \leq r \leq t$, and in particular, for all $r \in [\tau + \varepsilon, t]$ we have that (from the above estimate)

$$\|v(r)\|_{\mathbb{L}^2(\Omega)}^2 \leq \frac{2}{\varepsilon} \left((2C_3 + 1) \left(t - \tau - \frac{\varepsilon}{2}\right) + 1\right) \int_{\tau+\varepsilon/2}^t \|v(\theta)\|_{\mathbb{L}^2(\Omega)}^2 d\theta + \int_{\tau}^t \|g'(\theta)\|_{\mathbb{L}^2(\Omega)}^2 d\theta. \quad (4.61)$$

On the other hand, multiplying the first equation in (1.2) by $v(r)$ and integrating over Ω , we deduce that

$$\|v(r)\|_{\mathbb{L}^2(\Omega)}^2 + \frac{\beta}{2} \frac{d}{dr} \|u(r)\|_{\mathbb{S}_0^1(\Omega)}^2 + \int_{\Omega} (f(u, v)) dX \leq \int_{\Omega} (g, v) dX, \quad (4.62)$$

where we have used (1.11). Using the Cauchy inequality and condition (H2bis), then (4.62) becomes

$$\|v(r)\|_{\mathbb{L}^2(\Omega)}^2 + \frac{d}{dr} \left(\beta \|u(r)\|_{\mathbb{S}_0^1(\Omega)}^2 + 2 \int_{\Omega} F(u(r)) dX \right) \leq \|g(r)\|_{\mathbb{L}^2(\Omega)}^2. \quad (4.63)$$

Integrating from $\tau + \varepsilon/2$ to t we have

$$\begin{aligned} \int_{\tau+\varepsilon/2}^t \|v(\theta)\|_{\mathbb{L}^2(\Omega)}^2 d\theta + \beta \|u(t)\|_{\mathbb{S}_0^1(\Omega)}^2 + 2 \int_{\Omega} F(u(t)) dX &\leq \beta \left\| u\left(\tau + \frac{\varepsilon}{2}\right) \right\|_{\mathbb{S}_0^1(\Omega)}^2 \\ &+ 2 \int_{\Omega} F\left(u\left(\tau + \frac{\varepsilon}{2}\right)\right) dX + \int_{\tau+\varepsilon/2}^t \|g(\theta)\|_{\mathbb{L}^2(\Omega)}^2, \end{aligned} \quad (4.64)$$

and hence because of (4.1), we get

$$\begin{aligned} \int_{\tau+\varepsilon/2}^t \|v(\theta)\|_{\mathbb{L}^2(\Omega)}^2 d\theta &\leq \beta \left\| u\left(\tau + \frac{\varepsilon}{2}\right) \right\|_{\mathbb{S}_0^1(\Omega)}^2 + 2\bar{C}_2 \left\| u\left(\tau + \frac{\varepsilon}{2}\right) \right\|_{\mathbb{L}^p(\Omega)}^p \\ &+ 4\bar{C}_0 |\Omega| + \int_{\tau}^t \|g(\theta)\|_{\mathbb{L}^2(\Omega)}^2. \end{aligned} \quad (4.65)$$

Now, substituting (4.65) into (4.61) we deduce

$$\begin{aligned} \|v(\theta)\|_{\mathbb{L}^2(\Omega)}^2 &\leq \frac{2}{\varepsilon} \left((2C_3 + 1) \left(t - \tau - \frac{\varepsilon}{2} \right) + 1 \right) \\ &\times \left(\beta \left\| u\left(\tau + \frac{\varepsilon}{2}\right) \right\|_{\mathbb{S}_0^1(\Omega)}^2 + 2\bar{C}_2 \left\| u\left(\tau + \frac{\varepsilon}{2}\right) \right\|_{\mathbb{L}^p(\Omega)}^p + 4\bar{C}_0 |\Omega| + \int_{\tau}^t \|g(\theta)\|_{\mathbb{L}^2(\Omega)}^2 d\theta \right) \\ &+ \int_{\tau}^t \|g'(\theta)\|_{\mathbb{L}^2(\Omega)}^2 d\theta, \end{aligned} \quad (4.66)$$

for all $r \in [\tau + \varepsilon, t]$. Finally, from (4.66) and (4.56) we obtain

$$\begin{aligned} \|G_s u(r)\|_{\mathbb{L}^2(\Omega)}^2 &\leq \frac{8}{\beta^2 \varepsilon} \left((2C_3 + 1) \left(t - \tau - \frac{\varepsilon}{2} \right) + 1 \right) \\ &\times \left(\beta \left\| u\left(\tau + \frac{\varepsilon}{2}\right) \right\|_{\mathbb{S}_0^1(\Omega)}^2 + 2\bar{C}_2 \left\| u\left(\tau + \frac{\varepsilon}{2}\right) \right\|_{\mathbb{L}^p(\Omega)}^p + 4\bar{C}_0 |\Omega| + \int_{\tau}^t \|g(\theta)\|_{\mathbb{L}^2(\Omega)}^2 d\theta \right) \\ &+ \frac{2}{\beta} \int_{\tau}^t \|g'(\theta)\|_{\mathbb{L}^2(\Omega)}^2 d\theta + \frac{2}{\beta^2} \left(C_3^2 \|u(r)\|_{\mathbb{L}^2(\Omega)}^2 + 2 \|g(r)\|_{\mathbb{L}^2(\Omega)}^2 \right), \quad \forall r \in [\tau + \varepsilon, t]. \end{aligned} \quad (4.67)$$

Because $\|u\|_{\mathbb{S}_0^2(\Omega)}^2 = \|G_s u\|_{\mathbb{L}^2(\Omega)}^2$ then from (4.67), the proof is complete. \square

4.3. Fractal Dimensional Estimates of the Pullback \mathfrak{D} -Attractor

Theorem 4.9. *Under assumptions (H1bis)–(H3bis), the process $U(t, \tau)$ possesses a pullback \mathfrak{D} -attractor $\mathcal{A}_{\mathbb{L}^2(\Omega)}$ which has a finite fractal dimension in $\mathbb{L}^2(\Omega)$ and*

$$\dim_f(A(t)) \leq k \log \left(1 + \frac{8 \cdot e^{2C_3}}{1 - \delta} \right) \left[\log \frac{2}{1 + \delta} \right]^{-1}, \quad \forall t \in \mathbb{R}, \quad (4.68)$$

where $\delta < 1, k \in \mathbb{N}$, and C_3 in (1.5).

Proof. Let $H_k = \text{span}\{e_1, e_2, \dots, e_k\} \subset \mathbb{L}^2(\Omega)$ and $P_k : \mathbb{L}^2(\Omega) \rightarrow H_k$ be the orthogonal projection, where $e_1, e_2, \dots, e_j, \dots$ are the eigenvectors of the operator $-G_s$ corresponding to eigenvalues $\{\lambda_j\}_{j=1}^\infty$ such that $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots \leq \lambda_j \leq \dots$ and $\lambda_j \rightarrow +\infty$ as $j \rightarrow +\infty$.

From (4.3), we can easily show that there exists a uniformly pullback absorbing set \mathcal{B} of process $U(t, \tau)$ in $\mathbb{S}_0^1(\Omega)$. We set $u_1(t) = U(t, \tau)u_{1\tau}$ and $u_2(t) = U(t, \tau)u_{2\tau}$ to be solutions associated to problem (1.2) with initial datum $u_{1\tau}, u_{2\tau} \in \mathcal{B}$.

Let $w = u_1 - u_2$, because u_1, u_2 being two solutions of (1.2) then we have

$$\frac{\partial w}{\partial t} - aG_s w + f(u_1) - f(u_2) = 0. \quad (4.69)$$

Multiplying (4.69) with w and integrating over Ω then we have

$$\frac{1}{2} \frac{d}{dt} \|w(t)\|_{\mathbb{L}^2(\Omega)}^2 + \beta \|w\|_{\mathbb{S}_0^1(\Omega)}^2 + \int_{\Omega} (f(u_1) - f(u_2), w) dX \leq 0, \quad (4.70)$$

here, we have used (1.11).

Using (1.5) then we have

$$\frac{d}{dt} \|w(t)\|_{\mathbb{L}^2(\Omega)}^2 \leq 2C_3 \|w(t)\|_{\mathbb{L}^2(\Omega)}^2. \quad (4.71)$$

Thus,

$$\|w(t)\|_{\mathbb{L}^2(\Omega)}^2 \leq e^{2C_3(t-\tau)} \|w(\tau)\|_{\mathbb{L}^2(\Omega)}^2. \quad (4.72)$$

Let $w(t) = w_1(t) + w_2(t)$ where $w_1(t) := P_k w(t)$ and $w_2(t) := (I - P_k)w(t)$. Therefore, by (4.72) we have

$$\|w_1(t)\|_{\mathbb{L}^2(\Omega)}^2 \leq e^{2C_3(t-\tau)} \|w(\tau)\|_{\mathbb{L}^2(\Omega)}^2. \quad (4.73)$$

Now, taking the inner product of (4.69) with w_2 in $\mathbb{L}^2(\Omega)$, we have

$$\frac{1}{2} \frac{d}{dt} \|w_2(t)\|_{\mathbb{L}^2(\Omega)}^2 + \beta \|w_2\|_{\mathbb{S}_0^1(\Omega)}^2 \leq - \int (f(u_1) - f(u_2), w_2) dX. \quad (4.74)$$

Using the Hölder inequality and (1.4), we have

$$\begin{aligned}
 -\int_{\Omega} (f(u_1) - f(u_2), w_2) dX &\leq \int_{\Omega} |f(u_1) - f(u_2)| |w_2| dX \\
 &\leq \left(\int_{\Omega} |f(u_1) - f(u_2)|^2 dX \right)^{1/2} \left(\int_{\Omega} |w_2|^2 dX \right)^{1/2} \\
 &\leq C \left(\int_{\Omega} (1 + |u_1|^{2p-2} + |u_2|^{2p-2}) dX \right)^{1/2} \|w_2\|_{\mathbb{L}^2(\Omega)} \\
 &\leq C \left(1 + \|u_1\|_{\mathbb{L}^{2p-2}(\Omega)}^{2p-2} + \|u_2\|_{\mathbb{L}^{2p-2}(\Omega)}^{2p-2} \right)^{1/2} \|w\|_{\mathbb{L}^2(\Omega)} \\
 &\leq C \left(1 + \|u_1\|_{\mathbb{L}^{2p-2}(\Omega)}^{2p-2} + \|u_2\|_{\mathbb{L}^{2p-2}(\Omega)}^{2p-2} \right) \|w\|_{\mathbb{L}^2(\Omega)}.
 \end{aligned} \tag{4.75}$$

Therefore, by (4.74), (4.75), and Proposition 4.6 we obtain

$$\begin{aligned}
 &\frac{1}{2} \frac{d}{dt} \|w_2(t)\|_{\mathbb{L}^2(\Omega)}^2 + \beta \|w_2\|_{\mathbb{S}_0^1(\Omega)}^2 \\
 &\leq C \left(1 + \|g(t)\|_{\mathbb{L}^2(\Omega)}^2 + e^{-at} \int_{-\infty}^t e^{\alpha s} (\|g(s)\|_{\mathbb{L}^2(\Omega)}^2 + \|g'(s)\|_{\mathbb{L}^2(\Omega)}^2) ds \right) \|w(t)\|_{\mathbb{L}^2(\Omega)}.
 \end{aligned} \tag{4.76}$$

Because $\|w_2(t)\|_{\mathbb{S}_0^1(\Omega)}^2 \geq \lambda_k \|w_2(t)\|_{\mathbb{L}^2(\Omega)}^2$, then (4.76) implies that

$$\begin{aligned}
 &\frac{d}{dt} \|w_2(t)\|_{\mathbb{L}^2(\Omega)}^2 + 2\beta\lambda_k \|w_2\|_{\mathbb{L}^2(\Omega)}^2 \\
 &\leq 2C \left(1 + \|g(t)\|_{\mathbb{L}^2(\Omega)}^2 + e^{-at} \int_{-\infty}^t e^{\alpha s} (\|g(s)\|_{\mathbb{L}^2(\Omega)}^2 + \|g'(s)\|_{\mathbb{L}^2(\Omega)}^2) ds \right) \|w(t)\|_{\mathbb{L}^2(\Omega)}.
 \end{aligned} \tag{4.77}$$

Now, multiplying (4.77) by $e^{\beta\lambda_k t}$ and integrating from τ to t , we get

$$\begin{aligned}
 \|w_2(t)\|_{\mathbb{L}^2(\Omega)}^2 &\leq e^{-\beta\lambda_k(t-\tau)} \|w_2(\tau)\|_{\mathbb{L}^2(\Omega)}^2 + 2C e^{-\beta\lambda_k t} \int_{\tau}^t e^{-\beta\lambda_k s} \\
 &\quad \times \left[1 + \|g(s)\|_{\mathbb{L}^2(\Omega)}^2 + e^{-at} \int_{-\infty}^s e^{ar} (\|g(r)\|_{\mathbb{L}^2(\Omega)}^2 + \|g'(r)\|_{\mathbb{L}^2(\Omega)}^2) dr \right] \\
 &\quad \times \|w(s)\|_{\mathbb{L}^2(\Omega)} ds.
 \end{aligned} \tag{4.78}$$

Using (4.73) we have

$$\begin{aligned}
 \|w_2(t)\|_{\mathbb{L}^2(\Omega)}^2 &\leq e^{-\beta\lambda_k(t-\tau)}\|w(\tau)\|_{\mathbb{L}^2(\Omega)}^2 + C\|w(\tau)\|_{\mathbb{L}^2(\Omega)}e^{-\beta\lambda_k t} \int_{\tau}^t e^{\beta\lambda_k s} e^{C_3(s-\tau)} \\
 &\quad \times \left[1 + \|g(s)\|_{\mathbb{L}^2(\Omega)}^2 + e^{-\alpha s} \int_{-\infty}^s e^{\alpha r} \left(\|g(r)\|_{\mathbb{L}^2(\Omega)}^2 + \|g'(r)\|_{\mathbb{L}^2(\Omega)}^2 \right) dr \right] ds \\
 &\leq e^{-\beta\lambda_k(t-\tau)}\|w(\tau)\|_{\mathbb{L}^2(\Omega)}^2 + C\|w(\tau)\|_{\mathbb{L}^2(\Omega)}e^{C_3(t-\tau)} \int_{-\infty}^t e^{-\beta\lambda_k(t-s)} \\
 &\quad \times \left[1 + \|g(s)\|_{\mathbb{L}^2(\Omega)}^2 + e^{-\alpha s} \int_{-\infty}^s e^{\alpha r} \left(\|g(r)\|_{\mathbb{L}^2(\Omega)}^2 + \|g'(r)\|_{\mathbb{L}^2(\Omega)}^2 \right) dr \right] ds \\
 &\leq e^{-\beta\lambda_k(t-\tau)}\|w(\tau)\|_{\mathbb{L}^2(\Omega)}^2 + C\|w(\tau)\|_{\mathbb{L}^2(\Omega)}e^{C_3(t-\tau)} \\
 &\quad \times \left[\frac{1}{\beta\lambda_k} + \int_{-\infty}^t e^{-\beta\lambda_k(t-s)} \|g(s)\|_{\mathbb{L}^2(\Omega)}^2 ds + \int_{-\infty}^t e^{-\beta\lambda_k(t-s)} e^{-\alpha s} \right. \\
 &\quad \left. \times \left(\int_{-\infty}^s e^{\alpha r} \left(\|g(r)\|_{\mathbb{L}^2(\Omega)}^2 + \|g'(r)\|_{\mathbb{L}^2(\Omega)}^2 \right) dr \right) ds \right].
 \end{aligned}
 \tag{4.79}$$

Now, because

$$\int_{-\infty}^t e^{\alpha s} \|g(s)\|_{\mathbb{L}^2(\Omega)}^2 ds < +\infty,
 \tag{4.80}$$

we can see that, for all $t \in \mathbb{R}$ (see, e.g., [6, Lemma 3.6]),

$$\int_{-\infty}^t e^{-\beta\lambda_k(t-s)} \|g(s)\|_{\mathbb{L}^2(\Omega)}^2 ds \longrightarrow 0 \quad \text{as } k \longrightarrow +\infty,
 \tag{4.81}$$

and we have

$$\begin{aligned}
 &\int_{-\infty}^t e^{-\beta\lambda_k(t-s)} e^{-\beta\lambda_1 s} \left(\int_{-\infty}^s e^{\alpha r} \left(\|g(r)\|_{\mathbb{L}^2(\Omega)}^2 + \|g'(r)\|_{\mathbb{L}^2(\Omega)}^2 \right) dr \right) ds \\
 &\leq \left(\int_{-\infty}^t e^{-\beta\lambda_k t + \beta(\lambda_k - \lambda_1) s} ds \right) \left(\int_{-\infty}^t e^{\alpha r} \left(\|g(r)\|_{\mathbb{L}^2(\Omega)}^2 + \|g'(r)\|_{\mathbb{L}^2(\Omega)}^2 \right) dr \right) \\
 &= \frac{e^{-\alpha t}}{\beta\lambda_k - \alpha} \int_{-\infty}^t e^{\alpha s} \left(\|g(r)\|_{\mathbb{L}^2(\Omega)}^2 + \|g'(r)\|_{\mathbb{L}^2(\Omega)}^2 \right) dr.
 \end{aligned}
 \tag{4.82}$$

Thus, for any $t \in \mathbb{R}$, from (4.82) we have

$$\int_{-\infty}^t e^{-\beta\lambda_k(t-s)} e^{-\beta\lambda_1 s} \left(\int_{-\infty}^s e^{\alpha r} \left(\|g(r)\|_{\mathbb{L}^2(\Omega)}^2 + \|g'(r)\|_{\mathbb{L}^2(\Omega)}^2 \right) dr \right) ds \longrightarrow 0 \quad \text{as } k \longrightarrow +\infty. \quad (4.83)$$

Combining (4.81), (4.83) and taking $T_0 = t - \tau = 1$, we get k is sufficient large then from (4.79) we deduce

$$\|w_2(t)\|_{\mathbb{L}^2(\Omega)}^2 \leq \delta \|w(\tau)\|_{\mathbb{L}^2(\Omega)}^2, \quad \text{here } 0 < \delta < 1. \quad (4.84)$$

From (4.73) and (4.31), we have

$$\|w_1(t)\|_{\mathbb{L}^2(\Omega)}^2 \leq l_0 \|w(\tau)\|_{\mathbb{L}^2(\Omega)}^2, \quad \|w_2(t)\|_{\mathbb{L}^2(\Omega)}^2 \leq \delta \|w(\tau)\|_{\mathbb{L}^2(\Omega)}^2, \quad \forall t \in \mathbb{R}. \quad (4.85)$$

Here, $l_0 = e^{2C_3}$; $0 < \delta < 1$; $T_0 = 1$. Thus, the process $U(t, \tau)$ associated to (1.2) satisfies all conditions of Theorem 2.10. This completes the proof. \square

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