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Research Article

The Laplace Likelihood Ratio Test for Heteroscedasticity

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It is shown that the likelihood ratio test for heteroscedasticity, assuming the Laplace distribution, gives good results for Gaussian and fat-tailed data. The likelihood ratio test, assuming normality, is very sensitive to any deviation from normality, especially when the observations are from a distribution with fat tails. Such a likelihood test can also be used as a robust test for a constant variance in residuals or a time series if the data is partitioned into groups.

1. Introduction

The likelihood ratio test for equal variances will be derived under the assumption of Laplace or double exponential distributed observations or residuals. The excess kurtosis of this distribution is three and it is leptokurtic. It is shown that the likelihood ratio test for the equality of variances when assuming the Laplace distribution for the residuals is more robust than the normal one. The distributional properties of this ratio are very similar to that when normality is assumed, but with a better approximation in the asymptotic chi-square approximation of the log-likelihood than the normal case.

One of the factors to consider when checking the fit of a model in time series is to see if the residuals are white noise. The use of volatility models for log returns attracted a lot of attention in the last few years, and ARCH and GARCH models are fitted when heteroscedasticity is present. A test for a constant variance after partitioning the residuals, suitable for observations from a distribution with fatter tails than a normal, can also be used to check for white noise.

The tests of Levene [1], Brown and Forsythe [2], and Gastwirth et al. [3] are often used as more robust tests than the normal likelihood ratio test for the equality of variances in general statistical tests for example, ANOVA's. Bootstrap methods can be helpful to investigate the distribution of test statistics in these problems. A review and some suggestions are given in the paper of Boos and Brownie [4]. A simulation study comparing the methods shows much more robust performance of the Laplace likelihood ratio than the normal likelihood ratio and the Levene test.

Assume, that a total of n independent observations $\{u_t\}$, $t = 1, \ldots, n$, from a normal distribution, are available. There are k groups with sample sizes $n_1, \ldots, n_k$. When equal sample sizes are under consideration, it will be assumed that $n = kn_0$. Let $n_j\widehat{\sigma}_j^2 = \sum_{j=1}^{n_j}(u_j - \overline{u}_j)^2$, $j = 1, \ldots, k$ denote the estimated variances for each partition, where $\overline{u}_j$ denotes the sample mean of the jth group or partition. The term $n_j\widehat{\sigma}_j^2/\sigma^2 \sim \chi^2_{n_j-1}$ is a gamma variable with parameters $(n_j - 1)/2, 1/2$.

In the case of equal variances, the σ^2's cancel in the likelihood ratio. The likelihood ratio λ_N [5] for the hypothesis of the equality of variances, $H_0 : \sigma_1^2 = \cdots = \sigma_k^2 = \sigma^2$, for normal data is

$$\lambda_N = \frac{\prod_{j=1}^{k}\left(\widehat{\sigma}_j^2\right)^{n_j/2}}{\left(\sum_{j=1}^{k} n_j\widehat{\sigma}_j^2/n\right)^{n/2}}. \tag{1.1}$$

A weak point of the statistic is that it is very sensitive to deviations from normality. The statistic and its asymptotic chi-square approximation, $-2\log(\lambda_N)$, was studied widely and many corrections were suggested to improve the approximation.

The ideas and results of Bartlett [6] and Box [7, 8] were the basis for many of the asymptotic corrections later derived for the statistic. They considered the statistic M_1:

$$\begin{gathered} M_1 = (n-k)\log\left(\sigma^2\right) - \sum_{j=1}^{k}(n_j - 1)\log\left(\sigma_j^2\right), \\ \widehat{\sigma}^2 = \frac{\sum_{j=1}^{\mathrm{k}}(n_j-1)\widehat{\sigma}_j^2}{(n-k)}, \quad (n_j - 1)\widehat{\sigma}_j^2 = \sum_{j}^{n_j}\left(u_j - \overline{u}_j\right)^2, \quad j = 1, \ldots, k. \end{gathered} \tag{1.2}$$

Let $\nu_j = n_j - 1$, $j = 1, \ldots, k$. M_1 can be denoted in the normal case as $M_1\,(0;\ \nu_1, \ldots, \nu_k)$, where the zero indicates excess kurtosis of zero. The statistic can be generalized to $M_1\,(\gamma_{21}, \ldots, \gamma_{2k}; \nu_1, \ldots, \nu_k)$, where each of the k variance estimates is from a population with a different kurtosis, γ_{2j}, $j = 1, \ldots, k$. It is shown [8] that if the kurtosis is equal to γ_2 for the k samples, the statistic is distributed as $\delta^{-1}M_1(0; \delta\nu_1, \ldots, \delta\nu_k)$, $\delta = (1 + (1/2)\gamma_2)^{-1}$, in large samples, for any distribution having finite cumulants. Or that the statistic is distributed as $(1 + (1/2)\gamma_2)\chi^2_{k-1}$ in large samples.

To find the moments of the log likelihood ratio was no problem, but the exact distributions of λ_N and also $\log(\lambda_N)$ are both extremely complex and not practical to use. The multivariate version of the normal likelihood tests concerning covariance matrices is covered in detail in the book of Muirhead [9]. The equal-sized partitioned series can be put in a sphericity test setting where the ellipticity statistic, which has a very similar form, is used.

The likelihood ratio λ_N is an interesting statistic, and $\lambda_N^{2/n}$ can be viewed as the ratio of the geometric mean of the estimated variances to the arithmetic mean of gamma variables, which is equal to one, only when the individual terms are independent and equal. The ratio of the geometric mean to the arithmetic mean of gamma variables was studied by Glaser [10]. Another way to look at λ_N is to notice that it can be written as the product of Dirichlet random variables, or in this work it will be considered as the product of beta random variables. Let

$$
\begin{aligned}
w_j &= \frac{n_j \widehat{\sigma}_j^2}{\sum_{j=1}^{k}\left(n_j \widehat{\sigma}_j^2\right)} \\
&= \frac{\widehat{\sigma}_j^2}{\sum_{j=1}^{k} \widehat{\sigma}_j^2} \quad \text{for equal sample sizes.}
\end{aligned}
\tag{1.3}
$$

This ratio has a beta distribution with parameters $\nu_j = (n_j - 1)/2$, $\nu = (n-k)/2$. The likelihood ratio can be expressed in terms of the product of beta random variables and

$$
\begin{aligned}
\lambda_N^2 &= \frac{\prod_{j=1}^{k}\left(\widehat{\sigma}_j^2\right)^{n_j}}{\left(\sum_{j=1}^{k} n_j \widehat{\sigma}_j^2 / n\right)^{n}} \\
&= \frac{\left(n^n / \prod_{j=1}^{k} n_j^{n_j}\right)\prod_{j=1}^{k}\left(n_j \widehat{\sigma}_j^2\right)^{n_j}}{\left(\sum_{j=1}^{k} n_j \widehat{\sigma}_j^2\right)^{n}} \\
&= \frac{k^n \prod_{j=1}^{k}\left(\widehat{\sigma}_j^2\right)^{n_j}}{\left(\sum_{j=1}^{k} \widehat{\sigma}_j^2\right)^{n}} \\
&= k^n \prod_{j=1}^{k} w_j^{n_0}.
\end{aligned}
\tag{1.4}
$$

The product of beta and Dirichlet random variables was studied by Springer and Thompson [11] and Rogers and Young [12]. The resulting density is complicated and expressed in terms of the Meijer's G and H-functions [13].

It will be shown that the likelihood test derived from residuals which have the Laplace distribution can also be expressed as a product of beta random variables for large sample sizes.

2. The Likelihood Ratio for Laplace Distributed Variables

The Laplace or double exponential density is given by

$$
p_X(x) = \frac{1}{2\phi}\exp\left(-\frac{|x-\theta|}{\phi}\right), \quad -\infty < x < \infty, \ \ \phi > 0. \tag{2.1}
$$

The variance is 2ϕ and the median of the observations is the maximum likelihood estimate of the mean θ. The maximum likelihood estimate of ϕ is $\sum_{j=1}^{n} |x_j - \widehat{\theta}|/n$ for a sample of size n, where $\widehat{\theta}$ denotes the estimated median. For θ known, $\sum_{j=1}^{n} |x_j - \theta|/n$ is distributed as a $(2n)^{-1}\phi\chi_{2n}^2$ variable. The properties of the Laplace distribution are reviewed in the book of Johnson et al. [14]. The variance of the median is $O(n^{-1})$, and the absolute deviations $|x_j - \widehat{\theta}|$, $j = 1, \ldots, n$, are asymptotically independent [15, page 335].

For the series $u_1, \ldots, u_n$ partitioned into k parts, the likelihood ratio λ_L, for the test $H_0 : \phi_1 = \cdots = \phi_k = \phi$, is

$$\lambda_L = \frac{\prod_{j=1}^{k} \widehat{\phi}_j^{n_j}}{\widehat{\phi}^{n}}, \tag{2.2}$$

with $\widehat{\phi}_j = \sum_{j=1}^{k} |x_j - \widehat{\theta}_j|/n_j$ and $\widehat{\phi} = (1/n)(n_1\widehat{\phi}_1 + \cdots + n_k\widehat{\phi}_k)$.

For equal sample sizes, n_0, the ratio is simplified to

$$\lambda_L = k^n \prod_{j=1}^{k} \left(\frac{\widehat{\phi}_j}{\sum_{j=1}^{k} \widehat{\phi}_j} \right)^{n_0}, \tag{2.3}$$

and $\lambda^{1/n}$ is proportional to the geometric mean of the ratios. If θ was known, the ratio $\widehat{\phi}_j / \sum_{j=1}^{k} \widehat{\phi}_j$ has a beta distribution with parameters n_j, $n - n_j$. This variance is approximately half that of the beta variable for the normal case.

Terms involving the distribution of the sum of the log of powers of beta random variables are found in the normal and Laplace likelihood ratio. The moment-generating function of the log of a beta variable with parameters n_j and $n - n_j$ to a power is

$$\begin{aligned} \varphi(t) &= E\left(\log\left(e^{t \log x^h}\right)\right) \\ &= E\left(x^{th}\right) \\ &= \frac{\Gamma(n)\Gamma(n_j + ht)}{\Gamma(n + ht)\Gamma(n_j)}, \end{aligned} \tag{2.4}$$

and the log of the moment-generating function of the sum of k such variables is

$$E(\log(\varphi(t))) = k \log(\Gamma(n)) - k \log(\Gamma(n_j)) + \sum_{j=1}^{k} \log(\Gamma(n_j + ht)) - k \log(\Gamma(n + ht)), \tag{2.5}$$

showing that the expected value of $-2\log(\lambda_L)$ found from the cumulant generating function is

$$\begin{aligned} E(-2\log(\lambda_L)) &= -2n\log(k) - 2\sum_{j=1}^{k} E\left(\log\left(x_j^{n_j}\right)\right) \\ &= -2n\log(k) - 2\sum_{j=1}^{k} \left(n_j\psi(n_j) - n_j\psi(n)\right), \end{aligned} \tag{2.6}$$

where x_j denotes a beta variable with parameters n_j, $n - n_j$, and ψ is the digamma function, the derivative of the log of the gamma function. By making use of the approximation of ψ [13], $\psi(n) \approx \log(n) - (1/n)(1/2 + 1/12n)$:

$$\begin{aligned} E(-2\log(\lambda_L)) &= k - 1 - 2n\log(k) + 2\sum_{j=1}^{k}(n_j\log(n) - n_j\log(n_j)) \\ &\quad + \frac{\sum_{j=1}^{k} 1}{(1+6n_j)} - \frac{1}{(1+6n)} \\ &\approx k - 1, \end{aligned} \tag{2.7}$$

for $n = kn_0$, $n_0 = n_1 = \cdots = n_k$, and large sample sizes, the expected value $-2\log(\lambda_L)$ is equal to that of a χ^2_{k-1} random variable. For the normal ratio, it would be $\nu_j = (n_j - 1)/2$ in place of the n_j's, showing that the large sample χ^2 approximation for the Laplace likelihood ratio would be a better approximation for the same sample size n, assuming the distributional assumption concerning θ.

The assumption of an expected value of zero for log returns is often made in financial time series of returns, but in most problems the expected value would be unknown, and the median of the observations of a specific partition would be used. The median is a maximum likelihood estimator and good approximations can be expected for reasonable large sample sizes.

In Figure 1, the histogram of simulated and expected frequencies of 1000 ratios is shown. The ratios were calculated using Gaussian white noise series. The $\hat{\phi}$'s are estimated using the median as the estimator of θ. A sample of 200 was partitioned into $k = 5$ equal parts and a histogram of 1000 ratios, $\hat{\phi}_1 / \sum_{j=1}^{k} \hat{\phi}_j$, is shown in Figure 1. The expected values are from a beta distribution with parameters $n_1 = 40$, $n - n_1 = 160$. The estimated mean was 0.1995 compared with the theoretical value of 0.2, and the estimated variance $7.9835e - 004$ compared with the theoretical value of $7.9602e - 004$.

The sample size of $n = 200$ is not very large for a time series. It can be seen that the ratios of the estimated ϕ_j's to the sum of the ϕ_j's are approximately beta distributed for this sample size.

A simulation study was conducted to check and compare the asymptotic χ^2 approximations of λ_N, λ_L of the likelihood ratios and the Brown-Forsythe test. Series of $n = 200$ observations, partitioned into $k = 5$, groups were simulated 1000 times and tested at the $\alpha = 0.05$ level. The median was used in the Brown-Forsythe variation of the Levene test. The expected values for the normal and Laplace χ^2 approximation are $k - 1 = 4$, and for the Levene statistic, the expected value of a $F_{4;196}$ variable is 1.0103.

The following data was generated: normal white noise, white noise from the Laplace distribution, and independent values from the stable distribution with index $\alpha = 1.9$ and 1.5.

These series were partitioned into 5 groups. Time series were generated to check the results when the test is used for checking a constant variance in residuals. The series generated were an AR(1) and a Garch series. The disturbance terms are normally distributed. The Garch series is the IGARCH(1,1) fitted by Tsay [16] to the excess returns of the SP&500. The results are shown in Table 1.

Both tests are sensitive to heteroskedasticity, but the Laplace test is less sensitive when testing for a constant variance for non-Gaussian white noise. It is interesting to note that both

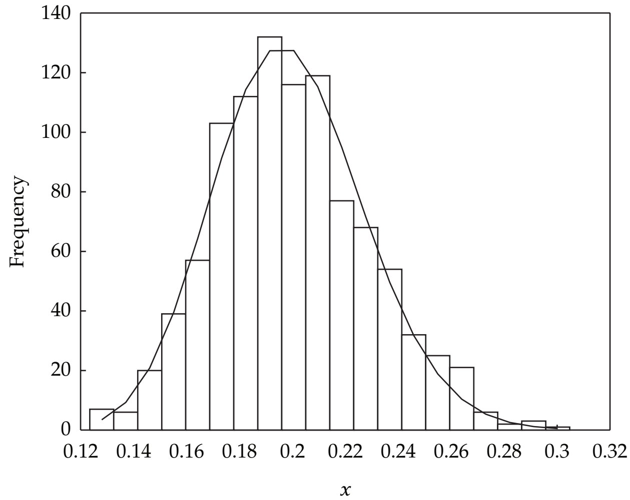

Figure 1: Simulated ratios and the expected frequencies of the beta density.

Table 1: Results of simulation study comparing the likelihood ratio tests and the Levene test.

	Laplace		Normal		Levene	
	Proportion rejected	Mean of $-2\log(\lambda_L)$	Proportion rejected	Mean of $-2\log(\lambda_N)$	Proportion rejected	Mean of Levene W
Gaussian White Noise	0.0020	2.3277	0.0500	4.0783	0.0430	0.9455
Laplace White Noise	0.0570	4.0606	0.4250	9.7604	0.0840	1.0904
t-distribution (df = 4) White Noise	0.0560	4.0253	0.5240	13.8947	0.1120	1.2181
Stable ($\alpha = 1.5$)	0.4880	19.2967	0.9370	107.9499	0.3410	3.5335
Stable ($\alpha = 1.9$)	0.0530	4.3012	0.3950	23.5564	0.1190	1.5190
AR(1) $\rho_1 = 0.1$	0.0040	2.3710	0.0630	4.1258	0.0450	0.9491
AR(2) $\rho_1 = 0.5$	0.0290	3.5989	0.2140	6.4760	0.1590	1.4636
IGARCH(1,1)	0.7890	34.6704	0.9410	83.8057	0.6970	7.9725

tests are sensitive to large autocorrelation in the series and also when the variance is infinite for stable data with $\alpha < 2.00$. The stable noise with index $\alpha = 1.9$ is close to Gaussian, but theoretically only $E(x^{\alpha})$ is finite. All the tests detect that the series with $\alpha = 1.5$ is not second-order stationary.

The tests are not very sensitive when autocorrelation is present as in the AR(1) models, but the sensitivity increases as the first-order autocorrelation increases. All the tests easily detect the heteroscedasticity in the IGARCH(1,1) model.

3. Conclusions and Suggestions

The Laplace likelihood test performs much better on data with heavier tails, and better than the Brown-Forsythe variation of the Levene test for example, in the case of the t-distributed series. The normal likelihood ratio test is only effective in the normal data case.

It seems that these tests are sensitive for serial correlation and heteroskedasticity in series and can be used as a check for white noise in residuals. The use of filtering [17] can be applied to investigate and improve results when testing for heteroscedasticity where serial correlation is present in time series models. An investigation into the use of size-adjustment can improve the effectiveness of this test, especially when working with GARCH type series.

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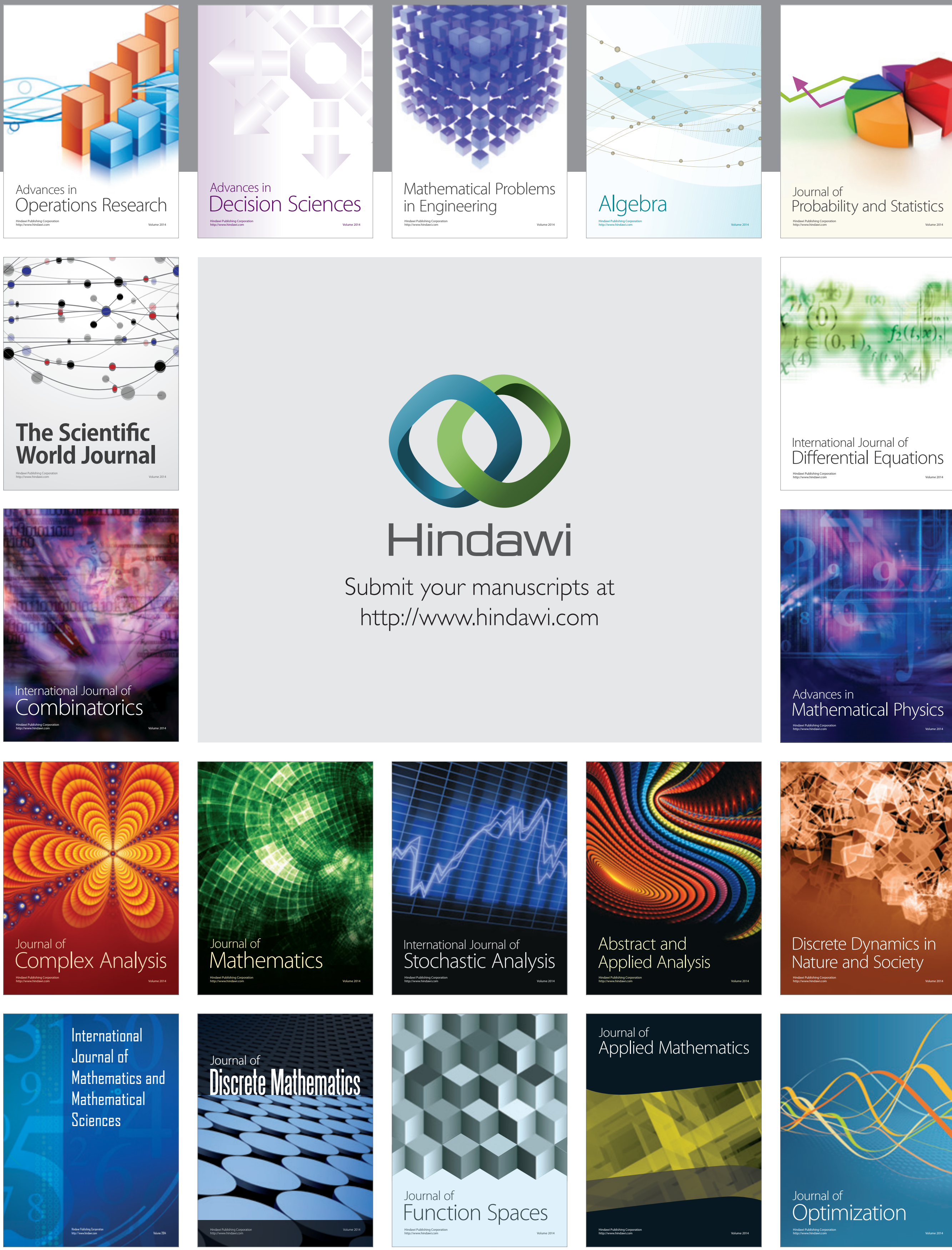
References

[1] H. Levene, "Robust tests for equality of variances," in *Contributions to Probability and Statistics*, I. Olkin, Ed., pp. 278–292, Stanford University Press, Palo Alto, CA, USA, 1960.
[2] M. B. Brown and A. B. Forsythe, "Robust test for equality of variances," *Journal of the American Statistical Association*, vol. 69, no. 346, pp. 364–367, 1974.
[3] J. L. Gastwirth, Y. R. Gel, and W. Miao, "The impact of Levene's test of equality of variances on statistical theory and practice," *Statistical Science.*, vol. 24, no. 3, pp. 343–360, 2009.
[4] D. D. Boos and C. Brownie, "Comparing variances and other measures of dispersion," *Statistical Science*, vol. 19, no. 4, pp. 571–578, 2004.
[5] A. M. Mood, F. A. Graybill, and D. C. Boes, *Introduction to the Theory of Statistics*, McGraw-Hill, New York, NY, USA, 1974.
[6] M. S. Bartlett, "Properties of sufficiency and statistical tests," *Proceedings of the Royal Society A*, vol. 160, no. 901, pp. 262–282, 1937.
[7] G. E. P. Box, "A general distribution theory for a class of likelihood criteria," *Biometrika*, vol. 36, pp. 317–346, 1949.
[8] G. E. P. Box, "Non-normality and tests on variances," *Biometrika*, vol. 40, pp. 318–335, 1953.
[9] R. J. Muirhead, *Aspects of Multivariate Statistical Theory*, John Wiley & Sons, New York, NY, USA, 1982, Wiley Series in Probability and Mathematical Statistic.
[10] R. E. Glaser, "The ratio of the geometric mean to the arithmetic mean for a random sample from a gamma distribution," *Journal of the American Statistical Association*, vol. 71, no. 354, pp. 480–487, 1976.
[11] M. D. Springer and W. E. Thompson, "The distribution of products of beta, gamma and Gaussian random variables," *SIAM Journal on Applied Mathematics*, vol. 18, pp. 721–737, 1970.
[12] G. S. Rogers and D. L. Young, "On the products of powers of generalized Dirichlet components with an application," *The Canadian Journal of Statistics*, vol. 1, no. 2, pp. 159–169, 1973.
[13] I. S. Gradshteyn and M. Y. Ryzhik, *Table of Integrals, Series and Products*, Academic Press, New York, NY, USA, 1980.
[14] N. L. Johnson, S. Kotz, and N. Balakrishnan, *Continuous Univariate Distributions*, vol. 2 of *Wiley Series in Probability and Mathematical Statistics: Applied Probability and Statistics*, John Wiley & Sons, New York, NY, USA, 2nd edition, 1995.
[15] M. Kendall, A. Stuart, and J. K. Ord, *Kendall's Advanced Theory of Statistics*, vol. 1, Charles Griffin and Company, London, UK, 1987.
[16] R. S. Tsay, *Analysis of Financial Time Series*, Wiley Series in Probability and Statistics, John Wiley & Sons, New York, NY, USA, 3rd edition, 2010.
[17] R. L. Lumsdaine and S. Ng, "Testing for ARCH in the presence of a possibly misspecified conditional mean," *Journal of Econometrics*, vol. 93, no. 2, pp. 257–279, 1999.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