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Research Article

Uniqueness of Meromorphic Functions and Differential Polynomials

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We study the uniqueness of meromorphic functions and differential polynomials sharing one value with weight and prove two main theorems which generalize and improve some results earlier given by M. L. Fang, S. S. Bhoosnurmath and R. S. Dyavanal, and so forth.

1. Introduction and Results

Let f be a nonconstant meromorphic function defined in the whole complex plane $\mathbb{C}$. It is assumed that the reader is familiar with the notations of the Nevanlinna theory such as $T(r,f)$, $m(r,f)$, $N(r,f)$, and $S(r,f)$, that can be found, for instance, in [1–3].

Let f and g be two nonconstant meromorphic functions. Let a be a finite complex number. We say that f and g share the value a CM (counting multiplicities) if $f-a$ and $g-a$ have the same zeros with the same multiplicities, and we say that f and g share the value a IM (ignoring multiplicities) if we do not consider the multiplicities. When f and g share 1 IM, let z_0 be a 1-point of f of order p and a 1-points of g of order q; we denote by $N_{11}(r,1/(f-1))$ the counting function of those 1-points of f and g, where $p=q=1$ and by $N_E^{(2)}(r,1/(f-1))$ the counting function of those 1-points of f and g, where $p=q\geq 2$. $\overline{N}_L(r,1/(f-1))$ is the counting function of those 1-points of both f and g, where $p>q$. In the same way, we can define $N_{11}(r,1/(g-1))$, $N_E^{(2)}(r,1/(g-1))$, and $\overline{N}_L(r,1/(g-1))$. If f and g share 1 IM, it is easy to see that

$$\begin{aligned}\overline{N}\left(r,\frac{1}{f-1}\right)&=N_{11}\left(r,\frac{1}{f-1}\right)+\overline{N}_L\left(r,\frac{1}{f-1}\right)+\overline{N}_L\left(r,\frac{1}{g-1}\right)+N_E^{(2)}\left(r,\frac{1}{f-1}\right)\\&=\overline{N}\left(r,\frac{1}{g-1}\right).\end{aligned}\tag{1.1}$$

Let f be a nonconstant meromorphic function. Let a be a finite complex number and k a positive integer; we denote by $N_{(k)}(r, 1/(f-a))$ (or $\overline{N}_{(k)}(r, 1/(f-a))$) the counting function for zeros of $f-a$ with multiplicity $\le k$ (ignoring multiplicities) and by $N_{(k)}(r, 1/(f-a))$ (or $\overline{N}_{(k)}(r, 1/(f-a))$) the counting function for zeros of $f-a$ with multiplicity at least k (ignoring multiplicities). Set

$$N_k\left(r, \frac{1}{f-a}\right) = \overline{N}\left(r, \frac{1}{f-a}\right) + \overline{N}_{(2)}\left(r, \frac{1}{f-a}\right) + \cdots + \overline{N}_{(k)}\left(r, \frac{1}{f-a}\right),$$
$$\Theta(a, f) = 1 - \overline{\lim_{r\to\infty}} \frac{\overline{N}(r, (1/f-a))}{T(r, f)}. \tag{1.2}$$

We further define

$$\delta_k(a, f) = 1 - \overline{\lim_{r\to\infty}} \frac{N_k(r, 1/(f-a))}{T(r, f)}. \tag{1.3}$$

In 2002, C. Y. Fang and M. L. Fang [4] proved the following result.

Theorem A (see [4]). *Let $f(z)$ and $g(z)$ be two nonconstant entire functions, and let $n(\ge 8)$ be a positive integer. If $[f^n(z)(f(z)-1)]f'(z)$ and $[g^n(z)(g(z)-1)]g'(z)$ share 1* CM, *then $f(z) \equiv g(z)$.*

Fang [5] proved the following result.

Theorem B (see [5]). *Let $f(z)$ and $g(z)$ be two nonconstant entire functions, and let n, k be two positive integers with $n > 2k+8$. If $[f^n(z)(f(z)-1)]^{(k)}$ and $[g^n(z)(g(z)-1)]^{(k)}$ share 1* CM, *then $f(z) \equiv g(z)$.*

In [6], for some general differential polynomials such as $[f^n(f-1)^m]^{(k)}$, Liu proved the following result.

Theorem C (see [6]). *Let $f(z)$ and $g(z)$ be two nonconstant entire functions, and let n, m, k be three positive integers such that $n > 5k+4m+9$. If $[f^n(z)(f(z)-1)^m]^{(k)}$ and $[g^n(z)(g(z)-1)^m]^{(k)}$ share 1* IM, *then either $f(z) \equiv g(z)$ or f and g satisfy the algebraic equation $R(f, g) \equiv 0$, where $R(\omega_1, \omega_2) = \omega_1^n(\omega_1-1)^m - \omega_2^n(\omega_2-1)^m$.*

The following example shows that Theorem A is not valid when f and g are two meromorphic functions.

Example 1.1. Let $f = (n+2)(h-h^{n+2})/(n+1)(1-h^{n+2})$, $g = (n+2)(1-h^{n+1})/(n+1)(1-h^{n+2})$, where $h = e^z$. Then $[f^n(z)(f(z)-1)]f'(z)$ and $[g^n(z)(g(z)-1)]g'(z)$ share 1 CM, but $f(z) \not\equiv g(z)$.

Lin and Yi [7] and Bhoosnurmath and Dyavanal [8] generalized the above results and obtained the following results.

Theorem D (see [7]). *Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions with $\Theta(\infty, f) > 2/(n+1)$, and let $n(\ge 12)$ be a positive integer. If $[f^n(z)(f(z)-1)]f'(z)$ and $[g^n(z)(g(z)-1)]g'(z)$ share 1* CM, *then $f(z) \equiv g(z)$.*

Theorem E (see [8]). *Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions satisfying $\Theta(\infty, f) > 3/(n+1)$, and let n, k be two positive integers with $n > 3k+13$. If $[f^n(z)(f(z)-1)]^{(k)}$ and $[g^n(z)(g(z)-1)]^{(k)}$ share 1* CM, *then $f(z) \equiv g(z)$.*

Naturally, one may ask the following question: is it really possible to relax in any way the nature of sharing 1 in the above results?

To state the next result, we require the following definition.

Definition 1.2 (see [9]). Let k be a nonnegative integer or infinity. For $a \in \overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ one denotes by $E_k(a, f)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a, f) = E_k(a, g)$, one says that f, g share the value a with weight k.

We write that f, g share (a, k) to mean that f, g share the value a with weight k; clearly if f, g share (a, k), then f, g share (a, p) for all integers p with $0 \leq p \leq k$. Also, we note that f, g share a value a IM or CM if and only if they share $(a, 0)$ or (a, ∞), respectively.

Recently, with the notion of weighted sharing of values, Xu et al. [10] improved the above results and proved the following theorem.

Theorem F (see [10]). *Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions, and let n, k be two positive integers with $n > 5k+11$. If $\Theta(\infty, f) + \Theta(\infty, g) > 4/n$, $[f^n(z)(f(z)-1)]^{(k)}$, and $[g^n(z)(g(z)-1)]^{(k)}$ share 1 $(1,2)$, then $f(z) \equiv g(z)$.*

Theorem G (see [10]). *Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions, and let n, k be two positive integers with $n > 7k+23/2$. If $\Theta(\infty, f) + \Theta(\infty, g) > 4/n$, $[f^n(z)(f(z)-1)]^{(k)}$, and $[g^n(z)(g(z)-1)]^{(k)}$ share 1 $(1,1)$, then $f(z) \equiv g(z)$.*

Remark 1.3. The proof of Theorem E contains some mistakes: for example, one cannot get formulas (6.9) and (6.10) in [8]. Therefore, the last inequality in page 1203 of [8] does not hold. So, Theorem E will not stand. Similarly, in [10], the proof of Case 1 in Theorem F is incorrect (see page 63 of paper [10]). Hence, the conclusion of Theorems F and G will not stand.

Now one may ask the following questions which are the motivations of the paper.

(i) What happens if the conclusions of Theorems E, F, and G can not stand?

(ii) Can the value n be further reduced in the above results?

In the paper, we investigate the solutions of the above questions. We improve and generalize the above related results by proving the following theorems.

Theorem 1.4. *Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions, and let n, k be two positive integers with $n > 3k+11$. If $\Theta(\infty, f) > 2/n$, $[f^n(z)(f(z)-1)]^{(k)}$, and $[g^n(z)(g(z)-1)]^{(k)}$ share 1 $(1,2)$, then $f(z) \equiv g(z)$ or $[f^n(z)(f(z)-1)]^{(k)} \cdot [g^n(z)(g(z)-1)]^{(k)} \equiv 1$.*

Theorem 1.5. *Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions, and let n, k be two positive integers with $n > 5k+14$. If $\Theta(\infty, f) > 2/n$, $[f^n(z)(f(z)-1)]^{(k)}$ and $[g^n(z)(g(z)-1)]^{(k)}$ share 1 $(1,1)$, then $f(z) \equiv g(z)$ or $[f^n(z)(f(z)-1)]^{(k)} \cdot [g^n(z)(g(z)-1)]^{(k)} \equiv 1$.*

2. Some Lemmas

For the proof of our result, we need the following lemmas.

Lemma 2.1 (see [1]). *Let f be nonconstant meromorphic function, and let $a_0, a_1, \ldots, a_n$ be finite complex numbers such that $a_n \neq 0$. Then*

$$T\left(r, a_n f^n + a_{n-1} f^{n-1} + \cdots + a_0\right) = nT(r, f) + S(r, f). \tag{2.1}$$

Lemma 2.2 (see [1]). *Let $f(z)$ be a nonconstant meromorphic function, k a positive integer, and c a nonzero finite complex number. Then*

$$\begin{aligned} T(r, f) &\leq \overline{N}(r, f) + N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f^{(k)} - c}\right) - N\left(r, \frac{1}{f^{(k+1)}}\right) + S(r, f) \\ &\leq \overline{N}(r, f) + N_{k+1}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{f^{(k)} - c}\right) - N_0\left(r, \frac{1}{f^{(k+1)}}\right) + S(r, f). \end{aligned} \tag{2.2}$$

Here $N_0(r, 1/f^{(k+1)})$ is the counting function which only counts those points such that $f^{(k+1)} = 0$ but $f(f^{(k)} - c) \neq 0$.

Lemma 2.3 (see [11]). *Let f be a nonconstant meromorphic function, and let k, p be two positive integers. Then*

$$N_p\left(r \frac{1}{f^{(k)}}\right) \leq N_{p+k}\left(r \frac{1}{f}\right) + k\overline{N}(r, f) + S(r, f). \tag{2.3}$$

Clearly $\overline{N}(r(1/f^{(k)})) = N_1(r(1/f^{(k)}))$.

Lemma 2.4 (see [1]). *Let $f(z)$ be a transcendental meromorphic function, and let $a_1(z)$, $a_2(z)$ be two meromorphic functions such that $T(r, a_i) = S(r, f)$, $i = 1, 2$. Then*

$$T(r, f) \leq \overline{N}(r, f) + \overline{N}\left(r, \frac{1}{f - a_1}\right) + \overline{N}\left(r, \frac{1}{f - a_2}\right) + S(r, f). \tag{2.4}$$

Lemma 2.5. *Let f and g be two nonconstant meromorphic functions, and let $k(\geq 1)$, $l(\geq 1)$ be two positive integers. Suppose that $f^{(k)}$ and $g^{(k)}$ share $(1, l)$.*

(i) *If $l = 2$ and*

$$\Delta_1 = 2\Theta(\infty, f) + (k+2)\Theta(\infty, g) + \Theta(0, f) + \Theta(0, g) + \delta_{k+1}(0, f) + \delta_{k+1}(0, g) > k + 7 \tag{2.5}$$

then either $f^{(k)} g^{(k)} \equiv 1$ or $f \equiv g$.

(ii) *If* $l=1$ *and*

$$\Delta_2=(k+3)\Theta(\infty,f)+(k+2)\Theta(\infty,g)+\Theta(0,f)+\Theta(0,g)+2\delta_{k+1}(0,f)+\delta_{k+1}(0,g)>2k+9 \tag{2.6}$$

then either $f^{(k)}g^{(k)}\equiv 1$ *or* $f\equiv g$.

Proof. Let

$$h(z)=\frac{f^{(k+2)}(z)}{f^{(k+1)}(z)}-2\frac{f^{(k+1)}(z)}{f^{(k)}(z)-1}-\frac{g^{(k+2)}(z)}{g^{(k+1)}(z)}+2\frac{g^{(k+1)}(z)}{g^{(k)}(z)-1}. \tag{2.7}$$

Suppose that $h\not\equiv 0$.

If z_0 is a common simple 1-point of $f^{(k)}$ and $g^{(k)}$, substituting their Taylor series at z_0 into (2.7), we see that z_0 is a zero of $h(z)$. Thus, we have

$$\begin{aligned} N_{11}\left(r,\frac{1}{f^{(k)}-1}\right)&=N_{11}\left(r,\frac{1}{g^{(k)}-1}\right)\\ &\le \overline{N}\left(r,\frac{1}{h}\right)\\ &\le T(r,h)+O(1)\\ &\le N(r,h)+S(r,f)+S(r,g). \end{aligned} \tag{2.8}$$

By our assumptions, $h(z)$ have poles only at zeros of $f^{(k+1)}$ and $g^{(k+1)}$ and poles of f and g, and those 1-points of $f^{(k)}$ and $g^{(k)}$ whose multiplicities are distinct from the multiplicities of correspond to 1-points of $g^{(k)}$ and $f^{(k)}$, respectively. Thus, we deduce from (2.7) that

$$\begin{aligned} N(r,h)\le{}&\overline{N}(r,f)+\overline{N}(r,g)+\overline{N}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{g}\right)+N_0\left(r,\frac{1}{f^{(k+1)}}\right)\\ &+N_0\left(r,\frac{1}{g^{(k+1)}}\right)+\overline{N}_L\left(r,\frac{1}{f^{(k)}-1}\right)+\overline{N}_L\left(r,\frac{1}{g^{(k)}-1}\right), \end{aligned} \tag{2.9}$$

here $N_0(r,1/f^{(k+1)})$ has the same meaning as in Lemma 2.2.

By Lemma 2.2, we have

$$\begin{aligned} T(r,f)&\le \overline{N}(r,f)+N_{k+1}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{f^{(k)}-1}\right)-N_0\left(r,\frac{1}{f^{(k+1)}}\right)+S(r,f),\\ T(r,g)&\le \overline{N}(r,g)+N_{k+1}\left(r,\frac{1}{g}\right)+\overline{N}\left(r,\frac{1}{g^{(k)}-1}\right)-N_0\left(r,\frac{1}{g^{(k+1)}}\right)+S(r,g). \end{aligned} \tag{2.10}$$

Since $f^{(k)}$ and $g^{(k)}$ share $(1,0)$, we get

$$\begin{aligned}\overline{N}\left(r,\frac{1}{f^{(k)}-1}\right)+\overline{N}\left(r,\frac{1}{g^{(k)}-1}\right)&=2N_{11}\left(r,\frac{1}{f-1}\right)+2\overline{N}_L\left(r,\frac{1}{f^{(k)}-1}\right)\\&\quad+2\overline{N}_L\left(r,\frac{1}{g^{(k)}-1}\right)+2N_E^{(2)}\left(r,\frac{1}{f^{(k)}-1}\right).\end{aligned}\tag{2.11}$$

We obtain from (2.8)–(2.11) that

$$\begin{aligned}T(r,f)+T(r,g)\leq{}&2\overline{N}(r,f)+2\overline{N}(r,g)+\overline{N}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{g}\right)+N_{11}\left(r,\frac{1}{f^{(k)}-1}\right)\\&+3\overline{N}_L\left(r,\frac{1}{f^{(k)}-1}\right)+3\overline{N}_L\left(r,\frac{1}{g^{(k)}-1}\right)+2N_E^{(2)}\left(r,\frac{1}{f^{(k)}-1}\right)\\&+N_{k+1}\left(r,\frac{1}{f}\right)+N_{k+1}\left(r,\frac{1}{g}\right)+S(r,f)+S(r,g).\end{aligned}\tag{2.12}$$

(i) If $l\geq 2$, it is easy to see that

$$\begin{aligned}&N_{11}\left(r,\frac{1}{f^{(k)}-1}\right)+3\overline{N}_L\left(r,\frac{1}{f^{(k)}-1}\right)+3\overline{N}_L\left(r,\frac{1}{g^{(k)}-1}\right)+2N_E^{(2)}\left(r,\frac{1}{f^{(k)}-1}\right)\\&\quad\leq N\left(r,\frac{1}{g^{(k)}-1}\right)+S(r,f)+S(r,g).\end{aligned}\tag{2.13}$$

Since

$$\begin{aligned}N\left(r,\frac{1}{g^{(k)}-1}\right)&\leq T\left(r,g^{(k)}\right)+S(r,g)=m\left(r,g^{(k)}\right)+N\left(r,g^{(k)}\right)+S(r,g)\\&\leq m(r,g)+m\left(r,\frac{g^{(k)}}{g}\right)+N(r,g)+k\overline{N}(r,g)+S(r,g)\\&\leq T(r,g)+k\overline{N}(r,g)+S(r,g),\end{aligned}\tag{2.14}$$

combining with (2.12), (2.13), and (2.14), we obtain

$$\begin{aligned}T(r,f)\leq{}&2\overline{N}(r,f)+(k+2)\overline{N}(r,g)+\overline{N}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{g}\right)\\&+N_{k+1}\left(r,\frac{1}{f}\right)+N_{k+1}\left(r,\frac{1}{g}\right)+S(r,f)+S(r,g).\end{aligned}\tag{2.15}$$

Without loss of generality, we suppose that there exists a set I with infinite measure such that $T(r,g) \leq T(r,f)$ for $r \in I$:

$$\begin{aligned} T(r,f) \leq \{ [(k+8) - 2\Theta(\infty,f) - (k+2)\Theta(\infty,g) - \Theta(0,f) - \Theta(0,g) \\ -\delta_{k+1}(0,f) - \delta_{k+1}(0,g)] + \varepsilon \} T(r,f) + S(r,F) \end{aligned} \tag{2.16}$$

for $r \in I$ and $0 < \varepsilon < \Delta_1 - (k+7)$, that is,

$$[\Delta_1 - (k+7) - \varepsilon] T(r,f) \leq S(r,f), \tag{2.17}$$

that is, $\Delta_1 \leq k+7$, which contradicts hypothesis (2.5).

Therefore, we have $h \equiv 0$, that is,

$$\frac{f^{(k+2)}(z)}{f^{(k+1)}(z)} - 2\frac{f^{(k+1)}(z)}{f^{(k)}(z)-1} = \frac{g^{(k+2)}(z)}{g^{(k+1)}(z)} - 2\frac{g^{(k+1)}(z)}{g^{(k)}(z)-1}. \tag{2.18}$$

By solving this equation, we obtain

$$g^{(k)} = \frac{(b+1)f^{(k)} + (a-b-1)}{bf^{(k)} + (a-b)}, \tag{2.19}$$

where $a(\neq 0), b$ are two constants. Next, we consider three cases.

Case 1 ($b \neq 0, -1$). For more details see the following subcases.

Subcase 1.1 ($a - b - 1 \neq 0$). Then, by (2.19), we have $\overline{N}(r, 1/(f^{(k)} - (a-b-1)/(b+1))) = \overline{N}(r, 1/g^{(k)})$.

By Lemma 2.2 and Lemma 2.3, we get

$$\begin{aligned} T(r,f) &\leq \overline{N}(r,f) + N_{k+1}\left(r,\frac{1}{f}\right) + \overline{N}\left(r,\frac{1}{f^{(k)} - (a-b-1)/(b+1)}\right) \\ &\quad - N_0\left(r,\frac{1}{f^{(k+1)}}\right) + S(r,f) \\ &\leq \overline{N}(r,f) + N_{k+1}\left(r,\frac{1}{f}\right) + \overline{N}\left(r,\frac{1}{g^{(k)}}\right) + S(r,f) \\ &\leq \overline{N}(r,f) + N_{k+1}\left(r,\frac{1}{f}\right) + N_{k+1}\left(r,\frac{1}{g}\right) + k\overline{N}(r,g) + S(r,f) \\ &\leq 2\overline{N}(r,f) + (k+2)\overline{N}(r,g) + \overline{N}\left(r,\frac{1}{f}\right) + \overline{N}\left(r,\frac{1}{g}\right) \\ &\quad + N_{k+1}\left(r,\frac{1}{f}\right) + N_{k+1}\left(r,\frac{1}{g}\right) + S(r,f) + S(r,g), \end{aligned} \tag{2.20}$$

that is, $T(r,f) \le (k+8-\Delta_1)T(r,f)+S(r,f)$. Thus, by (2.5) we deduce that $T(r,f)\le S(r,f)$ for $r\in I$, a contradiction.

Subcase 1.2 ($a-b-1=0$). Then, by (2.19), we have $g^{(k)}=(b+1)f^{(k)}/(bf^{(k)}+1)$.

Therefore $\overline{N}(r,1/(f^{(k)}+(1/b)))=\overline{N}(r,g^{(k)})$.

By Lemma 2.2 and Lemma 2.3, we get

$$\begin{aligned}
T(r,f) &\le \overline{N}(r,f)+N_{k+1}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{f^{(k)}+(1/b)}\right)-N_0\left(r,\frac{1}{f^{(k+1)}}\right)+S(r,f)\\
&\le \overline{N}(r,f)+N_{k+1}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,g^{(k)}\right)+S(r,f)\\
&\le 2\overline{N}(r,f)+(k+2)\overline{N}(r,g)+\overline{N}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{g}\right)\\
&\quad+N_{k+1}\left(r,\frac{1}{f}\right)+N_{k+1}\left(r,\frac{1}{g}\right)+S(r,f)+S(r,g),
\end{aligned} \tag{2.21}$$

that is, $T(r,f) \le (k+8-\Delta_1)T(r,f)+S(r,f)$. Thus, by (2.5) we deduce that $T(r,f)\le S(r,f)$ for $r\in I$, a contradiction.

Case 2 ($b=-1$). For more details see the following subcases.

Subcase 2.1 ($a+1\neq 0$). Then by (2.19), we get $a/((a+1)-f^{(k)})=g^{(k)}$. So, we have $\overline{N}(r,1/(f^{(k)}-(a+1)))=\overline{N}(r,g^{(k)})$. We can deduce a contradiction as in Case 1.

Subcase 2.2 ($a+1=0$). Then by (2.19), we get $f^{(k)}g^{(k)}\equiv 1$.

Case 3 ($b=0$). For more details see the following subcases.

Subcase 3.1 ($a-1\neq 0$). Then by (2.19), we get $(f^{(k)}+a-1)/a=g^{(k)}$. So, we have $\overline{N}(r,1/(f^{(k)}-(1-a)))=\overline{N}(r,g^{(k)})$. We can deduce a contradiction as in Case 1.

Subcase 3.2 ($a-1=0$). Then by (2.19), we get $f^{(k)}\equiv g^{(k)}$. From this, we obtain $f=g+P(z)$, where $P(z)$ is a polynomial, and so $T(r,f)=T(r,g)+S(r,f)$. If $P(z)\not\equiv 0$, then, by Lemma 2.4, we get

$$\begin{aligned}
T(r,f) &\le \overline{N}(r,f)+\overline{N}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{f-P}\right)+S(r,f)\\
&\le \overline{N}(r,f)+\overline{N}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{g}\right)+S(r,f)\\
&\le 2\overline{N}(r,f)+(k+2)\overline{N}(r,g)+\overline{N}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{g}\right)\\
&\quad+N_{k+1}\left(r,\frac{1}{f}\right)+N_{k+1}\left(r,\frac{1}{g}\right)+S(r,f)+S(r,g).
\end{aligned} \tag{2.22}$$

Thus, by (2.5) we deduce that $T(r,f) \le S(r,f)$ for $r \in I$, a contradiction. Therefore, we conclude that $P(z) \equiv 0$, that is, $f \equiv g$.

(ii) If $l = 1$, it is easy to see that

$$\begin{aligned} &N_{11}\left(r,\frac{1}{f^{(k)}-1}\right)+2\overline{N}_L\left(r,\frac{1}{f^{(k)}-1}\right)+3\overline{N}_L\left(r,\frac{1}{g^{(k)}-1}\right)+2N_E^{(2)}\left(r,\frac{1}{f^{(k)}-1}\right)\\ &\quad\le N\left(r,\frac{1}{g^{(k)}-1}\right)+S(r,f)+S(r,g). \end{aligned} \tag{2.23}$$

By Lemma 2.3, we can get

$$\begin{aligned} \overline{N}_L\left(r,\frac{1}{f^{(k)}-1}\right) &\le N\left(r,\frac{1}{f^{(k)}-1}\right)+\overline{N}\left(r,\frac{1}{f^{(k)}-1}\right)\\ &\le N\left(r,\frac{f^{(k)}}{f^{(k+1)}}\right)\le N\left(r,\frac{f^{(k+1)}}{f^{(k)}}\right)+S(r,f)\\ &\le \overline{N}\left(r,\frac{1}{f^{(k)}}\right)+\overline{N}(r,f)+S(r,f)\\ &\le (k+1)\overline{N}(r,f)+N_{k+1}\left(r,\frac{1}{f}\right)+S(r,f). \end{aligned} \tag{2.24}$$

Combining with (2.12), (2.24), and (2.14), we obtain

$$\begin{aligned} T(r,f) &\le (k+3)\overline{N}(r,f)+(k+2)\overline{N}(r,g)+\overline{N}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{g}\right)+2N_{k+1}\left(r,\frac{1}{f}\right)\\ &\quad+N_{k+1}\left(r,\frac{1}{g}\right)+S(r,f)+S(r,g). \end{aligned} \tag{2.25}$$

Without loss of generality, we suppose that there exists a set I with infinite measure such that $T(r,g) \le T(r,f)$ for $r \in I$:

$$\begin{aligned} T(r,f) \le \{&[(2k+10)-(k+3)\Theta(\infty,f)-(k+2)\Theta(\infty,g)-\Theta(0,f)\\ &-\Theta(0,g)-2\delta_{k+1}(0,f)-\delta_{k+1}(0,g)]+\varepsilon\}T(r,f)+S(r,F) \end{aligned} \tag{2.26}$$

for $r \in I$ and $0 < \varepsilon < \Delta_2 - (2k+9)$, that is,

$$[\Delta_2-(2k+9)-\varepsilon]T(r,f) \le S(r,f), \tag{2.27}$$

that is, $\Delta_2 \le 2k+9$, which contradicts hypothesis (2.6).

Therefore, we have $h \equiv 0$, that is,

$$\frac{f^{(k+2)}(z)}{f^{(k+1)}(z)} - 2\frac{f^{(k+1)}(z)}{f^{(k)}(z) - 1} = \frac{g^{(k+2)}(z)}{g^{(k+1)}(z)} - 2\frac{g^{(k+1)}(z)}{g^{(k)}(z) - 1}. \tag{2.28}$$

By solving this equation, we obtain

$$g^{(k)} = \frac{(b+1)f^{(k)} + (a-b-1)}{bf^{(k)} + (a-b)}, \tag{2.29}$$

where $a(\neq 0), b$ are two constants. Using the argument in (i), we can obtain $f^{(k)}g^{(k)} \equiv 1$ or $f \equiv g$. We here omit the details.

The proof of Lemma 2.5 is completed. □

3. Proof of Theorem 1.4

Proof. Let $F(z) = f^n(f-1)$ and $G(z) = g^n(g-1)$. We have

$$\Delta_1 = 2\Theta(\infty, f) + (k+2)\Theta(\infty, g) + \Theta(0, f) + \Theta(0, g) + \delta_{k+1}(0, f) + \delta_{k+1}(0, g). \tag{3.1}$$

Since

$$\begin{aligned} \Theta(0, F) &= 1 - \overline{\lim_{r\to\infty}} \frac{\overline{N}(r, 1/F)}{T(r, F)} = 1 - \overline{\lim_{r\to\infty}} \frac{\overline{N}(r, 1/(f^n(f-1)))}{(n+1)T(r, f)} \\ &= 1 - \overline{\lim_{r\to\infty}} \frac{\overline{N}(r, 1/f) + \overline{N}(r, 1/(f-1))}{(n+1)T(r, f)} \geq 1 - \overline{\lim_{r\to\infty}} \frac{2T(r, f)}{(n+1)T(r, f)} \geq \frac{n-1}{n+1}, \end{aligned} \tag{3.2}$$

similarly,

$$\begin{aligned} \Theta(0, G) &\geq \frac{n-1}{n+1}, \\ \Theta(\infty, F) &\geq \frac{n}{n+1}, \\ \Theta(\infty, G) &\geq \frac{n}{n+1}. \end{aligned} \tag{3.3}$$

Next, by Lemma 2.1, we have

$$\begin{aligned} \delta_{k+1}(0, F) &= 1 - \overline{\lim_{r\to\infty}} \frac{N_{k+1}(r, 1/F)}{T(r, F)} \geq 1 - \overline{\lim_{r\to\infty}} \frac{(k+1)\overline{N}(r, 1/F)}{T(r, F)} \\ &\geq 1 - \overline{\lim_{r\to\infty}} \frac{(k+2)T(r, f)}{(n+1)T(r, f)} \geq 1 - \frac{k+2}{n+1} = \frac{n-(k+1)}{n+1}. \end{aligned} \tag{3.4}$$

Similarly

$$\delta_{k+1}(0,G) \geq 1 - \frac{k+2}{n+1} = \frac{n-(k+1)}{n+1}. \tag{3.5}$$

From (3.2)–(3.5), we get

$$\Delta_1 \geq 2\frac{n}{n+1} + (k+2)\frac{n}{n+1} + \frac{n-1}{n+1} + \frac{n-1}{n+1} + \frac{n-(k+1)}{n+1} + \frac{n-(k+1)}{n+1}. \tag{3.6}$$

Since $n > 3k+11$, we get $\Delta_1 > k+7$. Considering that $F^{(k)}$ and $G^{(k)}$ share $(1,2)$, then, by Lemma 2.5, we deduce that either $F^{(k)}G^{(k)} \equiv 1$ or $F \equiv G$.

Next, we consider the following two cases.

Case 1. $F^{(k)}G^{(k)} \equiv 1$, that is, $[f^n(z)(f(z)-1)]^{(k)} \cdot [g^n(z)(g(z)-1)]^{(k)} \equiv 1$.

Case 2. $F \equiv G$, that is,

$$f^n(f-1) = g^n(g-1). \tag{3.7}$$

Suppose that $f \not\equiv g$; then we consider following two cases.

(i) Let $h = f/g$ be a constant. Then from (3.7) it follows that $h \neq 1, h^n \neq 1, h^{n+1} \neq 1$ and $g = (1-h^n)/(1-h^{n+1}) = \textit{constant}$, which leads to contradiction.

(ii) Let $h = f/g$ be not a constant. Since $f \not\equiv g$, we have $h \not\equiv 1$, and hence we deduce that $g = (1-h^n)/(1-h^{n+1})$ and $g = ((1-h^n)/(1-h^{n+1}))h = (1+h+h^2+\cdots+h^{n-1})h/(1+h+h^2+\cdots+h^n)$, where h is a nonconstant meromorphic function. It follows that

$$T(r,f) = T(r,gh) = nT(r,h) + S(r,f). \tag{3.8}$$

On the other hand, by the second fundamental theorem, we get

$$\overline{N}(r,f) = \sum_{j=1}^{n}\overline{N}\left(r,\frac{1}{h-\alpha_j}\right) \geq (n-2)T(r,h) + S(r,f), \tag{3.9}$$

where $\alpha_j(\neq 1)(j=1,2,\ldots,n)$ are distinct roots of the algebraic equation $h^{n+1}=1$.

So we have

$$\Theta(\infty,f) = 1 - \overline{\lim_{r\to\infty}}\frac{\overline{N}(r,f)}{T(r,f)} \leq 1 - \overline{\lim_{r\to\infty}}\frac{(n-2)T(r,h)+S(r,f)}{nT(r,h)+S(r,f)} \leq 1 - \frac{n-2}{n} = \frac{2}{n}, \tag{3.10}$$

which contradicts the assumption that $\Theta(\infty,f) > 2/n$. Thus $f \equiv g$. This completes the proof of Theorem 1.4. □

4. Proof of Theorem 1.5

Proof. From (3.2)–(3.5), we get

$$\Delta_2 \geq (k+3)\frac{n}{n+1} + (k+2)\frac{n}{n+1} + \frac{n-1}{n+1} + \frac{n-1}{n+1} + 2\frac{n-(k+1)}{n+1} + \frac{n-(k+1)}{n+1}. \tag{4.1}$$

Since $n > 5k+14$, we get $\Delta_2 > 2k+9$. Considering that $F^{(k)}$ and $G^{(k)}$ share $(1,2)$, then, by Lemma 2.5, we deduce that either $F^{(k)}G^{(k)} \equiv 1$ or $F \equiv G$.

Next, by using the argument in Theorem 1.4, we obtain the conclusion of Theorem 1.5. We here omit the details. □

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