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Research Article

Invariant Points and ε-Simultaneous Approximation

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We generalize and extend Brosowski-Meinardus type results on invariant points from the set of best approximation to the set of ε-simultaneous approximation. As a consequence some results on ε-approximation and best approximation are also deduced. The results proved in this paper generalize and extend some of the known results on the subject.

1. Introduction and Preliminaries

Fixed point theory has gained impetus, due to its wide range of applicability, to resolve diverse problems emanating from the theory of nonlinear differential equations, theory of nonlinear integral equations, game theory, mathematical economics, control theory, and so forth. For example, in theoretical economics, such as general equilibrium theory, a situation arises where one needs to know whether the solution to a system of equations necessarily exists; or, more specifically, under what conditions will a solution necessarily exist. The mathematical analysis of this question usually relies on fixed point theorems. Hence finding necessary and sufficient conditions for the existence of fixed points is an interesting aspect.

Fixed point theorems have been used in many instances in best approximation theory. It is pertinent to say that in best approximation theory, it is viable, meaningful, and potentially productive to know whether some useful properties of the function being approximated is inherited by the approximating function. The idea of applying fixed point theorems to approximation theory was initiated by Meinardus [1]. Meinardus introduced the notion of invariant approximation in normed linear spaces. Brosowski [2] proved the following theorem on invariant approximation using fixed point theory by generalizing the result of Meinardus [1].

Theorem 1.1. *Let T be a linear and nonexpansive operator on a normed linear space E. Let C be a T-invariant subset of E and x a T-invariant point. If the set $P_C(x)$ of best C-approximants to x is nonempty, compact, and convex, then it contains a T-invariant point.*

Subsequently, various generalizations of Brosowski's results appeared in the literature. Singh [3] observed that the linearity of the operator T and convexity of the set $P_C(x)$ in Theorem 1.1 can be relaxed and proved the following.

Theorem 1.2. *Let $T : E \to E$ be a nonexpansive self-mapping on a normed linear space E. Let C be a T-invariant subset of E and x a T-invariant point. If the set $P_C(x)$ is nonempty, compact, and star shaped, then it contains a T-invariant point.*

Singh [4] further showed that Theorem 1.2 remains valid if T is assumed to be nonexpansive only on $P_C(x) \cup \{x\}$. Since then, many results have been obtained in this direction (see Chandok and Narang [5, 6], Mukherjee and Som [7], Mukherjee and Verma [8], Narang and Chandok [9–11], Rao and Mariadoss [12], and references cited therein).

In this paper we prove some similar types of results on T-invariant points for the set of ε-simultaneous approximation in a metric space (X, d). Some results on T-invariant points for the set of ε-approximation and best approximation are also deduced. The results proved in the paper generalize and extend some of the results of [6, 8–13] and of few others.

Let G be a nonempty subset of a metric space (X, d) and let F be a nonempty bounded subset of X. For $x \in X$, let $d_F(x) = \sup\{d(y, x) : y \in F\}$, $D(F, G) = \inf\{d_F(x) : x \in G\}$ and $P_G(F) = \{g_0 \in G : d_F(g_0) = D(F, G)\}$. An element $g_0 \in P_G(F)$ is said to be a *best simultaneous approximation* of F with respect to G.

For $\varepsilon > 0$, we define $P_{G(\varepsilon)}(F) = \{g_0 \in G : d_F(g_0) \leq D(F, G) + \varepsilon\} = \{g_0 \in G : \sup_{y\in F} d(y, g_0) \leq \inf_{g\in G} \sup_{y\in F} d(y, g) + \varepsilon\}$. An element $g_0 \in P_{G(\varepsilon)}(F)$ is said to be a *ε-simultaneous approximation* of F with respect to G.

It can be easily seen that for $\varepsilon > 0$, the set $P_{G(\varepsilon)}(F)$ is always a nonempty bounded set and is closed if G is closed.

In case $F = \{p\}$, $p \in X$, then elements of $P_G(p)$ are called *best approximations* to p in G and of $P_{G(\varepsilon)}(p)$ are called *$\boldsymbol{\varepsilon}$-approximation* to p in G.

A sequence $\langle y_n \rangle$ in G is called a *$\boldsymbol{\varepsilon}$-minimizing sequence* for F, if $\lim \sup_{x\in F} d(x, y_n) \leq D(F, G) + \varepsilon$. The set G is said to be *ε-simultaneous approximatively compact with respect to F* if for every $x \in F$, each ε-minimizing sequence $\langle y_n \rangle$ in G has a subsequence $\langle y_{n_i} \rangle$ converging to an element of G.

Let (X, d) be a metric space. A continuous mapping $W : X \times X \times [0, 1] \to X$ is said to be a *convex structure* on X if for all $x, y \in X$ and $\lambda \in [0, 1]$,

$$d(u, W(x, y, \lambda)) \leq \lambda d(u, x) + (1 - \lambda) d(u, y) \tag{1.1}$$

holds for all $u \in X$. The metric space (X, d) together with a convex structure is called a *convex metric space* [14].

A convex metric space (X, d) is said to satisfy *Property (I)* [15] if for all $x, y, p \in X$ and $\lambda \in [0, 1]$,

$$d(W(x, p, \lambda), W(y, p, \lambda)) \leq \lambda d(x, y). \tag{1.2}$$

A normed linear space and each of its convex subset are simple examples of convex metric spaces with W given by $W(x, y, \lambda) = \lambda x + (1 - \lambda)y$ for $x, y \in X$ and $0 \leq \lambda \leq 1$. There are many convex metric spaces which are not normed linear spaces (see [14]). Property (I) is always satisfied in a normed linear space.

A subset K of a convex metric space (X, d) is said to be

(i) a *convex set* [14] if $W(x, y, \lambda) \in K$ for all $x, y \in K$ and $\lambda \in [0, 1]$;

(ii) *p-star shaped* [16] where $p \in K$, provided $W(x, p, \lambda) \in K$ for all $x \in K$ and $\lambda \in [0, 1]$;

(iii) *star shaped* if it is p-star shaped for some $p \in K$.

Clearly, each convex set is star shaped but not conversely.

A self-map T on a metric space (X, d) is said to be

(i) *contraction* if there exists k, $0 \leq k < 1$ such that $d(Tx, Ty) \leq kd(x, y)$ for all $x, y \in X$;

(ii) *nonexpansive* if $d(Tx, Ty) \leq d(x, y)$ for all $x, y \in X$;

(iii) *quasi-nonexpansive* if the set $F(T)$ of fixed points of T is nonempty and $d(Tx, p) \leq d(x, p)$ for all $x \in X$ and $p \in F(T)$.

A nonexpansive mapping T on X with $F(T) \neq \emptyset$ is quasi-nonexpansive, but not conversely. A linear quasi-nonexpansive mapping on a Banach space is nonexpansive. But there exist continuous and discontinuous nonlinear quasi-nonexpansive mappings that are not nonexpansive.

2. Main Results

To start with, we prove the following proposition on ε-simultaneous approximation which will be used in the sequel.

Proposition 2.1. *Let F be a nonempty bounded subset of a metric space (X, d), and let C be a nonempty subset of X. If C is ε-simultaneous approximatively compact with respect to F, then the set $P_{C(\varepsilon)}(F)$ is a nonempty compact subset of C.*

Proof. Since $\varepsilon > 0$, $P_{C(\varepsilon)}(F)$ is nonempty. We now show that $P_{C(\varepsilon)}(F)$ is compact. Let $\langle y_n \rangle$ be a sequence in $P_{C(\varepsilon)}(F)$. Then $\limsup_{x \in F} d(x, y_n) \leq D(F, C) + \varepsilon$, that is, $\langle y_n \rangle$ is an ε-minimizing sequence for C. Since C is ε-simultaneous approximatively compact with respect to F, there is a subsequence $\langle y_{n_i} \rangle$ such that $\langle y_{n_i} \rangle \to y \in C$. Consider

$$\begin{aligned} \sup_{x \in F} d(x, y) &= \sup_{x \in F} d\left(x, \lim y_{n_i}\right) \\ &= \lim \sup_{x \in F} d\left(x, y_{n_i}\right) \\ &\leq D(F, C) + \varepsilon. \end{aligned} \tag{2.1}$$

This implies that $y \in P_{C(\varepsilon)}(F)$. Thus we get a subsequence $\langle y_{n_i} \rangle$ of $\langle y_n \rangle$ converging to an element $y \in P_{C(\varepsilon)}(F)$. Hence $P_{C(\varepsilon)}(F)$ is compact. □

For $F = \{x\}$, we have the following result on the set of ε-approximation.

Corollary 2.2 (see [9]). *If C is an ε-approximatively compact set in a metric space (X, d) then $P_{C(\varepsilon)}(x)$ is a nonempty compact set.*

For $F = \{x\}$ and $\varepsilon = 0$, we have the following result on the set of best approximation.

Corollary 2.3 (see [10]). *Let C be a nonempty approximatively compact subset of a metric space (X, d), $x \in X$, and P_C be the metric projection of X onto C defined by $P_C(x) = \{y \in C : d(x, y)\} \equiv d(x, C)$. Then $P_C(x)$ is a nonempty compact subset of C.*

We will be using the following result of Hardy and Rogers [17] in proving our first theorem.

Lemma 2.4. *Let T be a mapping from a complete metric space (X, d) into itself satisfying*

$$d(Tx, Ty) \le a\left[d(x, Tx) + d(y, Ty)\right] + b\left[d(y, Tx) + d(x, Ty)\right] + cd(x, y), \tag{2.2}$$

for any $x, y \in X$, where a, b, and c are nonnegative numbers such that $2a + 2b + c \le 1$. Then T has a unique fixed point u in X. In fact for any $x \in X$, the sequence $\{T^n x\}$ converges to u.

Theorem 2.5. *Let T be a continuous self-map on a complete convex metric space (X, d) with Property (I) and satisfying inequality (2.2), let C be a T-invariant subset of X, and let F be a nonempty bounded subset of X such that $Tx = x$ for all $x \in F$. If $P_{C(\varepsilon)}(F)$ is compact, and star shaped, then it contains a T-invariant point.*

Proof. Let $z \in P_{C(\varepsilon)}(F)$ be arbitrary. Then by (2.2), we have for all $x \in F$

$$\begin{aligned} d(x, Tz) &= d(Tx, Tz) \\ &\le a[d(x, Tx) + d(z, Tz)] + b[d(z, Tx) + d(x, Tz)] + cd(x, z) \\ &= a[d(z, Tz)] + b[d(z, Tx) + d(x, Tz)] + cd(x, z) \\ &\le a[d(z, x) + d(x, Tz)] + b[d(z, x) + d(x, Tx) + d(x, Tz)] + cd(x, z) \\ &= (a + b + c)d(x, z) + (a + b)[d(x, Tz)]. \end{aligned} \tag{2.3}$$

This gives

$$\begin{gathered} (1 - a - b)d(x, Tz) \le (a + b + c)d(x, z) \\ d(x, Tz) \le d(x, z) \end{gathered} \tag{2.4}$$

since $2a + 2b + c \le 1$. Therefore, using definition of $P_{C(\varepsilon)}(F)$, we get

$$\sup_{x \in F}\{d(x, Tz)\} \le \sup_{x \in F}\{d(x, z)\} \le D(F, C) + \varepsilon. \tag{2.5}$$

Hence $Tz \in P_{C(\varepsilon)}(F)$. Therefore T is a self-map on $P_{C(\varepsilon)}(F)$.

Let q be the star-center of $P_{C(\varepsilon)}(F)$. Define $T_n : P_{C(\varepsilon)}(F) \to P_{C(\varepsilon)}(F)$ as $T_n x = W(Tx, q, \lambda_n)$, $x \in P_{C(\varepsilon)}(F)$ where $\langle y_n \rangle$ is a sequence in $(0,1)$ such that $\lambda_n \to 1$. Consider

$$\begin{aligned} d(T_n x, T_n y) &= d(W(Tx, q, \lambda_n), W(Ty, q, \lambda_n)) \\ &\leq \lambda_n d(Tx, Ty) \\ &\leq \lambda_n [a[d(x, Tx) + d(y, Ty)] + b[d(y, Tx) + d(x, Ty)] + cd(x, y)] \\ &\leq a[d(x, Tx) + d(y, Ty)] + b[d(y, Tx) + d(x, Ty)] + cd(x, y), \end{aligned} \tag{2.6}$$

where $(2a+2b+c) \leq 1$. Therefore by Lemma 2.4, each T_n has a unique fixed point z_n in $P_{C(\varepsilon)}(F)$. Since $P_{C(\varepsilon)}(F)$ is compact, there is a subsequence $\langle z_{n_i} \rangle$ of $\langle z_n \rangle$ such that $z_{n_i} \to z_\circ \in P_{C(\varepsilon)}(F)$. We claim that $Tz_\circ = z_\circ$. Consider $z_{n_i} = \mathrm{T}_{n_i} z_{n_i} = W(Tz_{n_i}, q, \lambda_{n_i}) \to Tz_\circ$, as T is continuous. Thus $z_{n_i} \to Tz_\circ$ and consequently, $Tz_\circ = z_\circ$, that is, $z_\circ \in P_{C(\varepsilon)}(F)$ is a T-invariant point. □

Since for an ε-simultaneous approximatively compact subset C of a metric space (X, d) the set of ε-simultaneous C-approximant is nonempty and compact (Proposition 2.1), we have the following result.

Corollary 2.6. *Let T be a continuous self-map on a complete convex metric space (X, d) with Property (I) and satisfying inequality* (2.2)*, let F be a nonempty bounded subset of X such that $Tx = x$ for all $x \in F$, and let C be a T-invariant subset of X. If C is ε-simultaneous approximatively compact with respect to F and $P_{C(\varepsilon)}(F)$ is star shaped, then it contains a T-invariant point.*

Corollary 2.7 (see [8]). *Let T be a continuous self-map on a Banach space X satisfying* (2.2)*, let C be an approximatively compact and T-invariant subset of X. Let $Tx_i = x_i (i = 1, 2)$ for some x_1, x_2 not in $cl(C)$. If the set of best simultaneous C-approximants to x_1, x_2 is star shaped, then it contains a T-invariant point.*

Corollary 2.8 (see [11]). *Let T be a mapping on a metric space (X, d), let C be a T-invariant subset of X and x a T-invariant point. If $P_C(x)$ is a nonempty, compact set for which there exists a contractive jointly continuous family $\mathfrak{F}$ of functions and T is nonexpansive on $P_C(x) \cup \{x\}$ then $P_C(x)$ contains a T-invariant point.*

Corollary 2.9. *Let T be a mapping on a convex metric space (X, d) with Property (I),let C be an approximatively compact, p-star shaped, T-invariant subset of X and let x be a T-invariant point. If T is nonexpansive on $P_C(x) \cup \{x\}$, then $P_C(x)$ contains a T-invariant point.*

Corollary 2.10 (see [10, Theorem 4]). *Let T be a quasi-nonexpansive mapping on a convex metric space (X, d) with Property (I), let C be a T-invariant subset of X, and let x be a T-invariant point. If $P_C(x)$ is nonempty, compact, and star shaped, and T is nonexpansive on $P_C(x)$, then $P_C(x)$ contains a T-invariant point.*

Corollary 2.11 (see [10, Theorem 5]). *Let T be a quasi-nonexpansive mapping on a convex metric space (X, d) with Property (I), let C be an approximatively compact, T-invariant subset of X, and let x be a T-invariant point. If $P_C(x)$ is star shaped and T is nonexpansive on $P_C(x)$, then $P_C(x)$ contains a T-invariant point.*

Remark 2.12. Theorem 2.5 improves and generalizes Theorem 1 of Narang and Chandok [9] and of Rao and Mariadoss [12].

Definition 2.13. A subset K of a metric space (X, d) is said to be *contractive* if there exists a sequence $\langle f_n \rangle$ of contraction mappings of K into itself such that $f_n y \to y$ for each $y \in K$.

Theorem 2.14. *Let T be a nonexpansive self-mapping on a metric space (X, d), let C be a T-invariant subset of X, and let F be a nonempty bounded subset of X such that $Tx = x$ for all $x \in F$. If the set $P_{C(\varepsilon)}(F)$ is compact and contractive, then the set $P_{C(\varepsilon)}(F)$ contains a T-invariant point.*

Proof. Proceeding as in Theorem 2.5, we can prove that T is a self-map of $P_{C(\varepsilon)}(F)$. Since $P_{C(\varepsilon)}(F)$ is contractive, there exists a sequence $\langle f_n \rangle$ of contraction mapping of $P_{C(\varepsilon)}(F)$ into itself such that $f_n z \to z$ for every $z \in P_{C(\varepsilon)}(F)$.

Clearly, $f_n T$ is a contraction on the compact set $P_{C(\varepsilon)}(F)$ for each n and so by Banach contraction principle, each $f_n T$ has a unique fixed point, say z_n in $P_{C(\varepsilon)}(F)$. Now the compactness of $P_{C(\varepsilon)}(F)$ implies that the sequence $\langle z_n \rangle$ has a subsequence $\langle z_{n_i} \rangle \to z_\circ \in D$. We claim that $z_\circ$ is a fixed point of T. Let $\varepsilon > 0$ be given. Since $z_{n_i} \to z_\circ$ and $f_n T z_\circ \to T z_\circ$, there exist a positive integer m such that for all $n_i \geq m$

$$d(z_{n_i}, z_\circ) < \frac{\varepsilon}{2}, \qquad d(f_{n_i} T z_\circ, T z_\circ) < \frac{\varepsilon}{2}. \tag{2.7}$$

Again,

$$d(f_{n_i} T z_{n_i}, f_{n_i} T z_\circ) \leq d(z_{n_i}, z_\circ) < \frac{\varepsilon}{2}. \tag{2.8}$$

Hence

$$\begin{aligned} d(f_{n_i} T z_{n_i}, T z_\circ) &\leq d(f_{n_i} T z_{n_i}, f_{n_i} T z_\circ) + d(f_{n_i} T z_\circ, T z_\circ) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}, \end{aligned} \tag{2.9}$$

that is, $d(f_{n_i} T z_{n_i}, T z_\circ) < \varepsilon$ for all $n_i \geq m$ and so $f_{n_i} T z_{n_i} \to T z_\circ$. But $f_{n_i} T z_{n_i} = z_{n_i} \to z_\circ$ and therefore $T z_\circ = z_\circ$. □

Using Proposition 2.1 we have the following result.

Corollary 2.15. *Let T be a nonexpansive self-mapping on a metric space (X, d), let C be a T-invariant subset of X, and let F be a nonempty bounded subset of X such that $Tx = x$ for all $x \in F$. If C is ε-simultaneous approximatively compact with respect to F and the set $P_{C(\varepsilon)}(F)$ is contractive, and T-invariant, then $P_{C(\varepsilon)}(F)$ contains a T-invariant point.*

Corollary 2.16 (see [9]). *Let T be a self-mapping on a metric space (X, d), let G be a T-invariant subset of X, and let x be a T-invariant point. If the set D of ε-approximant to x is compact, contractive and T is nonexpansive on $D \cup \{x\}$, then D contains a T-invariant point.*

Remark 2.17. Theorem 2.14 also improves and generalizes the corresponding results of Brosowski [2], Mukherjee and Verma [8, 13], Chandok and Narang [9], Rao and Mariadoss [12], and of Singh [3].

Definition 2.18. For each bounded subset G of a metric space (X, d), the *Kuratowski's measure of noncompactness* of G, $\alpha[G]$ is defined as

$$\begin{aligned} \alpha[G] = \inf\{\varepsilon > 0 : &G \text{ is covered by a finite number of closed} \\ &\text{balls centered at points of } X \text{ of radius} \leq \varepsilon\}. \end{aligned} \tag{2.10}$$

A mapping $T : X \to X$ is called *condensing* if for all bounded sets $G \subset X$, $\alpha[T(G)] \leq \alpha[G]$.

We will be using the following result of [18] on fixed points of nonexpansive condensing maps.

Lemma 2.19. *Let X be a complete contractive metric space with contractions $\{f_n\}$. Let C be a closed bounded subsets of X and $T : C \to C$ is nonexpansive and condensing, then T has a fixed point in C.*

Using the above lemma and Theorem 2.5, we now prove the following result.

Theorem 2.20. *Let (X, d) be a complete, contractive metric space with contractions f_n. Let G be a closed and bounded subset of X and let F be a nonempty bounded subset of X. If T is a nonexpansive and condensing self-map on X such that $Tx = x$ for all $x \in F$, then $P_{G(\varepsilon)}(F)$ has a T-invariant point.*

Proof. As G is closed and bounded, $P_{G(\varepsilon)}(F)$ is nonempty, closed and bounded. Using Theorem 2.5, we can prove that T is a self-map of $P_{G(\varepsilon)}(F)$. Now a direct application of Lemma 2.19, gives a T-invariant point in $P_{G(\varepsilon)}(F)$. □

Corollary 2.21 (see [8, Theorem 3.1]). *Let X be a complete, contractive metric space with contractions f_n. Let G be a closed and bounded subset of X. If T is a nonexpansive and condensing self-map on X such that $Tx_1 = x_1$ and $Tx_2 = x_2$ for some $x_1, x_2 \in X$, and $D = P_G(x_1, x_2)$ is nonempty, then it has a T-invariant point.*

Corollary 2.22 (see [12, Theorem 4]). *Let X be a complete, contractive metric space with contractions f_n. Let G be a closed and bounded subset of X. If T is a nonexpansive and condensing self-map on X such that $Tx = x$ for some $x \in X$, and $P_G(x)$ is nonempty, then it has a T-invariant point.*

Definition 2.23. A mapping T on a metric space (X, d) is called a *Kannan mapping* [19] if there exists $\alpha \in (0, 1/2)$ such that

$$d(Tx, Ty) \leq \alpha\left[d(x, Tx) + d(y, Ty)\right] \tag{2.11}$$

for all $x, y \in X$.

Kannan [19] proved that if X is complete, then every Kannan mapping has a unique fixed point.

For ε-simultaneous approximation, we have the following result.

Theorem 2.24. *Let G be a nonempty subset of a complete metric space (X, d) and let F be a nonempty bounded subset of X. Let T be a self-map on X with $Tx = x$ for all $x \in F$ and T^m satisfies,*

$$d(T^m y, T^m z) \le \alpha\left[d(y, T^m y) + d(z, T^m z)\right], \tag{2.12}$$

for some positive integer m, all $y, z \in X$ and some fixed $0 < \alpha < 1/2$. If $D = P_{G(\varepsilon)}(F)$ is compact, then it has a unique fixed point of T.

Proof. As $Tx = x$, $T^n x = x$ for all positive integers n. Let $y_0 \in D$. Then, for $0 < \alpha < 1/2$,

$$\begin{aligned} d(x, T^m y_0) &= d(T^m x, T^m y_0) \\ &\le \alpha\left[d(x, T^m x) + d(y_0, T^m y_0)\right] \\ &= \alpha d(y_0, T^m y_0) \\ &\le \alpha\left[d(y_0, x) + d(x, T^m y_0)\right], \end{aligned} \tag{2.13}$$

which implies that

$$d(x, T^m y_0) \le \frac{\alpha}{1-\alpha} d(y_0, x). \tag{2.14}$$

Further, we have

$$\sup_{x \in F} d(x, T^m y_0) \le \sup_{x \in F}\{d(y_0, x)\} \le D(F, G) + \varepsilon. \tag{2.15}$$

Therefore, $T^m y_0 \in D$, $T^m(D) \subset D$. Since T^m satisfies the conditions of Kannan map, T^m has a unique fixed point x_0 in D. Now, $T^m(Tx_0) = T(T^m x_0) = Tx_0$, implies that Tx_0 is a fixed point of T^m. But the fixed point of T^m is unique and equals x_0. Therefore $Tx_0 = x_0$ and hence x_0 is a unique fixed point of T in D. □

Corollary 2.25. *Let F be a nonempty bounded subset of a complete metric space (X, d) and G a subset of X. Let G be ε-simultaneous approximatively compact with respect to F, and T a self map on X with $T_x = x$ for all $x \in F$, and T^m satisfies*

$$d(T^m y, T^m z) \le \alpha\left[d(y, T^m y) + d(z, T^m z)\right], \tag{2.16}$$

for some positive integer m, all $y, z \in X$ and some fixed $0 < \alpha < 1/2$ then $D = P_{G(\varepsilon)}(F)$ has a unique fixed point of T.

Remarks 2.26. Theorem 2.24 extends and generalizes Theorem 3.2 of Mukherjee and Verma [8] and Theorem 5 of Rao and Mariadoss [12] from the set of best simultaneous approximation and best approximation, respectively, to ε-simultaneous approximation.

For $\varepsilon > 0$, we define $R_{G(\varepsilon)}(F) = \{g_0 \in G : \sup_{g \in G} d(g, g_0) + \varepsilon \le \inf_{g \in G} \sup_{y \in F} d(y, g)\}$. An element $g_0 \in R_{G(\varepsilon)}(F)$ is said to be a *ε-simultaneous coapproximation* of F with respect to G.

A mapping $T : X \to X$ satisfies *condition (A)* (see [13]) if $d(Tx, y) \leq d(x, y)$ for all $x, y \in X$.

We now prove a result for T-invariant points from the set of ε-simultaneous coapproximations.

Theorem 2.27. *Let T be a self-map satisfying condition (A) and inequality* (2.2) *on a convex metric space (X, d) satisfying Property (I), let G be a subset of X, and let F be a nonempty bounded subset of X such that $R_{G(\varepsilon)}(F)$ is compact and star shaped. Then $R_{G(\varepsilon)}(F)$ contains a T-invariant point.*

Proof. Let $g_\circ \in R_{G(\varepsilon)}(F)$. Consider

$$d(Tg_\circ, g) + \varepsilon \leq d(g_\circ, g) + \varepsilon \leq \inf_{g \in G} \sup_{y \in F} d(y, g), \tag{2.17}$$

and so $Tg_\circ \in R_{G(\varepsilon)}(F)$, that is, $T : R_{G(\varepsilon)}(F) \to R_{G(\varepsilon)}(F)$. Since $R_{G(\varepsilon)}(F)$ is star shaped, there exists $p \in R_{G(\varepsilon)}(F)$ such that $W(z, p, \lambda) \in R_{G(\varepsilon)}(F)$ for all $z \in R_{G(\varepsilon)}(F)$, $\lambda \in [0, 1]$. Let $\langle k_n \rangle, 0 \leq k_n < 1$, be a sequence of real numbers such that $k_{\mathrm{n}} \to 1$ as $n \to \infty$. Define T_n as $T_n(z) = W(Tz, p, k_n)$, $z \in R_{G(\varepsilon)}(F)$. Since T is a self-map on $R_{G(\varepsilon)}(F)$ and $R_{G(\varepsilon)}(F)$ is star shaped, each T_n is a well defined and maps $R_{G(\varepsilon)}(F)$ into $R_{G(\varepsilon)}(F)$. Moreover,

$$\begin{aligned} d(T_n y, T_n z) &= d(W(Ty, p, k_n), W(Tz, p, k_n)) \\ &\leq k_n d(Ty, Tz) \\ &\leq k_n [a[d(y, Ty) + d(z, Tz)] + b[d(z, Ty) + d(y, Tz)] + cd(y, z)], \end{aligned} \tag{2.18}$$

where $k_n[2a + 2b + c] \leq 1$. So by Lemma 2.4 each T_n has a unique fixed point $x_n \in R_{G(\varepsilon)}(F)$, that is, $T_n x_n = x_n$ for each n. Since $R_{G(\varepsilon)}(F)$ is compact, $\langle x_n \rangle$ has a subsequence $x_{n_i} \to x \in R_{G(\varepsilon)}(F)$. Now, we claim that $Tx = x$. As $d(x_{n_i}, Tx) \leq d(x_{n_i}, x)$. On letting $n \to \infty$, we have $x_{n_i} \to Tx$. Therefore $Tx = x$, that is, x is T-invariant, hence the result. □

For $F = \{x\}$, $x \in X$, we have the following result on the set of ε-coapproximation.

Corollary 2.28 (see [9, Theorem 4]). *Let T be a self-map satisfying condition (A) on a convex metric space (X, d) satisfying Property (I), let G be a subset of X such that $R_{G(\varepsilon)}(x)$ is nonempty compact, star shaped, and let T be nonexpansive on $R_{G(\varepsilon)}(x)$. Then there exists a $g_\circ \in R_{G(\varepsilon)}(x)$ such that $Tg_\circ = g_\circ$.*

Remarks 2.29. (i) Theorem 2.27 also improves and generalizes Theorem 4.1 of Mukherjee and Verma [13] from the set of best approximation to ε-simultaneous approximation.

(ii) By taking $F = \{x_1, x_2\}$, $x_1, x_2 \in X$, the set $P_{G(\varepsilon)}(F)$ (respectively, $R_{G(\varepsilon)}(F)$) is the set of ε-simultaneous approximation (respectively, ε-simultaneous coapproximation) to the pair of points x_1, x_2 and so the results of this paper generalize and extend the corresponding results proved in [6].

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