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Research Article

No Null-Helix Mannheim Curves in the Minkowski Space $\mathbb{E}_1^3$

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We study a null Mannheim curve with time-like or space-like Mannheim partner curve in the Minkowski 3-space $\mathbb{E}_1^3$. We get the characterization of a null Mannheim curve. Then, we investigate there is no null-helix Mannheim curve in $\mathbb{E}_1^3$.

1. Introduction

In the study of the fundamental theory and the characterizations of space curves, the related curves for which there exist corresponding relations between the curves are very interesting and important problems. The most fascinating examples of such curve are associated curves, the curves for which at the corresponding points of them one of the Frenet vectors of a curve coincides with the one of Frenet vectors of the other curve. The well-known associated curve is Bertrand curve which is characterized as a kind of corresponding relation between the two curves. The relation is that the principal normal of a curve is the principal normal of the other curve, that is, the Bertrand curve is a curve which shares the normal line with the other curve [1].

Furthermore, Bertrand curves are not only the example of associated curves. Recently, a new definition of the associated curves was given by Liu and Wang [2]. They called these new curves as Mannheim partner curves. They showed that the curve γ_1 is the Mannheim partner of the other curve γ if and only if the curvature κ_1 and τ_1 of γ_1 satisfy the following equation:

$$\dot{\tau} = \frac{\kappa_1}{\lambda}\left(1 + \lambda^2 {\tau_1}^2\right), \tag{1.1}$$

for some nonzero constant λ. They also study the Mannheim curves in Minkowski 3-space. Some different characterizations of Mannheim partner curves are given by Orbay and Kasap [3]. Another example is null Mannheim curves from Öztekin and Ergüt [4]. Since a null vector and a nonnull vector are linearly independent in the Minkowski space $\mathbb{E}_1^3$, they have noticed that the Mannheim partner curve of a null curve cannot be a null curve. They defined the null Mannheim curves whose Mannheim partner curves are either time-like or space-like.

In this paper, we get the necessary and sufficient conditions for the null Mannheim curves. Then, we investigate there exists no null-helix Mannheim curve in the Minkowski 3-space $\mathbb{E}_1^3$.

2. Preliminaries

Let $\mathbb{E}_1^3$ be a 3-dimensional Lorentzian space and C a smooth null curve in $\mathbb{E}_1^3$, given by

$$\gamma(t) = \left(\gamma_1(t), \gamma_2(t), \gamma_3(t)\right), \quad t \in I \subset \mathbb{R}. \tag{2.1}$$

Then, the tangent vector field $l = \gamma'$ in $\mathbb{E}_1^3$ satisfies

$$\langle l, l \rangle = 0. \tag{2.2}$$

Denote by TC the tangent bundle of C and $TC^{\perp}$ the TC perpendicular. Clearly, $TC^{\perp}$ is a vector bundle over C of rank 2. Since ξ is null, the tangent bundle TC of C is a subbundle of $TC^{\perp}$ of rank 1. This implies that $TC^{\perp}$ is not complementary of TC in $\mathbb{E}_1^3|_C$. Thus, we must find a complementary vector bundle to TC of C in $\mathbb{E}_1^3$ which will play the role of the normal bundle $TC^{\perp}$ consistent with the classical non-degenerate theory.

Suppose $S(TC^{\perp})$ denotes the complementary vector subbundle to TC in $TC^{\perp}$; that is, we have

$$TC^{\perp} = TC \perp S\left(TC^{\perp}\right), \tag{2.3}$$

where $\perp$ means the orthogonal direct sum. It follows that $S(TC^{\perp})$ is a nondegenerate vector subbundle of $\mathbb{E}_1^3$, of rank of 1. We call $S(TC^{\perp})$ *a screen vector bundle of* C, which being non-degenerate, and we have

$$\mathbb{E}_1^3|_C = S\left(TC^{\perp}\right) \perp S\left(TC^{\perp}\right)^{\perp}, \tag{2.4}$$

where $S(TC^{\perp})^{\perp}$ is a complementary orthogonal vector subbundle to $S(TC^{\perp})$ in $\mathbb{E}_1^3|_C$ of rank 2.

We denote by $F(C)$ the algebra of smooth functions on C and by $\Gamma(E)$ the $F(C)$ module of smooth sections of a vector bundle F over C. We use the same notation for any other vector bundle.

Theorem 2.1 (see [5, 6]). *Let C be a null curve of a Lorentzian space $\mathbb{E}_1^3$ and $S(TC^{\perp})$ a screen vector bundle of C. Then, there exists a unique vector bundle* $\mathrm{ntr}(C)$ *over C of rank 1 such that there is a unique section $n \in \Gamma(\mathrm{ntr}(C))$ satisfying*

$$\langle l, n\rangle = 1, \qquad \langle n, n\rangle = \langle n, X\rangle = 0, \quad \forall X \in \Gamma\left(S\left(TC^{\perp}\right)\right). \tag{2.5}$$

We call the vector bundle $\mathrm{ntr}(C)$ the null transversal bundle of C with respect to $S(TC^{\perp})$. Next consider the vector bundle

$$\mathrm{tr}(C) = \mathrm{ntr}(C) \perp S\left(TC^{\perp}\right), \tag{2.6}$$

which from (2.5) is complementary but not orthogonal to TC in $\mathbb{E}_1^3|_C$.

More precisely, we have

$$\mathbb{E}_1^3|_C = TC \oplus \mathrm{tr}(C) = (TC \oplus \mathrm{ntr}(C)) \perp S\left(TC^{\perp}\right). \tag{2.7}$$

One calls $\mathrm{tr}(C)$ the transversal vector bundle of C with respect to $S(TC^{\perp})$. The vector field n in Theorem 2.1 is called the null transversal vector field of C with respect to ξ. As $\{\xi, n\}$ is a null basis of $\Gamma(TC \oplus \mathrm{ntr}(C))$ satisfying (2.5), any screen vector bundle $S(TC^{\perp})$ of C is Euclidean.

Note that for any arbitrary parameter t on C and a screen vector bundle $S(TC^{\perp})$ one finds a distinguished parameter given by

$$p = \int_{t_0^*}^{t^*} \exp\left(\int_{s_0}^{s} \langle \gamma'', n\rangle dt^*\right) ds, \tag{2.8}$$

where n is the null transversal vector field with respect to $S(TC^{\perp})$ and γ'.

Let $C = C(p)$ be a smooth null curve, parametrized by the distinguished parameter p instead of t such that $\|\gamma''\| = \kappa \neq 0$ ([6]). Using (2.5) and (2.7) and taking into account that the screen vector bundle $S(TC^{\perp})$ is Euclidean of rank 1, one obtains the following Frenet equations [1]:

$$\frac{d}{dp}\begin{bmatrix} l \\ n \\ u \end{bmatrix} = \begin{bmatrix} 0 & 0 & \kappa \\ 0 & 0 & \tau \\ -\tau & -\kappa & 0 \end{bmatrix}\begin{bmatrix} l \\ n \\ u \end{bmatrix}. \tag{2.9}$$

Definition 2.2. Let γ be a curve in the Minkowski 3-space $\mathbb{E}_1^3$ and γ' a velocity of vector of γ. The curve γ is called *time-like (or space-like)* if $\langle \gamma', \gamma'\rangle < 0$ (or if $\langle \gamma', \gamma'\rangle > 0$).

Let T, N, B be the tangent, the principal normal, and the binormal of γ, respectively. Then, there are two cases for the Frenet formulae.

Case 1. T and B are space-like vectors, and N is a time-like vector

$$\frac{d}{ds}\begin{bmatrix} T \\ N \\ B \end{bmatrix} = \begin{bmatrix} & \kappa & 0 \\ \kappa & 0 & \tau \\ 0 & \tau & 0 \end{bmatrix}\begin{bmatrix} T \\ N \\ B \end{bmatrix}. \tag{2.10}$$

Case 2. T is a time-like vector, and N and B are space-like vectors

$$\frac{d}{d\tilde{s}}\begin{bmatrix} T \\ N \\ B \end{bmatrix} = \begin{bmatrix} & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix}\begin{bmatrix} T \\ N \\ B \end{bmatrix}, \tag{2.11}$$

where κ and τ are called the dual curvature and dual torsion of γ, respectively [5].

3. Null Mannheim Curves in $\mathbb{E}_1^3$

Definition 3.1. Let $C : \gamma(p)$ be a Cartan framed null curve and $C^* : \gamma^*(p^*)$ a time-like or space-like curve in the Minkowski space $\mathbb{E}_1^3$. If there exists a corresponding relationship between the space curves C and C^* such that the principal normal lines of C coincides with the binormal lines of C^* at the corresponding points of the curves, then C called a *null Mannheim curve* and C^* is called a *time-like or space-like Mannheim partner curve of* C. The pair of $\{C, C^*\}$ is said to be a *null Mannheim pair* [2, 4].

Theorem 3.2. *Let $C : \gamma(p)$ be a null Mannheim curve with time-like Mannheim partner curve $C^* : \gamma^*(p^*)$, and let $\{l(p), n(p), u(p)\}$ be the Cartan frame field along C and $\{T(p^*), N(p^*), B(p^*)\}$ the Frenet frame field along C^*. Then, C^* is the time-like Mannheim partner curve of γ if and only if its torsion τ^* is constant such that $\tau^* = \mp(1/\mu)$, where μ is nonzero constant.*

Proof. Assume that γ is a null Mannheim curve with time-like Mannheim partner curve γ^*. Then, by Definition 3.1, we can write

$$\gamma(p(p^*)) = \gamma^*(p^*) + \mu(p^*)B(p^*), \tag{3.1}$$

for some function $\mu(p^*)$. By taking the derivative of (3.1) with respect to p^* and applying the Frenet formulae, we have

$$l\frac{dp}{dp^*} = T + \mu' B + \mu(-\tau^* N). \tag{3.2}$$

Since u coincides with B, we get

$$\mu' = 0, \tag{3.3}$$

which means that μ is a nonzero constant. Thus, we have

$$l\frac{dp}{dp^*} = T - \mu\tau^* N. \tag{3.4}$$

Since l is null and from (3.4), we obtain

$$-1 + (\mu\tau^*)^2 = 0 \longrightarrow \tau^* = \mp\frac{1}{\mu}, \tag{3.5}$$

which means that γ^* is a time-like curve with constant torsion.

Conversely, let the torsion $\tau^*$of the time-like curve C^* be a constant with $\tau^* = \mp(1/\mu)$ for some nonzero constant μ. By considering a null curve $C : \gamma(p)$ defined by

$$\gamma(p^*) = \gamma^*(p^*) + \mu(p^*)B(p^*), \tag{3.6}$$

we prove that γ is a null Mannheim and γ^* is the time-like Mannheim partner curve of γ. By differentiating (3.6) with respect to p^*, we get

$$l\frac{dp}{dp^*} = T - \mu\tau^* N. \tag{3.7}$$

If we use $\tau^* = \mp 1/\mu$ in (3.7), we obtain

$$l\frac{dp}{dp^*} = T \mp N, \tag{3.8}$$

which means that l lies in the plane which is spanned by T and N, hence $l \perp B$. The proof is complete. □

Theorem 3.3. *A Cartan framed null curve γ in $\mathbb{E}^3_1$ is a null Mannheim curve with time-like Mannheim partner curve γ^* if and only if the torsion τ of γ is nonzero constant.*

Proof. Let $\gamma = \gamma(p)$ be a null Mannheim curve in $\mathbb{E}^3_1$. Suppose that $\gamma^* = \gamma^*(p^*)$ is a time-like curve whose binormal direction coincides with the principal normal of γ. Then, $B(p^*) = \mp u(p)$. Therefore, we can write

$$\gamma^*(p) = \gamma(p) + \mu(p)u(p), \tag{3.9}$$

for some function $\mu(p) \neq 0$. Differentiating (3.9) with respect to p, we obtain

$$T\frac{dp^*}{dp} = (1 - \mu\tau)l - \mu\kappa n + \mu' u. \tag{3.10}$$

Since the binormal direction of γ^* coincides with the principal normal of γ, we get $\langle T, u\rangle = 0$. Therefore, we have $\mu' = 0$ and μ is constant. By taking the derivative of (3.10), we get

$$\kappa^* N\left(\frac{dp^*}{dp}\right)^2 + T\frac{d^2p^*}{dp^2} = -\mu\tau' l - \mu\kappa' n + (1-2\mu\tau)\kappa u. \tag{3.11}$$

Since u is in the binormal direction of γ^*, we have

$$(1-2\mu\tau)\kappa = 0, \tag{3.12}$$

and hence

$$\tau = \frac{1}{2\mu} = \text{const.} \tag{3.13}$$

Conversely, similar to the proof of Theorem 3.2, we easily get a null Mannheim curve with time-like Mannheim partner curve. □

Proposition 3.4. *If $\gamma = \gamma(p)$ be a generalized null-helix in $\mathbb{E}_1^3$, then, the curve can not be a Mannheim curve.*

Proof. Suppose that $\gamma = \gamma(p)$ is a Mannheim curve in $\mathbb{E}_1^3$. Then, there exists the Mannheim partner curve $\gamma^* = \gamma^*(p^*)$ of $\gamma = \gamma(p)$ in $\mathbb{E}_1^3$. From Theorems 3.2 and 3.3, the torsions of the Mannheim pair $\{\gamma, \gamma^*\}$, τ and τ^*, are nonzero-constant. Since $\gamma = \gamma(p)$ be a generalized null-helix, κ/τ is constant, and thus κ is constant. Using (3.10), (3.13), and the fact that T is time-like, we have

$$\left(\frac{dp^*}{dp}\right)^2 = \mu\kappa = \text{const.} \tag{3.14}$$

From (3.11), we have

$$\kappa^* N\left(\frac{dp^*}{dp}\right)^2 = 0, \tag{3.15}$$

and thus, we get $\kappa^* = 0$. This shows that $\gamma^* = \gamma^*(p^*)$ is a straight line with nonzero torsion in $\mathbb{E}_1^3$, which is impossible. Therefore, $\gamma = \gamma(p)$ cannot be a dual Mannheim curve in $\mathbb{E}_1^3$. □

Corollary 3.5. (1) *If a Cartan framed null curve γ in $\mathbb{E}_1^3$ is a null Mannheim curve with time-like Mannheim partner curve γ^*, the signs of κ and τ are the same.*

(2) *If a Cartan framed null curve γ in $\mathbb{E}_1^3$ is a null Mannheim curve with space-like Mannheim partner curve γ^*, the signs of κ and τ are opposite.*

Proof. From (3.10) and (3.13),

$$\left(\frac{dp^*}{dp}\right)^2 = \frac{\kappa}{2\tau} \quad \text{if Mannheim partner curve is time-like,} \tag{3.16}$$

$$\left(\frac{dp^*}{dp}\right)^2 = -\frac{\kappa}{2\tau} \quad \text{if Mannh eimpartner curve is space-like.} \tag{3.17}$$

The proof is complete. □

Remarks. (a) Theorems hold for null dual Mannheim curve with space-like dual Mannheim partner curve.

(b) Some results in [4] unfortunately are not correct. For example, Theorem 3.3 gave necessary and sufficient conditions for null Mannheim curve, which implies that the null Mannheim curve should be a null-helix from Proposition 3.4. Moreover, Propositions in [4] are related with a null-helix partner curve.

References

[1] D. H. Jin, "Natural Frenet equations of null curves," *Journal of Dongguk University*, vol. 18, pp. 203–212, 1999.

[2] H. Liu and F. Wang, "Mannheim partner curves in 3-space," *Journal of Geometry*, vol. 88, no. 1-2, pp. 120–126, 2008.

[3] K. Orbay and E. Kasap, "On mannheim partner curves in $\mathbb{E}^3$," *International Journal of Physical Sciences*, vol. 4, no. 5, pp. 261–264, 2009.

[4] H. B. Öztekin and M. Ergüt, "Null mannheim curves in the minkowski 3-space $\mathbb{E}^3_1$," *Turkish Journal of Mathematics*, vol. 35, no. 1, pp. 107–114, 2011.

[5] K. L. Duggal and A. Bejancu, *Lightlike Submanifolds of Semi-Riemannian Manifolds and Applications*, Mathematics and its Applications, Kluwer Academic Publishes, Dordrecht, 1996.

[6] K. L. Duggal and D. H. Jin, *Null Curves and Hypersurfaces of Semi-Riemannian Manifolds*, World Scientific, 2007.