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Research Article

Adaptive Wavelet Estimation of a Biased Density for Strongly Mixing Sequences

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The estimation of a biased density for exponentially strongly mixing sequences is investigated. We construct a new adaptive wavelet estimator based on a hard thresholding rule. We determine a sharp upper bound of the associated mean integrated square error for a wide class of functions.

1. Introduction

In the standard density estimation problem, we observe n random variables $X_1, \ldots, X_n$ with common density function f. The goal is to estimate f from $X_1, \ldots, X_n$. However, in some applications, $X_1, \ldots, X_n$ are not accessible; we only have n random variables $Z_1, \ldots, Z_n$ with the common density

$$g(x) = \mu^{-1} w(x) f(x), \tag{1.1}$$

where w denotes a known positive function and μ is the unknown normalization parameter: $\mu = \int w(y) f(y) dy$. Our goal is to estimate the "biased density" f from $Z_1, \ldots, Z_n$. Practical examples can be found in, for example, [1–3] and the survey by the author of [4].

The standard i.i.d. case has been investigated in several papers. See, for example, [5–9]. To the best of our knowledge, the dependent case has only been examined in [10] for associated (positively or negatively) $Z_1, \ldots, Z_n$. In this paper, we study another dependent (and realistic) structure which has not been addressed earlier: we suppose that $Z_1, \ldots, Z_n$ is a sample of a strictly stationary and exponentially strongly mixing process $(Z_i)_{i \in \mathbb{Z}}$ (to be defined in Section 2). Such a dependence condition arises for a wide class of GARCH-type time series models classically encountered in finance. See, for example, [11, 12] for an overview.

We focus our attention on the wavelet methods because they provide a coherent set of procedures that are spatially adaptive and near optimal over a wide range of function spaces. See, for example, [13, 14] for a detailed coverage of wavelet theory in statistics. We develop two new wavelet estimators: a linear nonadaptive based on projections and a nonlinear adaptive using the hard thresholding rule introduced by [15]. We measure their performances by determining upper bounds of the mean integrated squared error (MISE) over Besov balls (to be defined in Section 3). We prove that our adaptive estimator attains a sharp rate of convergence, close to the one attained by the linear wavelet estimator (constructed in a nonadaptive fashion to minimize the MISE).

The rest of the paper is organized as follows. Section 2 is devoted to the assumptions on the model. In Section 3, we present wavelets and Besov balls. The considered wavelet estimators are defined in Section 4. Section 5 is devoted to the results. The proofs are postponed in Section 6.

2. Assumptions on the Model

We assume that $Z_1, \ldots, Z_n$ coming from a strictly stationary process $(Z_i)_{i\in\mathbb{Z}}$. For any $m \in \mathbb{Z}$, we define the mth strongly mixing coefficient of $(Z_i)_{i\in\mathbb{Z}}$ by

$$a_m = \sup_{(A,B)\in\mathcal{F}^Z_{-\infty,0}\times\mathcal{F}^Z_{m,\infty}} |\mathbb{P}(A\cap B) - \mathbb{P}(A)\mathbb{P}(B)|, \tag{2.1}$$

where, for any $u \in \mathbb{Z}$, $\mathcal{F}^Z_{-\infty,u}$ is the σ-algebra generated by the random variables $\ldots, Z_{u-1}, Z_u$ and $\mathcal{F}^Z_{u,\infty}$ is the σ-algebra generated by the random variables $Z_u, Z_{u+1}, \ldots$.

We consider the exponentially strongly mixing case, that is, there exist three known constants, $\gamma > 0$, $c > 0$, and $\theta > 0$, such that, for any $m \in \mathbb{Z}$,

$$a_m \le \gamma \exp\left(-c|m|^{\theta}\right). \tag{2.2}$$

This assumption is satisfied by a large class of GARCH processes. See, for example, [11, 12, 16, 17].

Note that, when $\theta \to \infty$, we are in the standard i.i.d. case.

W.o.l.g., the support of the functions f, and w are $[0,1]$.

There exist two constants, $c > 0$ and $C > 0$, such that

$$c \le \inf_{x\in[0,1]} w(x), \qquad \sup_{x\in[0,1]} w(x) \le C. \tag{2.3}$$

There exists a (known) constant $C > 0$ such that

$$\sup_{x\in[0,1]} f(x) \le C. \tag{2.4}$$

For any $m \in \{1,\ldots,n\}$, let $g_{(Z_0,Z_m)}$ be the density of (Z_0, Z_m). There exists a constant $C > 0$ such that

$$\sup_{m\in\{1,\ldots,n\}} \sup_{(x,y)\in[0,1]^2} \left|g_{(Z_0,Z_m)}(x,y) - g(x)g(y)\right| \leq C. \tag{2.5}$$

The two first boundedness assumptions are standard in the estimation of biased densities. See, for example, [6–8].

3. Wavelets and Besov Balls

Let N be an integer ϕ and ψ be the initial wavelets of dbN (so $\mathrm{supp}(\phi) = \mathrm{supp}(\psi) = [1-N, N]$). Set

$$\phi_{j,k}(x) = 2^{j/2}\phi\left(2^j x - k\right), \qquad \psi_{j,k}(x) = 2^{j/2}\psi\left(2^j x - k\right). \tag{3.1}$$

With an appropriate treatments at the boundaries, there exists an integer τ satisfying $2^\tau \geq 2N$ such that the collection $\mathcal{B} = \{\phi_{\tau,k}(\cdot),\ k \in \{0,\ldots,2^\tau - 1\};\ \psi_{j,k}(\cdot);\ j \in \mathbb{N} - \{0,\ldots,\tau-1\},\ k \in \{0,\ldots,2^j - 1\}\}$, is an orthonormal basis of $\mathbb{L}^2([0,1])$ (the space of square-integrable functions on $[0,1]$). See [18].

For any integer $\ell \geq \tau$, any $h \in \mathbb{L}^2([0,1])$ can be expanded on $\mathcal{B}$ as

$$h(x) = \sum_{k=0}^{2^\ell-1} \alpha_{\ell,k}\phi_{\ell,k}(x) + \sum_{j=\ell}^{\infty}\sum_{k=0}^{2^j-1} \beta_{j,k}\psi_{j,k}(x), \quad x \in [0,1], \tag{3.2}$$

where $\alpha_{j,k}$ and $\beta_{j,k}$ are the wavelet coefficients of h defined by

$$\alpha_{j,k} = \int_0^1 h(x)\phi_{j,k}(x)dx, \qquad \beta_{j,k} = \int_0^1 h(x)\psi_{j,k}(x)dx. \tag{3.3}$$

Let $M > 0$, $s > 0$, $p \geq 1$, and $r \geq 1$. A function h belongs to $B^s_{p,r}(M)$ if and only if there exists a constant $M^* > 0$ (depending on M) such that the associated wavelet coefficients (3.3) satisfy

$$2^{\tau(1/2-1/p)}\left(\sum_{k=0}^{2^\tau-1}|\alpha_{\tau,k}|^p\right)^{1/p} + \left(\sum_{j=\tau}^{\infty}\left(2^{j(s+1/2-1/p)}\left(\sum_{k=0}^{2^j-1}\left|\beta_{j,k}\right|^p\right)^{1/p}\right)^r\right)^{1/r} \leq M^*. \tag{3.4}$$

In this expression, s is a smoothness parameter and p and r are norm parameters. For a particular choice of s, p, and r, $B^s_{p,r}(M)$ contains some classical sets of functions as the Hölder and Sobolev balls. See [19].

4. Estimators

Firstly, we consider the following estimator for μ:

$$\widehat{\mu} = \left(\frac{1}{n} \sum_{i=1}^{n} \frac{1}{w(Z_i)} \right)^{-1}. \tag{4.1}$$

It is obtained by the method of moments (see Proposition 6.2 below).

Then, for any integer $j \geq \tau$ and any $k \in \{0, \ldots, 2^j - 1\}$, we estimate the unknown wavelet coefficient

(i) $\alpha_{j,k} = \int_0^1 f(x)\phi_{j,k}(x)dx$ by

$$\widehat{\alpha}_{j,k} = \frac{\widehat{\mu}}{n} \sum_{i=1}^{n} \frac{\phi_{j,k}(Z_i)}{w(Z_i)}, \tag{4.2}$$

(ii) $\beta_{j,k} = \int_0^1 f(x)\psi_{j,k}(x)dx$ by

$$\widehat{\beta}_{j,k} = \frac{\widehat{\mu}}{n} \sum_{i=1}^{n} \frac{\psi_{j,k}(Z_i)}{w(Z_i)}. \tag{4.3}$$

Note that they are those considered in the i.i.d. case (see, e.g., [8, 9]). Their statistical properties, with our dependent structure, are investigated in Propositions 6.2, 6.3, and 6.4 below.

Assuming that $f \in B^s_{p,r}(M)$ with $p \geq 2$, we define the linear estimator $\widehat{f}^L$ by

$$\widehat{f}^L(x) = \sum_{k=0}^{2^{j_0}-1} \widehat{\alpha}_{j_0,k}\phi_{j_0,k}(x), \quad x \in [0,1], \tag{4.4}$$

where $\widehat{\alpha}_{j,k}$ is defined by (4.2) and j_0 is the integer satisfying

$$\frac{1}{2} n^{1/(2s+1)} < 2^{j_0} \leq n^{1/(2s+1)}. \tag{4.5}$$

For a survey on wavelet linear estimators for various density models, we refer the reader to [20]. For the consideration of strongly mixing sequences, see, for example, [21, 22].

We define the hard thresholding estimator $\widehat{f}^H$ by

$$\widehat{f}^H(x) = \sum_{k=0}^{2^\tau-1} \widehat{\alpha}_{\tau,k}\phi_{\tau,k}(x) + \sum_{j=\tau}^{j_1} \sum_{k=0}^{2^j-1} \widehat{\beta}_{j,k} \mathbb{I}_{\{|\widehat{\beta}_{j,k}| \geq \kappa\lambda_n\}} \psi_{j,k}(x), \tag{4.6}$$

$x \in [0,1]$, where $\widehat{\alpha}_{\tau,k}$ is defined by (4.2) and $\widehat{\beta}_{j,k}$ by (4.3), for any random event $\mathcal{A}$, $\mathbb{I}_{\mathcal{A}}$ is the indicator function on $\mathcal{A}$, j_1 is the integer satisfying

$$\frac{1}{2}\frac{n}{(\ln n)^{1+1/\theta}} < 2^{j_1} \leq \frac{n}{(\ln n)^{1+1/\theta}}, \tag{4.7}$$

θ is the one in (2.2), κ is a large enough constant (the one in Proposition 6.4 below) and λ_n is the threshold

$$\lambda_n = \sqrt{\frac{(\ln n)^{1+1/\theta}}{n}}. \tag{4.8}$$

The feature of the hard thresholding estimator is to only estimate the "large" unknown wavelet coefficients of f which contain his main characteristics.

For the construction of hard thresholding wavelet estimators in the standard density model, see, for example, [15, 23].

5. Results

Theorem 5.1 (upper bound for $\widehat{f}^L$). *Consider* (1.1) *under the assumptions of Section 2. Suppose that $f \in B^s_{p,r}(M)$ with $s > 0$, $p \geq 2$, and $r \geq 1$. Let $\widehat{f}^L$ be* (4.4). *Then there exists a constant $C > 0$ such that*

$$\mathbb{E}\left(\int_0^1 \left(\widehat{f}^L(x) - f(x)\right)^2 dx\right) \leq C n^{-2s/(2s+1)}. \tag{5.1}$$

The proof of Theorem 5.1 uses a suitable decomposition of the MISE and a moment inequality on (4.2) (see Proposition 6.3 below).

Note that $n^{-2s/(2s+1)}$ is the optimal rate of convergence (in the minimax sense) for the standard density model in the independent case (see, e.g., [14, 23]).

Theorem 5.2 (upper bound for $\widehat{f}^H$). *Consider* (1.1) *under the assumptions of Section 2. Let $\widehat{f}^H$ be* (4.6). *Suppose that $f \in B^s_{p,r}(M)$ with $r \geq 1$, $\{p \geq 2$ and $s > 0\}$ or $\{p \in [1,2)$ and $s > 1/p\}$. Then there exists a constant $C > 0$ such that*

$$\mathbb{E}\left(\int_0^1 \left(\widehat{f}^H(x) - f(x)\right)^2 dx\right) \leq C\left(\frac{(\ln n)^{1+1/\theta}}{n}\right)^{2s/(2s+1)}. \tag{5.2}$$

The proof of Theorem 5.2 uses a suitable decomposition of the MISE, some moment inequalities on (4.2) and (4.3) (see Proposition 6.3 below), and a concentration inequality on (4.3) (see Proposition 6.4 below).

Theorem 5.2 shows that, besides being adaptive, $\widehat{f}^H$ attains a rate of convergence close to the one of $\widehat{f}^L$. The only difference is the logarithmic term $(\ln n)^{(1+1/\theta)(2s/(2s+1))}$.

Note that, if we restrict our study to the independent case, that is, $\theta \to \infty$, the rate of convergence attained by $\widehat{f}^H$ becomes the standard one: $(\log n/n)^{2s/(2s+1)}$. See, for example, [14, 15, 23].

6. Proofs

In this section, we consider (1.1) under the assumptions of Section 2. Moreover, C denotes any constant that does not depend on j, k and n. Its value may change from one term to another and may depends on ϕ or ψ.

6.1. Auxiliary Results

Lemma 6.1. *For any integer $j \geq \tau$ and any $k \in \{0,\ldots,2^j-1\}$, let $\widehat{\alpha}_{j,k}$ be* (4.2) *and $\alpha_{j,k} = \int_0^1 f(x)\phi_{j,k}(x)dx$. Then, under the assumptions of* Section 2*, there exists a constant $C>0$ such that*

$$\left|\widehat{\alpha}_{j,k}-\alpha_{j,k}\right| \leq C\left(\left|\frac{\mu}{n}\sum_{i=1}^{n}\frac{\phi_{j,k}(Z_i)}{w(Z_i)}-\alpha_{j,k}\right|+\left|\frac{1}{\widehat{\mu}}-\frac{1}{\mu}\right|\right). \tag{6.1}$$

This inequality holds for ψ instead of ϕ (and, a fortiori, $\widehat{\beta}_{j,k}$ defined by (4.3) *instead of $\widehat{\alpha}_{j,k}$ and $\beta_{j,k} = \int_0^1 f(x)\psi_{j,k}(x)dx$ instead of $\alpha_{j,k}$).*

Proof of Lemma 6.1. We have

$$\widehat{\alpha}_{j,k}-\alpha_{j,k} = \frac{\widehat{\mu}}{\mu}\left(\frac{\mu}{n}\sum_{i=1}^{n}\frac{\phi_{j,k}(Z_i)}{w(Z_i)}-\alpha_{j,k}\right)+\alpha_{j,k}\widehat{\mu}\left(\frac{1}{\mu}-\frac{1}{\widehat{\mu}}\right). \tag{6.2}$$

Due to (2.3), we have $|\widehat{\mu}| \leq C$ and $|\widehat{\mu}/\mu| \leq C$. Therefore

$$\left|\widehat{\alpha}_{j,k}-\alpha_{j,k}\right| \leq C\left(\left|\frac{\mu}{n}\sum_{i=1}^{n}\frac{\phi_{j,k}(Z_i)}{w(Z_i)}-\alpha_{j,k}\right|+\left|\alpha_{j,k}\right|\left|\frac{1}{\widehat{\mu}}-\frac{1}{\mu}\right|\right). \tag{6.3}$$

Using (2.4) and the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}\left|\alpha_{j,k}\right| &\leq \int_0^1 f(x)\left|\phi_{j,k}(x)\right|dx \leq C\int_0^1\left|\phi_{j,k}(x)\right|dx\\ &\leq C\left(\int_0^1\left(\phi_{j,k}(x)\right)^2dx\right)^{1/2} = C.\end{aligned} \tag{6.4}$$

Hence

$$\left|\widehat{\alpha}_{j,k}-\alpha_{j,k}\right| \leq C\left(\left|\frac{\mu}{n}\sum_{i=1}^{n}\frac{\phi_{j,k}(Z_i)}{w(Z_i)}-\alpha_{j,k}\right|+\left|\frac{1}{\widehat{\mu}}-\frac{1}{\mu}\right|\right). \tag{6.5}$$

Lemma 6.1 is proved. □

Proposition 6.2. *For any integer $j \geq \tau$ such that $2^j \leq n$ and any $k \in \{0, \ldots, 2^j - 1\}$, let $\alpha_{j,k} = \int_0^1 f(x)\phi_{j,k}(x)dx$ and $\widehat{\mu}$ be* (4.1)*. Then,*

(1) *one has*

$$\mathbb{E}\left(\frac{\mu}{n}\sum_{i=1}^{n}\frac{\phi_{j,k}(Z_i)}{w(Z_i)}\right) = \alpha_{j,k}, \qquad \mathbb{E}\left(\frac{1}{\widehat{\mu}}\right) = \frac{1}{\mu}, \tag{6.6}$$

(2) *there exists a constant $C > 0$ such that*

$$\mathbb{V}\left(\frac{\mu}{n}\sum_{i=1}^{n}\frac{\phi_{j,k}(Z_i)}{w(Z_i)}\right) \leq C\frac{1}{n}, \tag{6.7}$$

(3) *there exists a constant $C > 0$ such that*

$$\mathbb{V}\left(\frac{1}{\widehat{\mu}}\right) \leq C\frac{1}{n}. \tag{6.8}$$

These results hold for ψ instead of ϕ (and, a fortiori, $\beta_{j,k} = \int_0^1 f(x)\psi_{j,k}(x)dx$ instead of $\alpha_{j,k}$).

Proof of Proposition 6.2. (1) We have

$$\begin{aligned}\mathbb{E}\left(\frac{\mu}{n}\sum_{i=1}^{n}\frac{\phi_{j,k}(Z_i)}{w(Z_i)}\right) &= \mu\mathbb{E}\left(\frac{\phi_{j,k}(Z_1)}{w(Z_1)}\right) = \mu\int_0^1\frac{\phi_{j,k}(x)}{w(x)}g(x)dx\\ &= \mu\int_0^1\frac{\phi_{j,k}(x)}{w(x)}\mu^{-1}w(x)f(x)dx = \int_0^1 f(x)\phi_{j,k}(x)dx = \alpha_{j,k}.\end{aligned} \tag{6.9}$$

Since f is a density, we obtain

$$\begin{aligned}\mathbb{E}\left(\frac{1}{\widehat{\mu}}\right) &= \mathbb{E}\left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{w(Z_i)}\right) = \mathbb{E}\left(\frac{1}{w(Z_1)}\right) = \int_0^1\frac{1}{w(x)}g(x)dx\\ &= \int_0^1\frac{1}{w(x)}\mu^{-1}w(x)f(x)dx = \frac{1}{\mu}\int_0^1 f(x)dx = \frac{1}{\mu}.\end{aligned} \tag{6.10}$$

(2) We have

$$\begin{aligned}
\mathbb{V}\left(\frac{\mu}{n}\sum_{i=1}^{n}\frac{\phi_{j,k}(Z_i)}{w(Z_i)}\right) &= \frac{\mu^2}{n^2}\sum_{v=1}^{n}\sum_{\ell=1}^{n}\mathbb{C}\left(\frac{\phi_{j,k}(Z_v)}{w(Z_v)},\frac{\phi_{j,k}(Z_\ell)}{w(Z_\ell)}\right) \\
&= \frac{\mu^2}{n}\mathbb{V}\left(\frac{\phi_{j,k}(Z_1)}{w(Z_1)}\right) + 2\frac{\mu^2}{n^2}\sum_{v=2}^{n}\sum_{\ell=1}^{v-1}\mathbb{C}\left(\frac{\phi_{j,k}(Z_v)}{w(Z_v)},\frac{\phi_{j,k}(Z_\ell)}{w(Z_\ell)}\right) \\
&\le \frac{\mu^2}{n}\mathbb{V}\left(\frac{\phi_{j,k}(Z_1)}{w(Z_1)}\right) + 2\frac{\mu^2}{n^2}\left|\sum_{v=2}^{n}\sum_{\ell=1}^{v-1}\mathbb{C}\left(\frac{\phi_{j,k}(Z_v)}{w(Z_v)},\frac{\phi_{j,k}(Z_\ell)}{w(Z_\ell)}\right)\right|.
\end{aligned} \tag{6.11}$$

Using (2.3) and (2.4), we have $\sup_{x\in[0,1]}g(x) \le C$. Hence,

$$\begin{aligned}
\mathbb{V}\left(\frac{\phi_{j,k}(Z_1)}{w(Z_1)}\right) &\le \mathbb{E}\left(\left(\frac{\phi_{j,k}(Z_1)}{w(Z_1)}\right)^2\right) \le C\mathbb{E}\left(\left(\phi_{j,k}(Z_1)\right)^2\right) \\
&= C\int_0^1 \left(\phi_{j,k}(x)\right)^2 g(x)dx \le C\int_0^1 \left(\phi_{j,k}(x)\right)^2 dx = C.
\end{aligned} \tag{6.12}$$

It follows from the stationarity of $(Z_i)_{i\in\mathbb{Z}}$ and $2^j \le n$ that

$$\begin{aligned}
\left|\sum_{v=2}^{n}\sum_{\ell=1}^{v-1}\mathbb{C}\left(\frac{\phi_{j,k}(Z_v)}{w(Z_v)},\frac{\phi_{j,k}(Z_\ell)}{w(Z_\ell)}\right)\right| &= \left|\sum_{m=1}^{n}(n-m)\mathbb{C}\left(\frac{\phi_{j,k}(Z_0)}{w(Z_0)},\frac{\phi_{j,k}(Z_m)}{w(Z_m)}\right)\right| \\
&\le n\sum_{m=1}^{n}\left|\mathbb{C}\left(\frac{\phi_{j,k}(Z_0)}{w(Z_0)},\frac{\phi_{j,k}(Z_m)}{w(Z_m)}\right)\right| = T_1 + T_2,
\end{aligned} \tag{6.13}$$

where

$$\begin{aligned}
T_1 &= n\sum_{m=1}^{2^j-1}\left|\mathbb{C}\left(\frac{\phi_{j,k}(Z_0)}{w(Z_0)},\frac{\phi_{j,k}(Z_m)}{w(Z_m)}\right)\right|, \\
T_2 &= n\sum_{m=2^j}^{n}\left|\mathbb{C}\left(\frac{\phi_{j,k}(Z_0)}{w(Z_0)},\frac{\phi_{j,k}(Z_m)}{w(Z_m)}\right)\right|.
\end{aligned} \tag{6.14}$$

Let us now bound T_1 and T_2.

Upper Bound for T_1

Using (2.5), (2.3), and doing the change a variables $y = 2^j x - k$, we obtain

$$\begin{aligned}\left|\mathbb{C}\left(\frac{\phi_{j,k}(Z_0)}{w(Z_0)}, \frac{\phi_{j,k}(Z_m)}{w(Z_m)}\right)\right| &= \left|\iint_0^1 \left(g_{(Z_0,Z_m)}(x,y) - g(x)g(y)\right)\frac{\phi_{j,k}(x)}{w(x)}\frac{\phi_{j,k}(y)}{w(y)}dx\,dy\right| \\ &\le \iint_0^1 \left|g_{(Z_0,Z_m)}(x,y) - g(x)g(y)\right|\left|\frac{\phi_{j,k}(x)}{w(x)}\right|\left|\frac{\phi_{j,k}(y)}{w(y)}\right|dx\,dy \\ &\le C\left(\int_0^1 \left|\phi_{j,k}(x)\right|dx\right)^2 = C\left(2^{-j/2}\int_0^1 |\phi(x)|dx\right)^2 = C2^{-j}.\end{aligned} \tag{6.15}$$

Therefore,

$$T_1 \le Cn2^{-j}2^j = Cn. \tag{6.16}$$

Upper Bound for T_2

By the Davydov inequality for strongly mixing processes (see [24]), for any $q \in (0,1)$, it holds that

$$\begin{aligned}\left|\mathbb{C}\left(\frac{\phi_{j,k}(Z_0)}{w(Z_0)}, \frac{\phi_{j,k}(Z_m)}{w(Z_m)}\right)\right| &\le 10a_m^q\left(\mathbb{E}\left(\left|\frac{\phi_{j,k}(Z_0)}{w(Z_0)}\right|^{2/(1-q)}\right)\right)^{1-q} \\ &\le 10a_m^q\left(\sup_{x\in[0,1]}\left|\frac{\phi_{j,k}(x)}{w(x)}\right|\right)^{2q}\left(\mathbb{E}\left(\left(\frac{\phi_{j,k}(Z_0)}{w(Z_0)}\right)^2\right)\right)^{1-q}.\end{aligned} \tag{6.17}$$

By (2.3), we have

$$\sup_{x\in[0,1]}\left|\frac{\phi_{j,k}(x)}{w(x)}\right| \le C\sup_{x\in[0,1]}\left|\phi_{j,k}(x)\right| \le C2^{j/2} \tag{6.18}$$

and, by (6.12),

$$\mathbb{E}\left(\left(\frac{\phi_{j,k}(Z_0)}{w(Z_0)}\right)^2\right) \le C. \tag{6.19}$$

Therefore,

$$\left|\mathbb{C}\left(\frac{\phi_{j,k}(Z_0)}{w(Z_0)}, \frac{\phi_{j,k}(Z_m)}{w(Z_m)}\right)\right| \le C2^{qj}a_m^q. \tag{6.20}$$

Since $\sum_{m=2^j}^{n} m^q a_m^q \leq \sum_{m=1}^{\infty} m^q a_m^q = \gamma^q \sum_{m=1}^{\infty} m^q \exp(-cqm^{\theta}) < \infty$, we have

$$T_2 \leq Cn2^{qj} \sum_{m=2^j}^{n} a_m^q \leq Cn \sum_{m=2^j}^{n} m^q a_m^q \leq Cn. \tag{6.21}$$

It follows from (6.13), (6.16), and (6.21) that

$$\left| \sum_{v=2}^{n} \sum_{\ell=1}^{v-1} \mathbb{C}\left(\frac{\phi_{j,k}(Z_v)}{w(Z_v)}, \frac{\phi_{j,k}(Z_\ell)}{w(Z_\ell)} \right) \right| \leq Cn. \tag{6.22}$$

Combining (6.11), (6.12), and (6.22), we obtain

$$\mathbb{V}\left(\frac{\mu}{n} \sum_{i=1}^{n} \frac{\phi_{j,k}(Z_i)}{w(Z_i)} \right) \leq C\frac{1}{n}. \tag{6.23}$$

(3) Proceeding in a similar fashion to 2-, we obtain

$$\begin{aligned} \mathbb{V}\left(\frac{1}{\widehat{\mu}} \right) &= \mathbb{V}\left(\frac{1}{n} \sum_{i=1}^{n} \frac{1}{w(Z_i)} \right) \\ &= \frac{1}{n}\mathbb{V}\left(\frac{1}{w(Z_1)} \right) + 2\frac{1}{n^2} \sum_{v=2}^{n} \sum_{\ell=1}^{v-1} \mathbb{C}\left(\frac{1}{w(Z_v)}, \frac{1}{w(Z_\ell)} \right) \\ &\leq \frac{1}{n}\mathbb{V}\left(\frac{1}{w(Z_1)} \right) + 2\frac{1}{n} \sum_{m=1}^{n} \left| \mathbb{C}\left(\frac{1}{w(Z_0)}, \frac{1}{w(Z_m)} \right) \right|. \end{aligned} \tag{6.24}$$

Using (2.3) (which implies $\sup_{x\in[0,1]}(1/w(x)) \leq C$) and applying the Davydov inequality, we obtain

$$\mathbb{V}\left(\frac{1}{\widehat{\mu}} \right) \leq C\frac{1}{n}\left(1 + \sum_{m=1}^{n} a_m^q \right) \leq C\frac{1}{n}. \tag{6.25}$$

The proof of Proposition 6.2 is complete. □

Proposition 6.3. *For any integer $j \geq \tau$ such that $2^j \leq n$ and any $k \in \{0,\ldots,2^j-1\}$, let $\alpha_{j,k} = \int_0^1 f(x)\phi_{j,k}(x)dx$ and $\widehat{\alpha}_{j,k}$ be* (4.2). *Then,*

(1) *there exists a constant $C > 0$ such that*

$$\mathbb{E}\left((\widehat{\alpha}_{j,k} - \alpha_{j,k})^2 \right) \leq C\frac{1}{n}; \tag{6.26}$$

(2) *there exists a constant $C > 0$ such that*

$$\mathbb{E}\left((\widehat{\alpha}_{j,k} - \alpha_{j,k})^4 \right) \leq C2^j\frac{1}{n}. \tag{6.27}$$

These inequalities hold for $\widehat{\beta}_{j,k}$ *defined by* (4.3) *instead of* $\widehat{\alpha}_{j,k}$, *and* $\beta_{j,k} = \int_0^1 f(x)\psi_{j,k}(x)dx$ *instead of* $\alpha_{j,k}$.

Proof of Proposition 6.3. (1) Applying Lemma 6.1 and Proposition 6.2, we have

$$\begin{aligned}\mathbb{E}\left((\widehat{\alpha}_{j,k}-\alpha_{j,k})^2\right) &\le C\left(\mathbb{E}\left(\left(\frac{\mu}{n}\sum_{i=1}^{n}\frac{\phi_{j,k}(Z_i)}{w(Z_i)}-\alpha_{j,k}\right)^2\right)+\mathbb{E}\left(\left(\frac{1}{\widehat{\mu}}-\frac{1}{\mu}\right)^2\right)\right)\\ &= C\left(\mathbb{V}\left(\frac{\mu}{n}\sum_{i=1}^{n}\frac{\phi_{j,k}(Z_i)}{w(Z_i)}\right)+\mathbb{V}\left(\frac{1}{\widehat{\mu}}\right)\right)\le C\frac{1}{n}.\end{aligned} \tag{6.28}$$

(2) We have

$$\left|\widehat{\alpha}_{j,k}-\alpha_{j,k}\right| \le \left|\widehat{\alpha}_{j,k}\right| + \left|\alpha_{j,k}\right|. \tag{6.29}$$

By (2.3), we have $|\widehat{\mu}| \le C$ and $\sup_{x\in[0,1]}(1/w(x)) \le C$. So,

$$\begin{aligned}\left|\frac{\widehat{\mu}}{n}\sum_{i=1}^{n}\frac{\phi_{j,k}(Z_i)}{w(Z_i)}\right| &\le C\frac{1}{n}\sum_{i=1}^{n}\left|\frac{\phi_{j,k}(Z_i)}{w(Z_i)}\right| \le C\sup_{x\in[0,1]}\left|\frac{\phi_{j,k}(x)}{w(x)}\right|\\ &\le C\sup_{x\in[0,1]}\left|\phi_{j,k}(x)\right| \le C2^{j/2}.\end{aligned} \tag{6.30}$$

By (6.4), we have $|\alpha_{j,k}| \le C$. Therefore

$$\left|\widehat{\alpha}_{j,k}-\alpha_{j,k}\right| \le C\left(2^{j/2}+1\right) \le C2^{j/2}. \tag{6.31}$$

It follows from (6.31) and (6.28) that

$$\mathbb{E}\left((\widehat{\alpha}_{j,k}-\alpha_{j,k})^4\right) \le C2^j\,\mathbb{E}\left((\widehat{\alpha}_{j,k}-\alpha_{j,k})^2\right) \le C2^j\frac{1}{n}. \tag{6.32}$$

The proof of Proposition 6.3 is complete. □

Proposition 6.4. *For any* $j \in \{\tau,\dots,j_1\}$ *and any* $k \in \{0,\dots,2^j-1\}$, *let* $\beta_{j,k} = \int_0^1 f(x)\psi_{j,k}(x)dx$, $\widehat{\beta}_{j,k}$ *be* (4.3) *and* λ_n *be* (4.8). *Then there exist two constants,* $\kappa > 0$ *and* $C > 0$, *such that*

$$\mathbb{P}\left(\left|\widehat{\beta}_{j,k}-\beta_{j,k}\right| \ge \frac{\kappa\lambda_n}{2}\right) \le C\frac{1}{n^4}. \tag{6.33}$$

Proof of Proposition 6.4. It follows from Lemma 6.1 that

$$\mathbb{P}\left(\left|\widehat{\beta}_{j,k}-\beta_{j,k}\right| \ge \frac{\kappa\lambda_n}{2}\right) \le P_1 + P_2, \tag{6.34}$$

where

$$P_1 = \mathbb{P}\left(\left|\frac{\mu}{n}\sum_{i=1}^{n}\frac{\psi_{j,k}(Z_i)}{w(Z_i)} - \beta_{j,k}\right| \geq \kappa C\lambda_n\right),$$
$$P_2 = \mathbb{P}\left(\left|\frac{1}{\widehat{\mu}} - \frac{1}{\mu}\right| \geq \kappa C\lambda_n\right). \tag{6.35}$$

In order to bound P_1 and P_2, let us present a Bernstein inequality for exponentially strongly mixing process. We refer to [25, 26].

Lemma 6.5 (see [25, 26]). *Let $\gamma > 0$, $c > 0$, $\theta > 1$ and $(Z_i)_{i\in\mathbb{Z}}$ be a stationary process such that, for any $m \in \mathbb{Z}$, the associated mth strongly mixing coefficient* (2.2) *satisfies $a_m \leq \gamma \exp(-c|m|^{\theta})$. Let $n \in \mathbb{N}^*$, $h : \mathbb{R} \to \mathbb{R}$ be a measurable function and, for any $i \in \mathbb{Z}$, $U_i = h(Z_i)$. One assumes that $\mathbb{E}(U_1) = 0$ and there exists a constant $M > 0$ satisfying $|U_1| \leq M < \infty$. Then, for any $m \in \{1,\ldots,n\}$ and any $\lambda > 4mM/n$, one has*

$$\mathbb{P}\left(\left|\frac{1}{n}\sum_{i=1}^{n}U_i\right| \geq \lambda\right) \leq 4\exp\left(-\frac{\lambda^2 n}{m\left(64\mathbb{E}\left(U_1^2\right) + 8\lambda M/3\right)}\right) + 4\gamma\frac{n}{m}\exp\left(-cm^{\theta}\right). \tag{6.36}$$

Upper Bound for P_1

For any $i \in \{1,\ldots,n\}$, set

$$U_i = \mu\frac{\psi_{j,k}(Z_i)}{w(Z_i)} - \beta_{j,k}. \tag{6.37}$$

Then $U_1,\ldots,U_n$ are identically distributed, depend on the stationary strongly mixing process $(Z_i)_{i\in\mathbb{Z}}$ which satisfies (2.2), Proposition 6.2 gives

$$\mathbb{E}(U_1) = 0, \qquad \mathbb{E}\left(U_1^2\right) \leq \mathbb{E}\left(\left(\mu\frac{\psi_{j,k}(Z_1)}{w(Z_1)}\right)^2\right) \leq C \tag{6.38}$$

and, by (2.3) and (6.4),

$$|U_1| \leq \mu \sup_{x\in[0,1]}\left|\frac{\psi_{j,k}(x)}{w(x)}\right| + \left|\beta_{j,k}\right| \leq C\left(\sup_{x\in[0,1]}\left|\psi_{j,k}(x)\right| + 1\right)$$
$$\leq C\left(2^{j/2} + 1\right) \leq C2^{j/2}. \tag{6.39}$$

It follows from Lemma 6.5 applied with $U_1,\dots,U_n$, $\lambda = \kappa C\lambda_n$, $\lambda_n = ((\ln n)^{1+1/\theta}/n)^{1/2}$, $m = (u\ln n)^{1/\theta}$ with $u > 0$ (chosen later), $M = C2^{j/2}$ and $2^j \le 2^{j_1} \le n/(\ln n)^{1+1/\theta}$, that

$$\begin{aligned}
P_1 &= \mathbb{P}\left(\left|\frac{1}{n}\sum_{i=1}^{n} U_i\right| \ge \kappa C\lambda_n\right) \\
&\le 4\exp\left(-C\frac{\kappa^2\lambda_n^2 n}{m(1+\kappa\lambda_n M)}\right) + 4\gamma\frac{n}{m}\exp\left(-cm^\theta\right) \\
&\le 4\exp\left(-C\frac{\kappa^2(\ln n)^{1+1/\theta}}{(u\ln n)^{1/\theta}\left(1+\kappa 2^{j/2}\left((\ln n)^{1+1/\theta}/n\right)^{1/2}\right)}\right) \\
&\quad + 4\gamma\frac{n}{(u\ln n)^{1/\theta}}\exp(-cu\ln n) \\
&\le C\left(n^{-C\kappa^2/(u^{1/\theta}(1+\kappa))} + n^{1-cu}\right).
\end{aligned} \tag{6.40}$$

Therefore, for large enough κ and u, we have

$$P_1 \le C\frac{1}{n^4}. \tag{6.41}$$

Upper Bound for P_2

For any $i \in \{1,\dots,n\}$, set

$$U_i = \frac{1}{w(Z_i)} - \frac{1}{\mu}. \tag{6.42}$$

Then $U_1,\dots,U_n$ are identically distributed, depend on the stationary strongly mixing process $(Z_i)_{i\in\mathbb{Z}}$ which satisfies (2.2), Proposition 6.2 gives

$$\mathbb{E}(U_1) = 0, \qquad \mathbb{E}\left(U_1^2\right) \le \mathbb{E}\left(\frac{1}{(w(Z_1))^2}\right) \le C. \tag{6.43}$$

By (2.3), we have

$$|U_1| \le \sup_{x\in[0,1]}\frac{1}{w(x)} + \left|\frac{1}{\mu}\right| \le C. \tag{6.44}$$

It follows from Lemma 6.5 applied with $U_1,\ldots,U_n$, $\lambda = \kappa C\lambda_n$, $\lambda_n = ((\ln n)^{1+1/\theta}/n)^{1/2}$, $m = (u\ln n)^{1/\theta}$ with $u > 0$ (chosen later) and $M = C$ that

$$\begin{aligned}
P_2 &= \mathbb{P}\left(\left|\frac{1}{n}\sum_{i=1}^{n}U_i\right| \geq \kappa C\lambda_n\right) \\
&\leq 4\exp\left(-C\frac{\kappa^2\lambda_n^2 n}{m(1+\kappa\lambda_n M)}\right) + 4\gamma\frac{n}{m}\exp\left(-cm^{\theta}\right) \\
&\leq 4\exp\left(-C\frac{\kappa^2(\ln n)^{1+1/\theta}}{(u\ln n)^{1/\theta}\left(1+\kappa\left((\ln n)^{1+1/\theta}/n\right)^{1/2}\right)}\right) \\
&\quad + 4\gamma\frac{n}{(u\ln n)^{1/\theta}}\exp(-cu\ln n) \\
&\leq C\left(n^{-C\kappa^2/u^{1/\theta}} + n^{1-cu}\right).
\end{aligned} \tag{6.45}$$

Therefore, for large enough κ and u, we have

$$P_2 \leq C\frac{1}{n^4}. \tag{6.46}$$

Putting (6.34), (6.41), and (6.46) together, this ends the proof of Proposition 6.4. □

6.2. Proofs of the Main Results

Proof of Theorem 5.1. We expand the function f on $\mathcal{B}$ as

$$f(x) = \sum_{k=0}^{2^{j_0}-1}\alpha_{j_0,k}\phi_{j_0,k}(x) + \sum_{j=j_0}^{\infty}\sum_{k=0}^{2^j-1}\beta_{j,k}\psi_{j,k}(x), \quad x \in [0,1], \tag{6.47}$$

where $\alpha_{j_0,k} = \int_0^1 f(x)\phi_{j_0,k}(x)dx$ and $\beta_{j,k} = \int_0^1 f(x)\psi_{j,k}(x)dx$.

We have, for any $x \in [0,1]$,

$$\widehat{f}^L(x) - f(x) = \sum_{k=0}^{2^{j_0}-1}\left(\widehat{\alpha}_{j_0,k} - \alpha_{j_0,k}\right)\phi_{j_0,k}(x) - \sum_{j=j_0}^{\infty}\sum_{k=0}^{2^j-1}\beta_{j,k}\psi_{j,k}(x). \tag{6.48}$$

Since $\mathcal{B}$ is an orthonormal basis of $\mathbb{L}^2([0,1])$, we have,

$$\mathbb{E}\left(\int_0^1\left(\widehat{f}^L(x) - f(x)\right)^2 dx\right) = \sum_{k=0}^{2^{j_0}-1}\mathbb{E}\left(\left(\widehat{\alpha}_{j_0,k} - \alpha_{j_0,k}\right)^2\right) + \sum_{j=j_0}^{\infty}\sum_{k=0}^{2^j-1}\beta_{j,k}^2. \tag{6.49}$$

Using Proposition 6.3, we obtain

$$\sum_{k=0}^{2^{j_0}-1} \mathbb{E}\Big(\left(\widehat{\alpha}_{j_0,k} - \alpha_{j_0,k}\right)^2 \Big) \leq C2^{j_0}\frac{1}{n} \leq Cn^{-2s/(2s+1)}. \tag{6.50}$$

Since $p \geq 2$, we have $B^s_{p,r}(M) \subseteq B^s_{2,\infty}(M)$. Hence

$$\sum_{j=j_0}^{\infty} \sum_{k=0}^{2^j-1} \beta^2_{j,k} \leq C2^{-2j_0 s} \leq Cn^{-2s/(2s+1)}. \tag{6.51}$$

Therefore,

$$\mathbb{E}\left(\int_0^1 \left(\widehat{f}^L(x) - f(x)\right)^2 dx \right) \leq Cn^{-2s/(2s+1)}. \tag{6.52}$$

The proof of Theorem 5.1 is complete. □

Proof of Theorem 5.2. We expand the function f on $\mathcal{B}$ as

$$f(x) = \sum_{k=0}^{2^\tau-1} \alpha_{\tau,k}\phi_{\tau,k}(x) + \sum_{j=\tau}^{\infty}\sum_{k=0}^{2^j-1} \beta_{j,k}\psi_{j,k}(x), \quad x \in [0,1], \tag{6.53}$$

where $\alpha_{\tau,k} = \int_0^1 f(x)\phi_{\tau,k}(x)dx$ and $\beta_{j,k} = \int_0^1 f(x)\psi_{j,k}(x)dx$.

We have, for any $x \in [0,1]$,

$$\begin{aligned} \widehat{f}^H(x) - f(x) = &\sum_{k=0}^{2^\tau-1} (\widehat{\alpha}_{\tau,k} - \alpha_{\tau,k})\phi_{\tau,k}(x) + \sum_{j=\tau}^{j_1}\sum_{k=0}^{2^j-1} \left(\widehat{\beta}_{j,k}\mathbb{I}_{\{|\widehat{\beta}_{j,k}| \geq \kappa\lambda_n\}} - \beta_{j,k} \right)\psi_{j,k}(x) \\ &- \sum_{j=j_1+1}^{\infty}\sum_{k=0}^{2^j-1} \beta_{j,k}\psi_{j,k}(x). \end{aligned} \tag{6.54}$$

Since $\mathcal{B}$ is an orthonormal basis of $\mathbb{L}^2([0,1])$, we have

$$\mathbb{E}\left(\int_0^1 \left(\widehat{f}^H(x) - f(x)\right)^2 dx \right) = R + S + T, \tag{6.55}$$

where

$$R = \sum_{k=0}^{2^{\tau}-1} \mathbb{E}\Big((\widehat{\alpha}_{\tau,k} - \alpha_{\tau,k})^2 \Big), \qquad S = \sum_{j=\tau}^{j_1} \sum_{k=0}^{2^j-1} \mathbb{E}\bigg(\Big(\widehat{\beta}_{j,k} \mathbb{I}_{\{|\widehat{\beta}_{j,k}| \geq \kappa\lambda_n\}} - \beta_{j,k} \Big)^2 \bigg),$$
$$T = \sum_{j=j_1+1}^{\infty} \sum_{k=0}^{2^j-1} \beta_{j,k}^2. \tag{6.56}$$

Let us bound R, T, and S, in turn.

Upper Bound for R

Using Proposition 6.3 and $2s/(2s+1) < 1$, we obtain

$$R \leq C 2^{\tau} \frac{1}{n} \leq C \left(\frac{(\ln n)^{1+1/\theta}}{n} \right)^{2s/(2s+1)}. \tag{6.57}$$

Upper Bound for T

For $r \geq 1$ and $p \geq 2$, we have $B^s_{p,r}(M) \subseteq B^s_{2,\infty}(M)$. Since $2s/(2s+1) < 2s$, we have

$$T \leq C \sum_{j=j_1+1}^{\infty} 2^{-2js} \leq C 2^{-2j_1 s} \leq C \left(\frac{(\ln n)^{1+1/\theta}}{n} \right)^{2s} \leq C \left(\frac{(\ln n)^{1+1/\theta}}{n} \right)^{2s/(2s+1)}. \tag{6.58}$$

For $r \geq 1$ and $p \in [1,2)$, we have $B^s_{p,r}(M) \subseteq B^{s+1/2-1/p}_{2,\infty}(M)$. Since $s > 1/p$, we have $s + 1/2 - 1/p > s/(2s+1)$. So

$$\begin{aligned} T &\leq C \sum_{j=j_1+1}^{\infty} 2^{-2j(s+1/2-1/p)} \leq C 2^{-2j_1(s+1/2-1/p)} \\ &\leq C \left(\frac{(\ln n)^{1+1/\theta}}{n} \right)^{2(s+1/2-1/p)} \leq C \left(\frac{(\ln n)^{1+1/\theta}}{n} \right)^{2s/(2s+1)}. \end{aligned} \tag{6.59}$$

Hence, for $r \geq 1$, $\{p \geq 2$ and $s > 0\}$ or $\{p \in [1,2)$ and $s > 1/p\}$, we have

$$T \leq C \left(\frac{(\ln n)^{1+1/\theta}}{n} \right)^{2s/(2s+1)}. \tag{6.60}$$

Upper Bound for S

Note that we can write the term S as

$$S = S_1 + S_2 + S_3 + S_4, \tag{6.61}$$

where

$$S_1 = \sum_{j=\tau}^{j_1} \sum_{k=0}^{2^j-1} \mathbb{E}\left(\left(\widehat{\beta}_{j,k} - \beta_{j,k}\right)^2 \mathbb{I}_{\{|\widehat{\beta}_{j,k}| \geq \kappa\lambda_n\}} \mathbb{I}_{\{|\beta_{j,k}| < \kappa\lambda_n/2\}} \right),$$
$$S_2 = \sum_{j=\tau}^{j_1} \sum_{k=0}^{2^j-1} \mathbb{E}\left(\left(\widehat{\beta}_{j,k} - \beta_{j,k}\right)^2 \mathbb{I}_{\{|\widehat{\beta}_{j,k}| \geq \kappa\lambda_n\}} \mathbb{I}_{\{|\beta_{j,k}| \geq \kappa\lambda_n/2\}} \right),$$
$$S_3 = \sum_{j=\tau}^{j_1} \sum_{k=0}^{2^j-1} \mathbb{E}\left(\beta_{j,k}^2 \mathbb{I}_{\{|\widehat{\beta}_{j,k}| < \kappa\lambda_n\}} \mathbb{I}_{\{|\beta_{j,k}| \geq 2\kappa\lambda_n\}} \right),$$
$$S_4 = \sum_{j=\tau}^{j_1} \sum_{k=0}^{2^j-1} \mathbb{E}\left(\beta_{j,k}^2 \mathbb{I}_{\{|\widehat{\beta}_{j,k}| < \kappa\lambda_n\}} \mathbb{I}_{\{|\beta_{j,k}| < 2\kappa\lambda_n\}} \right). \tag{6.62}$$

Let us investigate the bounds of S_1, S_2, S_3, and S_4 in turn.

Upper Bounds for S_1 and S_3

We have

$$\left\{ \left|\widehat{\beta}_{j,k}\right| < \kappa\lambda_n, \left|\beta_{j,k}\right| \geq 2\kappa\lambda_n \right\} \subseteq \left\{ \left|\widehat{\beta}_{j,k} - \beta_{j,k}\right| > \frac{\kappa\lambda_n}{2} \right\},$$
$$\left\{ \left|\widehat{\beta}_{j,k}\right| \geq \kappa\lambda_n, \left|\beta_{j,k}\right| < \frac{\kappa\lambda_n}{2} \right\} \subseteq \left\{ \left|\widehat{\beta}_{j,k} - \beta_{j,k}\right| > \frac{\kappa\lambda_n}{2} \right\},$$
$$\left\{ \left|\widehat{\beta}_{j,k}\right| < \kappa\lambda_n, \left|\beta_{j,k}\right| \geq 2\kappa\lambda_n \right\} \subseteq \left\{ \left|\beta_{j,k}\right| \leq 2\left|\widehat{\beta}_{j,k} - \beta_{j,k}\right| \right\}. \tag{6.63}$$

So,

$$\max(S_1, S_3) \leq C \sum_{j=\tau}^{j_1} \sum_{k=0}^{2^j-1} \mathbb{E}\left(\left(\widehat{\beta}_{j,k} - \beta_{j,k}\right)^2 \mathbb{I}_{\{|\widehat{\beta}_{j,k} - \beta_{j,k}| > \kappa\lambda_n/2\}} \right). \tag{6.64}$$

It follows from the Cauchy-Schwarz inequality, Propositions 6.3 and 6.4, and $2^j \leq 2^{j_1} \leq n$ that

$$\mathbb{E}\left(\left(\widehat{\beta}_{j,k} - \beta_{j,k}\right)^2 \mathbb{I}_{\{|\widehat{\beta}_{j,k} - \beta_{j,k}| > \kappa\lambda_n/2\}} \right) \leq \left(\mathbb{E}\left(\left(\widehat{\beta}_{j,k} - \beta_{j,k}\right)^4 \right) \right)^{1/2} \left(\mathbb{P}\left(\left|\widehat{\beta}_{j,k} - \beta_{j,k}\right| > \frac{\kappa\lambda_n}{2} \right) \right)^{1/2}$$
$$\leq C\left(2^j \frac{1}{n}\right)^{1/2} \left(\frac{1}{n^4}\right)^{1/2} \leq C\frac{1}{n^2}. \tag{6.65}$$

Since $2s/(2s+1) < 1$, we have

$$\max(S_1, S_3) \leq C\frac{1}{n^2}\sum_{j=\tau}^{j_1} 2^j \leq C\frac{1}{n^2}2^{j_1} \leq C\frac{1}{n} \leq C\left(\frac{(\ln n)^{1+1/\theta}}{n}\right)^{2s/(2s+1)}. \tag{6.66}$$

Upper Bound for S_2

Using again Proposition 6.3, we obtain

$$\mathbb{E}\left(\left(\widehat{\beta}_{j,k} - \beta_{j,k}\right)^2\right) \leq C\frac{1}{n} \leq C\frac{(\ln n)^{1+1/\theta}}{n}. \tag{6.67}$$

Hence,

$$S_2 \leq C\frac{(\ln n)^{1+1/\theta}}{n}\sum_{j=\tau}^{j_1}\sum_{k=0}^{2^j-1}\mathbb{I}_{\{|\beta_{j,k}|>\kappa\lambda_n/2\}}. \tag{6.68}$$

Let j_2 be the integer defined by

$$\frac{1}{2}\left(\frac{n}{(\ln n)^{1+1/\theta}}\right)^{1/(2s+1)} < 2^{j_2} \leq \left(\frac{n}{(\ln n)^{1+1/\theta}}\right)^{1/(2s+1)}. \tag{6.69}$$

We have

$$S_2 \leq S_{2,1} + S_{2,2}, \tag{6.70}$$

where

$$\begin{aligned} S_{2,1} &= C\frac{(\ln n)^{1+1/\theta}}{n}\sum_{j=\tau}^{j_2}\sum_{k=0}^{2^j-1}\mathbb{I}_{\{|\beta_{j,k}|>\kappa\lambda_n/2\}}, \\ S_{2,2} &= C\frac{(\ln n)^{1+1/\theta}}{n}\sum_{j=j_2+1}^{j_1}\sum_{k=0}^{2^j-1}\mathbb{I}_{\{|\beta_{j,k}|>\kappa\lambda_n/2\}}. \end{aligned} \tag{6.71}$$

We have

$$S_{2,1} \leq C\frac{(\ln n)^{1+1/\theta}}{n}\sum_{j=\tau}^{j_2} 2^j \leq C\frac{(\ln n)^{1+1/\theta}}{n}2^{j_2} \leq C\left(\frac{(\ln n)^{1+1/\theta}}{n}\right)^{2s/(2s+1)}. \tag{6.72}$$

For $r \geq 1$ and $p \geq 2$, since $B^s_{p,r}(M) \subseteq B^s_{2,\infty}(M)$,

$$S_{2,2} \leq C\frac{(\ln n)^{1+1/\theta}}{n\lambda_n^2}\sum_{j=j_2+1}^{j_1}\sum_{k=0}^{2^j-1}\beta_{j,k}^2 \leq C\sum_{j=j_2+1}^{\infty}\sum_{k=0}^{2^j-1}\beta_{j,k}^2 \leq C2^{-2j_2 s} \leq C\left(\frac{(\ln n)^{1+1/\theta}}{n}\right)^{2s/(2s+1)}. \tag{6.73}$$

For $r \geq 1$, $p \in [1,2)$ and $s > 1/p$, using $\mathbb{I}_{\{|\beta_{j,k}|>\kappa\lambda_n/2\}} \leq C|\beta_{j,k}|^p/\lambda_n^p$, $B^s_{p,r}(M) \subseteq B^{s+1/2-1/p}_{2,\infty}(M)$ and $(2s+1)(2-p)/2 + (s+1/2-1/p)p = 2s$, we have

$$S_{2,2} \leq C\frac{(\ln n)^{1+1/\theta}}{n\lambda_n^p}\sum_{j=j_2+1}^{j_1}\sum_{k=0}^{2^j-1}|\beta_{j,k}|^p \leq C\left(\frac{(\ln n)^{1+1/\theta}}{n}\right)^{(2-p)/2}\sum_{j=j_2+1}^{\infty}2^{-j(s+1/2-1/p)p} \leq C\left(\frac{(\ln n)^{1+1/\theta}}{n}\right)^{(2-p)/2}2^{-j_2(s+1/2-1/p)p} \leq C\left(\frac{(\ln n)^{1+1/\theta}}{n}\right)^{2s/(2s+1)}. \tag{6.74}$$

So, for $r \geq 1$, $\{p \geq 2$ and $s > 0\}$ or $\{p \in [1,2)$ and $s > 1/p\}$, we have

$$S_2 \leq C\left(\frac{(\ln n)^{1+1/\theta}}{n}\right)^{2s/(2s+1)}. \tag{6.75}$$

Upper Bound for S_4

We have

$$S_4 \leq \sum_{j=\tau}^{j_1}\sum_{k=0}^{2^j-1}\beta_{j,k}^2\mathbb{I}_{\{|\beta_{j,k}|<2\kappa\lambda_n\}}. \tag{6.76}$$

Let j_2 be the integer (6.69). Then

$$S_4 \leq S_{4,1} + S_{4,2}, \tag{6.77}$$

where

$$S_{4,1} = \sum_{j=\tau}^{j_2}\sum_{k=0}^{2^j-1}\beta_{j,k}^2\mathbb{I}_{\{|\beta_{j,k}|<2\kappa\lambda_n\}}, \qquad S_{4,2} = \sum_{j=j_2+1}^{j_1}\sum_{k=0}^{2^j-1}\beta_{j,k}^2\mathbb{I}_{\{|\beta_{j,k}|<2\kappa\lambda_n\}}. \tag{6.78}$$

We have

$$S_{4,1} \leq C\sum_{j=\tau}^{j_2}2^j\lambda_n^2 = C\frac{(\ln n)^{1+1/\theta}}{n}\sum_{j=\tau}^{j_2}2^j \leq C\frac{(\ln n)^{1+1/\theta}}{n}2^{j_2} \leq C\left(\frac{(\ln n)^{1+1/\theta}}{n}\right)^{2s/(2s+1)}. \tag{6.79}$$

For $r \geq 1$ and $p \geq 2$, since $B^s_{p,r}(M) \subseteq B^s_{2,\infty}(M)$, we have

$$S_{4,2} \leq \sum_{j=j_2+1}^{\infty} \sum_{k=0}^{2^j-1} \beta_{j,k}^2 \leq C 2^{-2j_2 s} \leq C\left(\frac{(\ln n)^{1+1/\theta}}{n}\right)^{2s/(2s+1)}. \tag{6.80}$$

For $r \geq 1$, $p \in [1,2)$ and $s > 1/p$, using $\beta_{j,k}^2 \mathbb{I}_{\{|\beta_{j,k}|<2\kappa\lambda_n\}} \leq C\lambda_n^{2-p}|\beta_{j,k}|^p$, $B^s_{p,r}(M) \subseteq B^{s+1/2-1/p}_{2,\infty}(M)$ and $(2s+1)(2-p)/2 + (s+1/2-1/p)p = 2s$, we have

$$\begin{aligned}
S_{4,2} &\leq C\lambda_n^{2-p} \sum_{j=j_2+1}^{j_1} \sum_{k=0}^{2^j-1} |\beta_{j,k}|^p = C\left(\frac{(\ln n)^{1+1/\theta}}{n}\right)^{(2-p)/2} \sum_{j=j_2+1}^{j_1} \sum_{k=0}^{2^j-1} |\beta_{j,k}|^p \\
&\leq C\left(\frac{(\ln n)^{1+1/\theta}}{n}\right)^{(2-p)/2} \sum_{j=j_2+1}^{\infty} 2^{-j(s+1/2-1/p)p} \\
&\leq C\left(\frac{(\ln n)^{1+1/\theta}}{n}\right)^{(2-p)/2} 2^{-j_2(s+1/2-1/p)p} \leq C\left(\frac{(\ln n)^{1+1/\theta}}{n}\right)^{2s/(2s+1)}.
\end{aligned} \tag{6.81}$$

So, for $r \geq 1$, $\{p \geq 2$ and $s > 0\}$ or $\{p \in [1,2)$ and $s > 1/p\}$, we have

$$S_4 \leq C\left(\frac{(\ln n)^{1+1/\theta}}{n}\right)^{2s/(2s+1)}. \tag{6.82}$$

It follows from (6.61), (6.66), (6.75), and (6.82) that

$$S \leq C\left(\frac{(\ln n)^{1+1/\theta}}{n}\right)^{2s/(2s+1)}. \tag{6.83}$$

Combining (6.55), (6.57), (6.60), and (6.83), we have, for $r \geq 1$, $\{p \geq 2$ and $s > 0\}$ or $\{p \in [1,2)$ and $s > 1/p\}$,

$$\mathbb{E}\left(\int_0^1 \left(\hat{f}^H(x) - f(x)\right)^2 dx\right) \leq C\left(\frac{(\ln n)^{1+1/\theta}}{n}\right)^{2s/(2s+1)}. \tag{6.84}$$

The proof of Theorem 5.2 is complete. □

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