Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2011, Article ID 615014, 18 pages
doi:10.1155/2011/615014

Research Article

The Order of Hypersubstitutions of Type (2,1)

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Received 30 December 2010; Accepted 27 March 2011

Academic Editor: H. Srivastava

Hypersubstitutions are mappings which map operation symbols to terms of the corresponding arities. They were introduced as a way of making precise the concept of a hyperidentity and generalizations to M-hyperidentities. A variety in which every identity is satisfied as a hyperidentity is called solid. If every identity is an M-hyperidentity for a subset M of the set of all hypersubstitutions, the variety is called M-solid. There is a Galois connection between monoids of hypersubstitutions and sublattices of the lattice of all varieties of algebras of a given type. Therefore, it is interesting and useful to know how semigroup or monoid properties of monoids of hypersubstitutions transfer under this Galois connection to properties of the corresponding lattices of M-solid varieties. In this paper, we study the order of each hypersubstitution of type (2,1), that is, the order of the cyclic subsemigroup of the monoid of all hypersubstitutions of type (2,1) generated by that hypersubstitution.

1. Preliminaries

Let $\mathbb{N}$ denote the set of all positive integers. Let $\tau = \{(f_i, n_i) \mid i \in I\}$ be a type. Let $X = \{x_1, x_2, x_3, \ldots\}$ be a countably infinite alphabet of variables such that the sequence of the operation symbols $(f_i)_{i\in I}$ is disjoint with X, and let $X_n = \{x_1, x_2, \ldots, x_n\}$ be an n-element alphabet where $n \in \mathbb{N}$. Here, f_i is n_i-ary for a natural number $n_i \geq 1$. An n-ary ($n \geq 1$) `term` of type τ is inductively defined as follows:

(i) every variable $x_j \in X_n$ is an n-ary term,

(ii) if $t_1, \ldots, t_{n_i}$ are n-ary terms and f_i is an n_i-ary operation symbol, then $f_i(t_1, \ldots, t_{n_i})$ is an n-ary term.

Let $W_\tau(X_n)$ be the smallest set containing $x_1, \ldots, x_n$ and being closed under finite application of (ii). The set of all terms of type τ over the alphabet X is defined as the disjoint union $W_\tau(X) := \bigcup_{n=1}^{\infty} W_\tau(X_n)$.

Any mapping $\sigma : \{f_i : i \in \mathrm{I}\} \rightarrow W_\tau(X)$ is called a `hypersubstitution` of type τ if $\sigma(f_i)$ is an n_i-ary term of type τ for every $i \in I$. Any hypersubstitution σ of type τ can be uniquely extended to a map $\widehat{\sigma}$ on $W_\tau(X)$ as follows:

(i) $\widehat{\sigma}[t] := t$ if $t \in X$,

(ii) $\widehat{\sigma}[t] := \sigma(f_i)(\widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_{n_i}])$ if $t = f_i(t_1, \ldots, t_{n_i})$.

A binary operation $\circ_h$ is defined on the set $\mathrm{Hyp}(\tau)$ of all hypersubstitutions of type τ by

$$(\sigma_1 \circ_h \sigma_2)(f_i) := \widehat{\sigma}_1[\sigma_2(f_i)], \tag{1.1}$$

for all n_i-ary operation symbols f_i. This binary associative operation makes $\mathrm{Hyp}(\tau)$ into a monoid, with the identity hypersubstitution σ_{id} which maps every f_i to $f_i(x_1, \ldots, x_{n_i})$ as an identity element. For a submonoid M of $\mathrm{Hyp}(\tau)$ an identity $s \approx t$ of a variety V of type (τ) is called an M-hyperidentity of V if for every hypersubstitution $\sigma \in M$, the equation $\widehat{\sigma}[s] \approx \widehat{\sigma}[t]$ holds in V. A variety V is called M-solid if every identity of V is an M-hyperidentity of V. If M is a submonoid of $\mathrm{Hyp}(\tau)$, then the collection of all M-solid varieties of type τ is a complete sublattice of the lattice of all varieties of type τ [1].

Let $\sigma \in \mathrm{Hyp}(\tau)$ and let $\langle\sigma\rangle := \{\sigma^n \mid n \in \mathbb{N}\}$ be the cyclic subsemigroup of $\mathrm{Hyp}(\tau)$ generated by σ. The order of $\sigma \in \mathrm{Hyp}(\tau)$ is defined as the order of the semigroup $\langle\sigma\rangle$. If $\langle\sigma\rangle$ is finite, then the order of the hypersubstition σ is finite, otherwise the order of σ is infinite. The hypersubstitution σ is idempotent if and only if the order of σ is 1.

The order of a hypersubstitution of type (2) is 1, 2, or infinite [2]. The order of a hypersubstitution of type (3) is 1, 2, 3, or infinite [3]. The order of a hypersubstitution of type (2,2) is 1, 2, 3, 4, or infinite [4]. We are interested in type (2,1). The main result is as follows.

Main Theorem. *Any hypersubstitution of type* $(2, 1)$ *has order either infinite or less than or equal to 3.*

Throughout this paper, let f and g be the binary operation symbols and the unary operation symbols of type $\tau = (2, 1)$, respectively. For a binary term a and an unary term b of type τ, the hypersubstitution which maps the operation symbol f to the term a and the operation symbol g to the term b will be denoted by $\sigma_{a,b}$.

For a binary term $t \in W_{(2,1)}(X_2)$, we introduce the following notations:

leftmost(t)—the first variable (from the left) occurring in t,

rightmost(t)—the last variable occurring in t,

var(t)—the set of all variable occurring in t,

op(t)—the total number of all operation symbols occurring in t,

ops(t)—the set of all operation symbols occurring in t,

firstops(t)—the first operation symbol (from the left) occurring in t.

For $t \in W_{(2,1)}(X_2)$, let $\mathrm{Lp}(t)$ denote the left path from the root to the leaf which is labelled by the leftmost variable in t and $\mathrm{Rp}(t)$ denote the right path from the root to the leaf which is labelled by the rightmost variable in t. The operation symbols occurring in $\mathrm{Lp}(t)$ and

$\mathrm{Rp}(t)$ will be denoted by $\mathrm{ops}(\mathrm{Lp}(t))$ and $\mathrm{ops}(\mathrm{Rp}(t))$, respectively. If $t \in W_{(2,1)}(X_2)$ such that $\mathrm{var}(t) = \{x_1\}$ or $\mathrm{var}(t) = \{x_2\}$, we define $t^1 = t$ and

$$t^n = t^{n-1}(t,t), \quad \text{if } n \geq 1. \tag{1.2}$$

For a and b of the hypersubstitution $\sigma_{a,b}$, we break our analysis into the following six cases, which cover all possibilities for a and b:

(I) $\mathrm{op}(a) \leq 1$ and $\mathrm{op}(b) \leq 1$,

(II) $\mathrm{op}(a) > 1$ and $\mathrm{op}(b) > 1$,

(III) $\mathrm{op}(a) = 1$ and $\mathrm{op}(b) > 1$,

(IV) $\mathrm{op}(a) > 1$ and $\mathrm{op}(b) = 1$,

(V) $\mathrm{op}(a) > 1$ and $\mathrm{op}(b) = 0$,

(VI) $\mathrm{op}(a) = 0$ and $\mathrm{op}(b) > 1$.

In case (I), we have term $a \in \{x_1, x_2, f(x_1,x_1), f(x_1,x_2), f(x_2,x_1), f(x_2,x_2), g(x_1), g(x_2)\}$ and $b \in \{x_1, g(x_1), f(x_1,x_1)\}$. For $\sigma_{a,b}$ we have twenty-four cases, for which the orders can be verified by simple calculations. The case $a = g(x_2)$ and $b = f(x_1,x_1)$ gives order 3, six other cases give order 2, and the remaining 17 cases give order 1.

2. Case II: $\mathrm{op}(a) > 1$ and $\mathrm{op}(b) > 1$

In this section, we consider the order of a hypersubstitution $\sigma_{a,b}$ where $\mathrm{op}(a) > 1$ and $\mathrm{op}(b) > 1$. We consider three subcases of Case II:

(II-1) $\mathrm{var}(a) = \{x_1, x_2\}$, $\mathrm{var}(b) = \{x_1\}$,

(II-2) $\mathrm{var}(a) = \{x_1\}$, $\mathrm{var}(b) = \{x_1\}$,

(II-3) $\mathrm{var}(a) = \{x_2\}$, $\mathrm{var}(b) = \{x_1\}$.

The following formula for the operation symbol count of the compound term $s(t_1,\ldots,t_n)$ for some $s, t_1, \ldots, t_n \in W_\tau(X)$ was proved in [5]:

$$\mathrm{op}(s(t_1,\ldots,t_n)) = \sum_{j=1}^{n} vb_j(s)\mathrm{op}(t_j) + \mathrm{op}(s), \quad s, t_1, \ldots, t_n \in W_{(\tau)}(X), \tag{2.1}$$

where $vb_j(s)$ = the number of occurrences of variable x_j in the term s.

Using the facts (see [5]) that $\mathrm{op}(\widehat{\sigma}[t]) > \mathrm{op}(t)$ for all $t \in W_\tau(X)$ if $\sigma \in \mathrm{Hyp}(\tau)$ is regular, that is, $\mathrm{var}(\sigma(f_i)) = X_n$, for all $i \in I$, and the formula above, we obtain the following theorem.

Theorem 2.1. *Let $a \in W_{(2,1)}(X_2)$, $b \in W_{(2,1)}(X_1)$. If $\mathrm{op}(a) > 1$, $\mathrm{op}(b) > 1$, $\mathrm{var}(a) = \{x_1, x_2\}$, $\mathrm{var}(b) = \{x_1\}$, then the order of $\sigma_{a,b}$ is infinite.*

Proof. Since $\sigma_{a,b}$ is regular, by induction we obtain $\mathrm{op}(\sigma_{a,b}^k(f)) < \mathrm{op}(\sigma_{a,b}^{k+1}(f))$. Then, the order of $\sigma_{a,b}$ is infinite. □

The following lemmas are easy to prove.

Lemma 2.2. *Let $a, t \in W_{(2,1)}(X_2)$, $b \in W_{(2,1)}(X_1)$ be such that* $\mathrm{op}(a) \geq 1$, $\mathrm{op}(b) \geq 1$. *If $t \notin X_2$, then $\widehat{\sigma}^{n}_{a,b}[t]$ is not a variable for all $n \in \mathbb{N}$.*

Lemma 2.3. *Let $a, t \in W_{(2,1)}(X_2)$, $b \in W_{(2,1)}(X_1)$. If* $\mathrm{var}(t) = \{x_i\}$ *for some $i \in \{1,2\}$, then* $\mathrm{var}(\widehat{\sigma}^{n}_{a,b}[t]) = \{x_i\}$ *for all $n \in \mathbb{N}$.*

Lemma 2.4. *Let $a \in W_{(2,1)}(X_2)$, $b \in W_{(2,1)}(X_1)$ be such that* $\mathrm{op}(a) \geq 1$, $\mathrm{op}(b) \geq 1$, *and $a = f(a_1, a_2)$, $b = g(b_1)$ for some $a_1, a_2 \in W_{(2,1)}(X_2)$, $b_1 \in W_{(2,1)}(X_1)$. Then, the following hold:*

(i) *if* $\mathrm{var}(a) = \{x_1\}$ *and $a_1 \neq x_1$, then the order of $\sigma_{a,b}$ is infinite;*

(ii) *if* $\mathrm{var}(b) = \{x_1\}$ *and $b_1 \neq x_1$, then the order of $\sigma_{a,b}$ is infinite.*

Proof. (i) Assume $\mathrm{var}(a) = \{x_1\}$ and $a_1 \neq x_1$. Then, $a_1 \notin X_2$. Let $n \in \mathbb{N}$. By Lemma 2.2, $\widehat{\sigma}^{n+1}_{a,b}[a_1] \notin X_2$. Since $\mathrm{var}(a) = \{x_1\}$, by Lemma 2.3, $\mathrm{var}(\widehat{\sigma}^{n}_{a,b}[a]) = \{x_1\}$. Now,

$$\begin{aligned}
\mathrm{op}\left(\widehat{\sigma}^{n+1}_{a,b}[a]\right) &= \mathrm{op}\left(\widehat{\sigma}^{n}_{a,b}\left[\widehat{\sigma}_{a,b}\left[f(a_1, a_2)\right]\right]\right) \\
&= \mathrm{op}\left(\widehat{\sigma}^{n}_{a,b}\left[\sigma_{a,b}(f)(\widehat{\sigma}_{a,b}[a_1], \widehat{\sigma}_{a,b}[a_2])\right]\right) \\
&= \mathrm{op}\left(\widehat{\sigma}^{n}_{a,b}\left[a(\widehat{\sigma}_{a,b}[a_1], \widehat{\sigma}_{a,b}[a_2])\right]\right) \\
&= \mathrm{op}\left(\widehat{\sigma}^{n}_{a,b}[a]\left(\widehat{\sigma}^{n+1}_{a,b}[a_1], \widehat{\sigma}^{n+1}_{a,b}[a_2]\right)\right) \\
&> \mathrm{op}\left(\widehat{\sigma}^{n}_{a,b}[a]\right).
\end{aligned} \tag{2.2}$$

This shows that $\mathrm{op}(\widehat{\sigma}^{n+1}_{a,b}[a]) > \mathrm{op}(\widehat{\sigma}^{n}_{a,b}[a])$ for all $n \in \mathbb{N}$. Hence, the order of $\sigma_{a,b}$ is infinite.

(ii) Assume $\mathrm{var}(b) = \{x_1\}$ and $b_1 \neq x_1$, then $b_1 \notin X_1$. Let $n \in \mathbb{N}$. By Lemma 2.2, $\widehat{\sigma}^{n+1}_{a,b}[b_1] \notin X_1$. Since $\mathrm{var}(b) = \{x_1\}$, by Lemma 2.3 $\mathrm{var}(\widehat{\sigma}^{n}_{a,b}[b]) = \{x_1\}$ for all $n \in \mathbb{N}$. Then,

$$\begin{aligned}
\mathrm{op}\left(\widehat{\sigma}^{n+1}_{a,b}[b]\right) &= \mathrm{op}\left(\widehat{\sigma}^{n}_{a,b}\left[\widehat{\sigma}_{a,b}\left[g(b_1)\right]\right]\right) \\
&= \mathrm{op}\left(\widehat{\sigma}^{n}_{a,b}[b]\left(\widehat{\sigma}^{n+1}_{a,b}[b_1]\right)\right) \\
&> \mathrm{op}\left(\widehat{\sigma}^{n}_{a,b}[b]\right).
\end{aligned} \tag{2.3}$$

Therefore, $\mathrm{op}(\widehat{\sigma}^{n+1}_{a,b}[b]) > \mathrm{op}(\widehat{\sigma}^{n}_{a,b}[b])$ for all $n \in \mathbb{N}$. Hence we have the claim. □

Throughout the rest of this paper, we assume that when a is not a variable term it has the form $f(a_1, a_2)$ or $g(a_1)$, for some terms a_1, a_2 and that when b is not a variable term it has the form $f(b_1, b_2)$ or $g(b_1)$, for some terms b_1, b_2.

In Case (II-2), we have $\mathrm{var}(a) = \{x_1\}$, $\mathrm{var}(b) = \{x_1\}$. We consider the following four subcases:

(2.1) $\mathrm{firstops}(a) = f$, $\mathrm{firstops}(b) = f$;

(2.2) $\mathrm{firstops}(a) = g$, $\mathrm{firstops}(b) = g$;

(2.3) $\operatorname{firstops}(a) = f$, $\operatorname{firstops}(b) = g$;

(2.4) $\operatorname{firstops}(a) = g$, $\operatorname{firstops}(b) = f$.

In Case (2.1), we can separate into four subcases:

(2.1.1) $a_1 \notin X_2$;

(2.1.2) $a_1 = x_1$, $b_1 = x_1$;

(2.1.3) $a_1 = x_1$, $b_1 \notin X_1$, $\operatorname{ops}(\operatorname{Lp}(b)) = \{f\}$;

(2.1.4) $a_1 = x_1$, $b_1 \notin X_1$, $\operatorname{ops}(\operatorname{Lp}(b)) \neq \{f\}$.

Theorem 2.5. *Let $a \in W_{(2,1)}(X_2)$, $b \in W_{(2,1)}(X_1)$ be such that $\operatorname{op}(a) > 1$, $\operatorname{op}(b) > 1$, $\operatorname{var}(a) = \{x_1\} = \operatorname{var}(b)$, and $a = f(a_1, a_2)$, $b = f(b_1, b_2)$ for some $a_1, a_2 \in W_{(2,1)}(X_2)$, $b_1, b_2 \in W_{(2,1)}(X_1)$. Then, the following hold:*

(i) *if a and b satisfy (2.1.1) or (2.1.4), then $\sigma_{a,b}$ has infinite order;*

(ii) *if a and b satisfy (2.1.2) or (2.1.3), then the order of $\sigma_{a,b}$ is less than or equal to 3.*

Proof. (i) Assume a and b satisfy (2.1.1). By Lemma 2.2, we have $\widehat{\sigma}^n_{a,b}[a_1] \notin X_2$ for all $n \in \mathbb{N}$. Since $\operatorname{var}(\widehat{\sigma}_{a,b}[a]) = \{x_1\}$, by Lemma 2.3 $\operatorname{var}(\widehat{\sigma}^n_{a,b}[a]) = \{x_1\}$ for all $n \in \mathbb{N}$. Therefore,

$$\begin{aligned}\operatorname{op}\left(\widehat{\sigma}^{k+1}_{a,b}[a]\right) &= \operatorname{op}\left(\widehat{\sigma}^{k}_{a,b}[a(\widehat{\sigma}_{a,b}[a_1], \widehat{\sigma}_{a,b}[a_2])]\right)\\ &= \operatorname{op}\left(\widehat{\sigma}^{k}_{a,b}[a]\left(\widehat{\sigma}^{k+1}_{a,b}[a_1], \widehat{\sigma}^{k+1}_{a,b}[a_2]\right)\right)\\ &> \operatorname{op}\left(\widehat{\sigma}^{k}_{a,b}[a]\right).\end{aligned} \tag{2.4}$$

This shows that $\operatorname{op}(\widehat{\sigma}^{n+1}_{a,b}[a]) > \operatorname{op}(\widehat{\sigma}^{n}_{a,b}[a])$ for all $n \in \mathbb{N}$. Hence, $\sigma_{a,b}$ has infinite order.

Assume a and b satisfy (2.1.4). Since $\operatorname{var}(\widehat{\sigma}_{a,b}[a]) = \{x_1\}$, $\widehat{\sigma}^n_{a,b}[a] = \widehat{\sigma}_{a,b}[a]$ for all $n \in \mathbb{N}$. Since $g \in \operatorname{ops}(\operatorname{Lp}(b))$, we have $\operatorname{op}(\widehat{\sigma}^{n+1}_{a,b}[b]) > \operatorname{op}(\widehat{\sigma}^{n}_{a,b}[b])$ for all $n \in \mathbb{N}$. Hence, the order of $\sigma_{a,b}$ is infinite.

(ii) Assume a and b satisfy (2.1.2). Since $\operatorname{var}(\widehat{\sigma}_{a,b}[a]) = \{x_1\}$, $a_1 = x_1$ and $\operatorname{firstops}(a) = f$, we obtain $\widehat{\sigma}_{a,b}[a] = a$. This gives

$$\widehat{\sigma}^3_{a,b}[f] = \widehat{\sigma}^2_{a,b}[a] = \widehat{\sigma}_{a,b}[a] = \widehat{\sigma}^2_{a,b}[f]. \tag{2.5}$$

Since $b_1 = x_1$, $\operatorname{var}(\widehat{\sigma}_{a,b}[a]) = \{x_1\}$ and $\operatorname{firstops}(b) = f$, we have $\widehat{\sigma}_{a,b}[b]) = a$. So,

$$\widehat{\sigma}^3_{a,b}[g] = \widehat{\sigma}^2_{a,b}[b] = \widehat{\sigma}_{a,b}[a] = a = \widehat{\sigma}_{a,b}[b] = \widehat{\sigma}^2_{a,b}[g]. \tag{2.6}$$

Hence, $\sigma^2_{a,b} = \sigma^3_{a,b}$. This shows that the order of $\sigma_{a,b}$ is less than or equal to 2.

Assume a and b satisfy (2.1.3). Because of (2.1.2), we have $\widehat{\sigma}_{a,b}[a] = a$, which implies $\widehat{\sigma}^3_{a,b}[f]) = \widehat{\sigma}^2_{a,b}[f]$. Since $\operatorname{ops}(\operatorname{Lp}(b)) = \{f\}$, we have

$$\widehat{\sigma}_{a,b}[b] = a'_1(a'_2(\cdots a'_{m-1}(a'_m(x_1, x_1), \widehat{\sigma}_{a,b}[t'_{m-1}]), \ldots, \widehat{\sigma}_{a,b}[t'_2]), \widehat{\sigma}_{a,b}[t'_1]), \tag{2.7}$$

for some $t'_1, \ldots, t'_{m-1} \in W_{(2,1)}(X_1)$ with $m = \mathrm{op}(\mathrm{Lp}(b))$ and $a'_1 = a'_2 = \cdots = a'_m = a$. It follows that $\widehat{\sigma}^2_{a,b}[b] = (\widehat{\sigma}_{a,b}[a])^m$. Since $\widehat{\sigma}_{a,b}[a] = a$, we obtain

$$\begin{aligned}\widehat{\sigma}^4_{a,b}[g] &= \widehat{\sigma}^3_{a,b}[b] = \widehat{\sigma}_{a,b}\left[(\widehat{\sigma}_{a,b}[a])^m\right] = \left(\widehat{\sigma}^2_{a,b}[a]\right)^m \\ &= (\widehat{\sigma}_{a,b}[\widehat{\sigma}_{a,b}[a]])^m = (\widehat{\sigma}_{a,b}[a])^m = \widehat{\sigma}^2_{a,b}[b] = \widehat{\sigma}^3_{a,b}[g].\end{aligned} \tag{2.8}$$

Thus, $\sigma^3_{a,b} = \sigma^4_{a,b}$. Hence, the order of $\sigma_{a,b}$ is less than or equal to 3. □

In Case (2.2), we have firstops$(a) = g$ and firstops$(b) = g$. Since $\mathrm{op}(a) > 1$ and $\mathrm{op}(b) > 1$, $a_1 \notin X_2$ and $b_1 \notin X_1$. Then, we obtain the following.

Theorem 2.6. *Let $a \in W_{(2,1)}(X_2)$, $b \in W_{(2,1)}(X_1)$ be such that $\mathrm{op}(a) > 1$, $\mathrm{op}(b) > 1$, $\mathrm{var}(a) = \{x_1\}$, $a = g(a_1)$ and $b = g(b_1)$ for some $a_1 \in W_{(2,1)}(X_2)$, $b_1 \in W_{(2,1)}(X_1)$. Then, $\sigma_{a,b}$ has infinite order.*

Proof. Assume that $a = g(a_1)$, $b = g(b_1)$, $\mathrm{op}(a) > 1$, $\mathrm{op}(b) > 1$, $\mathrm{var}(a) = \{x_1\}$. Then, $a_1 \notin X_2$, $b_1 \notin X_1$. It can be proved, as in the proof of Theorem 2.5(i), that the order of $\sigma_{a,b}$ is infinite. □

In Case (2.3), we have firstops$(a) = f$ and firstops$(b) = g$. Since $\mathrm{op}(b) > 1$, $b_1 \notin X_1$. Then, we obtain the following.

Theorem 2.7. *Let $a \in W_{(2,1)}(X_2)$, $b \in W_{(2,1)}(X_1)$ be such that $\mathrm{op}(a) > 1$, $\mathrm{op}(b) > 1$, $\mathrm{var}(a) = \{x_1\}$, $a = f(a_1, a_2)$ and $b = g(b_1)$ for some $a_1, a_2 \in W_{(2,1)}(X_2)$, $b_1 \in W_{(2,1)}(X_1) \setminus X_1$. Then, $\sigma_{a,b}$ has infinite order.*

Proof. By $\mathrm{op}(b) > 1$, $b_1 \notin X_1$. Since firstops$(b) = g$, $\mathrm{var}(b) = \{x_1\}$ and Lemma 2.4(ii), we have $\sigma_{a,b}$ has infinite order. □

In Case (2.4), we have firstops$(a) = g$ and firstops$(b) = f$. Since $\mathrm{op}(a) > 1$ and $\mathrm{op}(b) > 1$, $a_1 \notin X_2$ and $b_1 \notin X_1$ or $b_2 \notin X_1$. We consider the following three subcases:

(2.4.1) $b_1 \notin X_1$;

(2.4.2) $b_1 = x_1$, $\mathrm{op}(a_1) = \{g\}$;

(2.4.3) $b_1 = x_1$, $\mathrm{op}(a_1) \neq \{g\}$.

Theorem 2.8. *Let $a \in W_{(2,1)}(X_2)$, $b \in W_{(2,1)}(X_1)$ be such that $\mathrm{op}(a) > 1$, $\mathrm{op}(b) > 1$, $\mathrm{var}(a) = \{x_1\}$, $\mathrm{var}(b) = \{x_1\}$, and $a = g(a_1)$, $b = f(b_1, b_2)$ for some $a_1 \in W_{(2,1)}(X_2)$, $b_1, b_2 \in W_{(2,1)}(X_1)$. Then, the following hold:*

(i) *if a and b satisfy (2.4.1) or (2.4.3), then $\sigma_{a,b}$ has infinite order;*

(ii) *if a and b satisfy (2.4.2), then the order of $\sigma_{a,b}$ is less than or equal to 2.*

Proof. (i) Assume that a and b satisfy (2.4.1). By Lemma 2.2, we have $\widehat{\sigma}^n_{a,b}[a_1] \notin X_2$ for all $n \in \mathbb{N}$. Since $\operatorname{var}(\widehat{\sigma}_{a,b}[a]) = \{x_1\}$, by Lemma 2.3 $\operatorname{var}(\widehat{\sigma}^n_{a,b}[a]) = \{x_1\}$ for all $n \in \mathbb{N}$. Therefore,

$$\begin{aligned}
\operatorname{op}\left(\widehat{\sigma}^{k+1}_{a,b}[a]\right) &= \operatorname{op}\left(\widehat{\sigma}^{k}_{a,b}[a(\widehat{\sigma}_{a,b}[a_1],\widehat{\sigma}_{a,b}[a_2])]\right) \\
&= \operatorname{op}\left(\widehat{\sigma}^{k}_{a,b}[a]\left(\widehat{\sigma}^{k+1}_{a,b}[a_1],\widehat{\sigma}^{k+1}_{a,b}[a_2]\right)\right) \\
&> \operatorname{op}\left(\widehat{\sigma}^{k}_{a,b}[a]\right).
\end{aligned} \tag{2.9}$$

This shows that $\operatorname{op}(\widehat{\sigma}^{n+1}_{a,b}[a]) > \operatorname{op}(\widehat{\sigma}^{n}_{a,b}[a])$ for all $n \in \mathbb{N}$. Hence, $\sigma_{a,b}$ has infinite order.

Assume that a and b satisfy (2.4.3). This case can be proved as in (2.4.1).

(ii) Assume that a and b satisfy (2.4.2). Then, $\widehat{\sigma}_{a,b}[a] = b$ and $\widehat{\sigma}_{a,b}[b] = a$. We have $\sigma^3_{a,b} = \sigma_{a,b}$ and $\sigma^4_{a,b} = \sigma^2_{a,b}$. Hence, the order of $\sigma_{a,b}$ is less than or equal to 3. □

In Case (II-3), we have $\operatorname{var}(a) = \{x_2\}$, $\operatorname{var}(b) = \{x_1\}$. We consider the following four subcases:

(3.1) $\operatorname{firstops}(a) = f$, $\operatorname{firstops}(b) = f$;

(3.2) $\operatorname{firstops}(a) = g$, $\operatorname{firstops}(b) = g$;

(3.3) $\operatorname{firstops}(a) = f$, $\operatorname{firstops}(b) = g$;

(3.4) $\operatorname{firstops}(a) = g$, $\operatorname{firstops}(b) = f$.

In Case (3.1), we can separate into four subcases:

(3.1.1) $a_2 \notin X_2$;

(3.1.2) $a_2 = x_2$, $b_2 = x_1$;

(3.1.3) $a_2 = x_2$, $b_2 \notin X_1$, $\operatorname{ops}(\operatorname{Rp}(b)) = \{f\}$;

(3.1.4) $a_2 = x_2$, $b_2 \notin X_1$, $\operatorname{ops}(\operatorname{Rp}(b)) \neq \{f\}$.

Theorem 2.9. *Let* $a \in W_{(2,1)}(X_2)$, $b \in W_{(2,1)}(X_1)$ *be such that* $\operatorname{op}(a) > 1$, $\operatorname{op}(b) > 1$, $\operatorname{var}(a) = \{x_2\}$, $\operatorname{var}(b) = \{x_1\}$, *and* $a = f(a_1, a_2)$, $b = f(b_1, b_2)$ *for some* $a_1, a_2 \in W_{(2,1)}(X_2)$, $b_1, b_2 \in W_{(2,1)}(X_1)$. *Then, the following hold:*

(i) *if a and b satisfy (3.1.1) or (3.1.4), then* $\sigma_{a,b}$ *has infinite order;*

(ii) *if a and b satisfy (3.1.2) or (2.1.3), then the order of* $\sigma_{a,b}$ *is less than or equal to 2.*

Proof. (i) Assume that a and b satisfy (3.1.1). By Lemma 2.2, we have $\widehat{\sigma}^n_{a,b}[a_1] \notin X_2$ for all n *in* $\mathbb{N}$. Since $\operatorname{var}(\widehat{\sigma}_{a,b}[a]) = \{x_2\}$, by Lemma 2.3 $\operatorname{var}(\widehat{\sigma}^n_{a,b}[a]) = \{x_2\}$ for all $n \in \mathbb{N}$. Therefore,

$$\begin{aligned}
\operatorname{op}\left(\widehat{\sigma}^{k+1}_{a,b}[a]\right) &= \operatorname{op}\left(\widehat{\sigma}^{k}_{a,b}[a(\widehat{\sigma}_{a,b}[a_1],\widehat{\sigma}_{a,b}[a_2])]\right) \\
&= \operatorname{op}\left(\widehat{\sigma}^{k}_{a,b}[a]\left(\widehat{\sigma}^{k+1}_{a,b}[a_1],\widehat{\sigma}^{k+1}_{a,b}[a_2]\right)\right) \\
&> \operatorname{op}\left(\widehat{\sigma}^{k}_{a,b}[a]\right).
\end{aligned} \tag{2.10}$$

This shows that $\operatorname{op}(\widehat{\sigma}^{n+1}_{a,b}[a]) > \operatorname{op}(\widehat{\sigma}^{n}_{a,b}[a])$ for all $n \in \mathbb{N}$. Hence, $\sigma_{a,b}$ has infinite order.

Assume that a and b satisfy (3.1.4). Then, $\widehat{\sigma}^n_{a,b}[a] = \widehat{\sigma}_{a,b}[a]$ for all $n \in \mathbb{N}$. Since $\operatorname{var}(\widehat{\sigma}_{a,b}[a]) = \{x_2\}$ and $b_2 \notin X_1$, we have

$$\begin{aligned}\operatorname{op}\left(\widehat{\sigma}^{k+1}_{a,b}[b]\right) &= \operatorname{op}\left(\widehat{\sigma}^{k}_{a,b}[b(\widehat{\sigma}_{a,b}[b_1],\widehat{\sigma}_{a,b}[b_2])]\right)\\ &= \operatorname{op}\left(\widehat{\sigma}^{k}_{a,b}[b]\left(\widehat{\sigma}^{k+1}_{a,b}[b_1],\widehat{\sigma}^{k+1}_{a,b}[b_2]\right)\right)\\ &> \operatorname{op}\left(\widehat{\sigma}^{k}_{a,b}[b]\right).\end{aligned} \tag{2.11}$$

This implies that $\operatorname{op}(\widehat{\sigma}^{n+1}_{a,b}[b]) > \operatorname{op}(\widehat{\sigma}^{n}_{a,b}[b])$ for all $n \in \mathbb{N}$. Hence, the order of $\sigma_{a,b}$ is infinite.

(ii) Assume that a and b satisfy (3.1.2). Then, we get $\widehat{\sigma}_{a,b}[a] = a$, $\widehat{\sigma}^2_{a,b}[b] = \widehat{\sigma}_{a,b}[b]$ or $\widehat{\sigma}_{a,b}[a] = a$, $\widehat{\sigma}_{a,b}[b] = b$. In the first, the order of $\sigma_{a,b}$ is equal to 2. For the latter, $\sigma_{a,b}$ is idempotent.

Assume that a and b satisfy (3.1.3). Then, $\widehat{\sigma}_{a,b}[a] = a$, $\widehat{\sigma}^2_{a,b}[b] = \widehat{\sigma}_{a,b}[b]$, which can be proved the same way as the first case of (ii). □

In Case (3.2), we have $\operatorname{firstops}(a) = g$ and $\operatorname{firstops}(b) = g$. Since $\operatorname{op}(a) > 1$ and $\operatorname{op}(b) > 1$, $a_1 \notin X_2$ and $b_1 \notin X_1$. Then, we obtain the following.

Theorem 2.10. *Let $a \in W_{(2,1)}(X_2)$, $b \in W_{(2,1)}(X_1)$ be such that $\operatorname{op}(a) > 1$, $\operatorname{op}(b) > 1$, $\operatorname{var}(a) = \{x_2\}$, $a = g(a_1)$ and $b = g(b_1)$ for some $a_1 \in W_{(2,1)}(X_2) \setminus X_2$, $b_1 \in W_{(2,1)}(X_1) \setminus X_1$. Then $\sigma_{a,b}$ has infinite order.*

Proof. Since $\operatorname{var}(a) = \{x_2\}$ and $b_1 \notin X_1$, we have

$$\begin{aligned}\operatorname{op}\left(\widehat{\sigma}^{k+1}_{a,b}[b]\right) &= \operatorname{op}\left(\widehat{\sigma}^{k}_{a,b}[b(\widehat{\sigma}_{a,b}[b_1],\widehat{\sigma}_{a,b}[b_2])]\right)\\ &= \operatorname{op}\left(\widehat{\sigma}^{k}_{a,b}[b]\left(\widehat{\sigma}^{k+1}_{a,b}[b_1],\widehat{\sigma}^{k+1}_{a,b}[b_2]\right)\right)\\ &> \operatorname{op}\left(\widehat{\sigma}^{k}_{a,b}[b]\right).\end{aligned} \tag{2.12}$$

Hence, the order of $\sigma_{a,b}$ is infinite. □

In Case (3.3), we have $\operatorname{firstops}(a) = f$ and $\operatorname{firstops}(b) = g$. Since $\operatorname{op}(b) > 1$, $b_1 \notin X_1$. Then, we obtain the following.

Theorem 2.11. *Let $a \in W_{(2,1)}(X_2)$, $b \in W_{(2,1)}(X_1)$ be such that $\operatorname{op}(a) > 1$, $\operatorname{op}(b) > 1$, $\operatorname{var}(a) = \{x_2\}$, $a = f(a_1, a_2)$ and $b = g(b_1)$ for some $a_1, a_2 \in W_{(2,1)}(X_2)$, $b_1 \in W_{(2,1)}(X_1) \setminus X_1$. Then $\sigma_{a,b}$ has infinite order.*

Proof. Since $b_1 \notin X_1$ $\operatorname{firstops}(b) = g$ and Lemma 2.4(ii), we have $\sigma_{a,b}$ has infinite order. □

In Case (3.4), we have $\text{firstops}(a) = g$ and $\text{firstops}(b) = f$. Since $\text{op}(a) > 1$ and $\text{op}(b) > 1$, $a_1 \notin X_2$ and $b_1 \notin X_1$. We have the following result.

Theorem 2.12. *Let* $a \in W_{(2,1)}(X_2)$, $b \in W_{(2,1)}(X_1)$ *be such that* $\text{op}(a) > 1$, $\text{op}(b) > 1$, $\text{var}(a) = \{x_2\}$, $\text{var}(b) = \{x_1\}$, *and* $a = g(a_1)$, $b = f(b_1, b_2)$ *for some* $a_1 \in W_{(2,1)}(X_2)$, $b_1, b_2 \in W_{(2,1)}(X_1)$. *Then,* $\sigma_{a,b}$ *has infinite order.*

Proof. Since $\text{op}(a) > 1$, $a_1 \notin X_2$. We have, by Lemma 2.2, $\widehat{\sigma}^n_{a,b}[a_1] \notin X_2$ for all n *in* $\mathbb{N}$. Since $\text{var}(\widehat{\sigma}_{a,b}[a]) = \{x_2\}$, by Lemma 2.3 $\text{var}(\widehat{\sigma}^n_{a,b}[a]) = \{x_2\}$ for all $n \in \mathbb{N}$. We obtain

$$\begin{aligned}\text{op}\left(\widehat{\sigma}^{k+1}_{a,b}[a]\right) &= \text{op}\left(\widehat{\sigma}^{k}_{a,b}[a(\widehat{\sigma}_{a,b}[a_1], \widehat{\sigma}_{a,b}[a_2])]\right) \\ &= \text{op}\left(\widehat{\sigma}^{k}_{a,b}[a]\left(\widehat{\sigma}^{k+1}_{a,b}[a_1], \widehat{\sigma}^{k+1}_{a,b}[a_2]\right)\right) \\ &> \text{op}\left(\widehat{\sigma}^{k}_{a,b}[a]\right).\end{aligned} \tag{2.13}$$

Hence, $\sigma_{a,b}$ has infinite order. □

3. Case III: $\text{op}(a) = 1$ and $\text{op}(b) > 1$

We consider three subcases:

(III-1) $\text{var}(a) = X_2$, $\text{var}(b) = \{x_1\}$;

(III-2) $\text{var}(a) = \{x_1\}$, $\text{var}(b) = \{x_1\}$;

(III-3) $\text{var}(a) = \{x_2\}$, $\text{var}(b) = \{x_1\}$.

In Case (III-1), we separate into the following subcases:

(1) $a = f(x_1, x_2)$, $\text{firstops}(b) = f$;

(2) $a = f(x_1, x_2)$, $\text{firstops}(b) = g$;

(3) $a = f(x_2, x_1)$, $\text{firstops}(b) = f$;

(4) $a = f(x_2, x_1)$, $\text{firstops}(b) = g$.

We have the following results. Subcases (2) and (4) give infinite order, by Lemma 2.4(ii). The next proposition deal with subcases (1) and (3).

Proposition 3.1. *Let* $b = f(b_1, b_2)$ *for some* $b_1, b_2 \in W_{(2,1)}(X_1)$ *be such that* $\text{var}(b) = \{x_1\}$ *and* $\text{op}(b) > 1$. *Then, the following hold:*

(i) *if* $a = f(x_1, x_2)$, *then the order of* $\sigma_{a,b}$ *is equal to 1 or is infinite;*

(ii) *if* $a = f(x_2, x_1)$, *then the order of* $\sigma_{a,b}$ *is less than or equal to 3 or infinite.*

Proof. (i) Assume that $a = f(x_1, x_2)$. Then, $\sigma_{a,b}[a] = a$. We consider the following.

(i′) If $\text{ops}(b) = \{f\}$, then $\sigma_{a,b}[b] = b$. We get that $\sigma_{a,b}$ is idempotent.

(i″) If $g \in \text{ops}(b)$, suppose that $g \in \text{ops}(b_1)$. Let $k \in \mathbb{N}$. Then $\text{op}(\widehat{\sigma}^{k+1}_{a,b}[b_1]) \geq \text{op}(\widehat{\sigma}^{k}_{a,b}[b_1])$.

Now,

$$\begin{aligned}
\operatorname{op}\left(\widehat{\sigma}_{a,b}^{k+1}[b]\right) &= \operatorname{op}\left(\widehat{\sigma}_{a,b}^{k}[\widehat{\sigma}_{a,b}[b]]\right) \\
&= \operatorname{op}\left(\widehat{\sigma}_{a,b}^{k}\left[\sigma_{a,b}(f)(\widehat{\sigma}_{a,b}[b_1],\widehat{\sigma}_{a,b}[b_2])\right]\right) \\
&= \operatorname{op}\left(\sigma_{a,b}^{k}(\sigma_{a,b}(f))\left(\widehat{\sigma}_{a,b}^{k+1}[b_1],\widehat{\sigma}_{a,b}^{k+1}[b_2]\right)\right) \\
&> \operatorname{op}\left(\sigma_{a,b}^{k}(f)\left(\widehat{\sigma}_{a,b}^{k}[b_1],\widehat{\sigma}_{a,b}^{k}[b_2]\right)\right) \\
&= \operatorname{op}\left(\widehat{\sigma}_{a,b}^{k}[b]\right).
\end{aligned} \tag{3.1}$$

A similar argument works for $g \in \operatorname{ops}(b_2)$. This shows that the order of $\sigma_{a,b}$ is infinite.

(ii) If $\operatorname{ops}(b) = \{f\}$, then $\widehat{\sigma}_{a,b}^{1}[a] = \widehat{\sigma}_{a,b}^{3}[a]$ and $\widehat{\sigma}_{a,b}[b] = \widehat{\sigma}_{a,b}^{3}[b]$. This gives $\widehat{\sigma}_{a,b}^{4} = \widehat{\sigma}_{a,b}^{2}$. Then, the order of $\sigma_{a,b}$ is less than or equal to 3. If $g \in \operatorname{ops}(b)$, then it can be proved as in (i$''$) that the order is infinite. □

In Case (III-2), we have $\operatorname{var}(a) = \{x_1\}$, $\operatorname{var}(b) = \{x_1\}$. We consider the following subcases:

(3.2.1) $a = f(x_1, x_1)$, $\operatorname{firstops}(b) = f$;

- (3.2.1.1) $b_1 = x_1$;
- (3.2.1.2) $b_1 \neq x_1$, $g \in \operatorname{ops}(\operatorname{Lp}(b))$;
- (3.2.1.3) $b_1 \neq x_1$, $g \notin \operatorname{ops}(\operatorname{Lp}(b))$;

(3.2.2) $a = g(x_1)$, $\operatorname{firstops}(b) = g$, then $b_1 \notin X_1$;

(3.2.3) $a = f(x_1, x_1)$, $\operatorname{firstops}(b) = g$, then $b_1 \notin X_1$;

(3.2.4) $a = g(x_1)$, $\operatorname{firstops}(b) = f$;

- (3.2.4.1) $b_1 = x_1$;
- (3.2.4.2) $b_1 \neq x_1$.

Proposition 3.2. *Let* $a = f(x_1, x_1)$, $\operatorname{firstops}(b) = f$, $\operatorname{op}(b) > 1$, *and* $\operatorname{var}(b) = \{x_1\}$. *Then, the following hold:*

(i) *if b satisfies* (3.2.1.1) *or* (3.2.1.3), *then the order of* $\sigma_{a,b}$ *is less than or equal to 2;*

(ii) *if b satisfies* (3.2.1.2), *then the order of* $\sigma_{a,b}$ *is infinite.*

Proof. (i) If a and b satisfy (3.2.1.1), then $\widehat{\sigma}_{a,b}[a] = a = \widehat{\sigma}_{a,b}[b]$. Assume that a and b satisfy (3.2.1.3). Since $a = f(x_1, x_1)$ and $g \notin \widehat{\sigma}_{a,b}[b]$, we have $\widehat{\sigma}_{a,b}^{2}[b] = \widehat{\sigma}_{a,b}[b]$. Clearly, $\widehat{\sigma}_{a,b}^{2}[a] = a$. Then, $\sigma_{a,b}^{3} = \sigma_{a,b}^{2}$. Hence, the order of $\sigma_{a,b}$ is less than or equal to 2.

(ii) Assume that a and b satisfy (3.2.1.2). Since $a = f(x_1, x_1)$, $\text{firstops}(b) = f$ and $g \in \text{ops}(\text{Lp}(b))$, it follows that $\widehat{\sigma}_{a,b}[b] = f(x_1, x_1)(b(v), b(v'))$ for some $v, v' \in W_{(2,1)}(X_1)$. For $n \in \mathbb{N}$, we have

$$\begin{aligned}
\text{op}\left(\widehat{\sigma}_{a,b}^{n+1}[b]\right) &= \text{op}(\widehat{\sigma}^n[\widehat{\sigma}_{a,b}[b]]) \\
&= \text{op}\left(\widehat{\sigma}_{a,b}^n\left[f(x_1, x_1)(b(v), b(v'))\right]\right) \\
&= \text{op}\left(f(x_1, x_1)\left(\widehat{\sigma}_{a,b}^n[b]\left(\widehat{\sigma}_{a,b}^n[v]\right), \widehat{\sigma}_{a,b}^n[b]\left(\widehat{\sigma}_{a,b}^n[v']\right)\right)\right) \\
&> \text{op}\left(\widehat{\sigma}_{a,b}^n[b]\right).
\end{aligned} \tag{3.2}$$

Hence, the order of $\sigma_{a,b}$ is infinite. □

Proposition 3.3. *Let* $a = g(x_1)$ *or* $a = f(x_1, x_1)$, $\text{firstops}(b) = g$, $\text{op}(b) > 1$, *and* $\text{var}(b) = \{x_1\}$. *Then, the order of* $\sigma_{a,b}$ *is infinite.*

Proof. Assume that $a = g(x_1)$. Since $\text{firstops}(b) = g$, $a = g(x_1)$, $\widehat{\sigma}_{a,b}[b] = b(g(x_1))$. For any natural number n, $\text{op}(\widehat{\sigma}_{a,b}^{n+1}[b]) > \text{op}(\widehat{\sigma}_{a,b}^n[b])$ makes $\sigma_{a,b}$ have infinite order.

Assume that $a = f(x_1, x_1)$. Since $\text{firstops}(b) = g$, $a = f(x_1, x_1)$, $\widehat{\sigma}_{a,b}[b] = b(f(x_1, x_1))$. For $n \in \mathbb{N}$, a similar proof to those above gives $\text{op}(\widehat{\sigma}_{a,b}^{n+1}[b]) > \text{op}(\widehat{\sigma}_{a,b}^n[b])$. Hence, the order of $\sigma_{a,b}$ is infinite. □

Proposition 3.4. *Let* $a = g(x_1)$, $\text{firstops}(b) = f$, $\text{op}(b) > 1$ *and* $\text{var}(b) = \{x_1\}$. *Then the following hold:*

(i) *if b satisfies* (3.2.4.1), *then the order of* $\sigma_{a,b}$ *is equal to 2;*

(ii) *if b satisfies* (3.2.4.2), *then the order of* $\sigma_{a,b}$ *is infinite.*

Proof. (i) If a and b satisfy (3.2.4.1), then $\widehat{\sigma}_{a,b}[a] = b$ and $\widehat{\sigma}_{a,b}[b] = a$. Hence, the order of $\sigma_{a,b}$ is equal to 2.

(ii) Assume that a and b satisfy (3.2.4.2). Since $a = g(x_1)$, $\text{firstops}(b) = f$, it follows that $\widehat{\sigma}_{a,b}[b] = g(x_1)b$. For $n \in \mathbb{N}$. As in the proof of Proposition 3.3, we have $\text{op}(\widehat{\sigma}_{a,b}^{n+1}[b]) > \text{op}(\widehat{\sigma}_{a,b}^n[b])$. Hence, the order of $\sigma_{a,b}$ is infinite. □

In Case (III-3), we have $\text{var}(a) = \{x_2\}$, $\text{var}(b) = \{x_1\}$. We consider the following subcases:

(3.3.1) $a = f(x_2, x_2)$, $\text{firstops}(b) = f$;

(3.3.1.1) $b_2 = x_1$;

(3.3.1.2) $b_2 \neq x_1$;

(3.3.2) $a = g(x_2)$, $\text{firstops}(b) = g$.

(3.3.3) $a = f(x_2, x_2)$, $\text{firstops}(b) = g$.

(3.3.4) $a = g(x_2)$, $\text{firstops}(b) = f$;

(3.3.4.1) $b_2 = x_1$;

(3.3.4.2) $b_2 \neq x_1$.

Proposition 3.5. *Let* $a = f(x_2, x_2)$, firstops$(b) = f$, op$(b) > 1$, *and* var$(b) = \{x_1\}$. *Then, the following hold:*

(i) *if* b *satisfies* (3.3.1.1), *then the order of* $\sigma_{a,b}$ *is equal to 2;*

(ii) *if* b *satisfies* (3.3.1.2), *then the order of* $\sigma_{a,b}$ *is infinite.*

Proof. (i) If a and b satisfy (3.3.1.1), then $\widehat{\sigma}_{a,b}[a] = a$ and $\widehat{\sigma}^2_{a,b}[b] = \widehat{\sigma}_{a,b}[b]$. Hence the order of $\sigma_{a,b}$ is equal to 2.

(ii) This can be proved in the same way as Proposition 3.2 (ii). □

Proposition 3.6. *Let* $a = g(x_2)$ *or* $a = f(x_2, x_2)$, firstops$(b) = g$, op$(b) > 1$, *and* var$(b) = \{x_1\}$. *Then, the order of* $\sigma_{a,b}$ *is infinite.*

Proof. Assume that $a = g(x_2)$. Since firstops$(b) = g$, $a = g(x_2)$, $\widehat{\sigma}_{a,b}[b] = b(g(x_2))$. The same argument works as in Proposition 3.3. Hence, the order of $\sigma_{a,b}$ is infinite. The case that $a = f(x_2, x_2)$ can be proved similarly. □

Proposition 3.7. *Let* $a = g(x_2)$, firstops$(b) = f$, op$(b) > 1$, *and* var$(b) = \{x_1\}$. *Then, the following hold:*

(i) *if* b *satisfies* (3.3.4.1), *then the order of* $\sigma_{a,b}$ *is equal to 2;*

(ii) *if* b *satisfies* (3.3.4.2), *then the order of* $\sigma_{a,b}$ *is infinite.*

Proof. (i) If a and b satisfy (3.3.4.1), then $\widehat{\sigma}^2_{a,b}[a] = a$ and $\widehat{\sigma}^2_{a,b}[b] = b$. Hence, the order of $\sigma_{a,b}$ is equal to 2.

(ii) Assume a and b satisfy (3.3.4.2), Since $b = g(x_2)$, firstops$(b) = f$, $b_2 \notin X_2$. By Lemma 2.2, we have $\widehat{\sigma}^n_{a,b}[b_1] \notin X_2$. Then,

$$\begin{aligned} \operatorname{op}(\widehat{\sigma}_{a,b}[b]) &= \operatorname{op}(\widehat{\sigma}_{a,b}[f(b_1, b_2)]) \\ &= \operatorname{op}(\sigma_{a,b}(f)(\widehat{\sigma}_{a,b}[b_1], \widehat{\sigma}_{a,b}[b_2])) \\ &> \operatorname{op}(f(b_1, b_2)) \\ &= \operatorname{op}(b). \end{aligned} \tag{3.3}$$

This shows that, $\operatorname{op}(\widehat{\sigma}_{a,b}[b]) > \operatorname{op}(b)$ for all $a \in W_{(2,1)}(X_1) \setminus X_1$. Consequently, $\operatorname{op}(\widehat{\sigma}^{k+1}_{a,b}[b]) = \operatorname{op}(\widehat{\sigma}_{a,b}[\widehat{\sigma}^k_{a,b}[b]])$ for all $k \in \mathbb{N}$. Then, order of $\sigma_{a,b}$ is infinite. □

4. Case IV: op$(a) > 1$ and op$(b) = 1$

We consider three subcases:

(IV-1) var$(a) = X_2$, var$(b) = \{x_1\}$;

(IV-2) var$(a) = \{x_1\}$, var$(b) = \{x_1\}$;

(IV-3) var$(a) = \{x_2\}$, var$(b) = \{x_1\}$.

In Case (IV-1), we separate into four cases:

(4.1.1) $b = f(x_1, x_1)$, firstops$(a) = f$;

(4.1.2) $b = f(x_1, x_1)$, $\text{firstops}(a) = g$;

(4.1.3) $b = g(x_1)$, $\text{firstops}(a) = f$;

(4.1.4) $b = g(x_1)$, $\text{firstops}(a) = g$.

Theorem 4.1. *Let* $a \in W_{(2,1)}(X_2)$, $b \in W_{(2,1)}(X_1)$ *be such that* $\text{op}(a) > 1$, $\text{op}(b) = 1$, $\text{var}(a) = \{x_1, x_2\}$, $\text{var}(b) = \{x_1\}$. *Then, the following hold:*

(i) *if b satisfies* (4.1.1) *or* (4.1.2) *or* (4.1.3), *then the order of* $\sigma_{a,b}$ *is infinite;*

(ii) *if b satisfies* (4.1.4), *then the order of* $\sigma_{a,b}$ *is equal to 1 or is infinite.*

Proof. (i) If a and b satisfy (4.1.1), then we let $a = f(a_1, a_2)$ for some $a_1, a_2 \in W_{(2,1)}(X_2)$. Since $\text{op}(a) > 1$, we may assume that $a_1 \notin X_2$. By Lemma 2.2, we have $\widehat{\sigma}^n_{a,b}[a_1] \notin X_2$. For any natural number n, $\text{op}(\widehat{\sigma}^{n+1}_{a,b}[a]) > \text{op}(\widehat{\sigma}^n_{a,b}[a])$ makes $\sigma_{a,b}$ have infinite order. Cases (4.1.2) and (4.1.3) can be proved similarly.

(ii) Assume that a and b satisfy (4.1.4). If $\text{ops}(a_1) = \{g\}$, then $\widehat{\sigma}_{a,b}[a] = a$ and $\widehat{\sigma}_{a,b}[b] = b$. Hence, $\sigma_{a,b}$ is idempotent. The case that $\text{op}(a_1) \neq \{g\}$ can be proved as in (i). □

In Case (IV-2), we have $\text{var}(a) = \{x_1\}$, $\text{var}(b) = \{x_1\}$. If $a = g(a_1)$ for some term a_1, then $\text{op}(a) > 1$ means that a_1 is not variable.

Proposition 4.2. *Let* $b = f(x_1, x_1)$ *or* $b = g(x_1)$, $\text{op}(a) > 1$, $a = g(a_1)$ *for some* a_1 *and* $\text{var}(a) = \{x_1\} = \text{var}(b)$. *Then, the order of* $\sigma_{a,b}$ *is infinite.*

Proof. Assume $\text{op}(a) > 1$, $a = g(a_1)$ for some a_1 and $\text{var}(a) = \{x_1\} = \text{var}(b)$. Since $a_1 \notin X_2$. By Lemma 2.2 we have $\widehat{\sigma}^n_{a,b}[a_1] \notin X_2$. It can be proved, as in the proof of Theorem 4.1(i), that the order of $\sigma_{a,b}$ is infinite. □

In Case (IV-3), we have $\text{var}(a) = \{x_2\}$, $\text{var}(b) = \{x_1\}$. If $\text{firstops}(a) = g$, then $a_2 \notin X_2$. We consider the following cases:

(4.3.1) $a_2 \notin X_2$, $b = f(x_1, x_1)$;

(4.3.2) $a_2 \notin X_2$, $b = g(x_1)$.

Proposition 4.3. *Let* $b = f(x_1, x_1)$, $\text{op}(a) > 1$, *and* $\text{var}(a) = \{x_2\}$, $\text{var}(b) = \{x_1\}$. *If a and b satisfy* (4.3.1), *then the order of* $\sigma_{a,b}$ *is infinite.*

Proof. The case (4.3.1) can be verified as in Proposition 4.2. □

Proposition 4.4. *Let* $b = g(x_1)$, $\text{op}(a) > 1$, *and* $\text{var}(a) = \{x_2\}$, $\text{var}(b) = \{x_1\}$. *If a and b satisfy* (4.3.2), *then the order of* $\sigma_{a,b}$ *is equal to 1 or is infinite.*

Proof. Assume that a and b satisfy (4.3.2). If $b = g(x_1)$, $\text{firstops}(a) = g$, and $\text{ops}(a_2) = \{g\}$, then $\widehat{\sigma}_{a,b}[a] = a$ and $\widehat{\sigma}_{a,b}[b] = b$. Hence, $\sigma_{a,b}$ is idempotent. Assume that $a_2 \notin X_2$. This case can be proved as in Proposition 4.3. □

5. Case V: $\mathrm{op}(a) > 1$ and $\mathrm{op}(b) = 0$

In this case, we have $a = f(a_1, a_2)$ or $a = g(a_1)$, $b = x_1$. We consider three subcases:

(V-1) $\mathrm{var}(a) = X_2$, $b = x_1$;

(V-2) $\mathrm{var}(a) = \{x_1\}$, $b = x_1$;

(V-3) $\mathrm{var}(a) = \{x_2\}$, $b = x_1$.

In Case (V-1), we separate into the following subcases:

(5.1.1) $\mathrm{firstops}(a) = f$, $a_1 \notin X_2$, $\mathrm{ops}(\mathrm{Lp}(a)) \neq \{g\}$;

(5.1.2) $\mathrm{firstops}(a) = f$, $a_1 \notin X_2$, $\mathrm{ops}(\mathrm{Lp}(a)) = \{g\}$;

(5.1.3) $\mathrm{firstops}(a) = f$, $a_1 \in X_2$, $\mathrm{ops}(\mathrm{Rp}(a)) \neq \{g\}$;

(5.1.4) $\mathrm{firstops}(a) = f$, $a_1 \in X_2$, $\mathrm{ops}(\mathrm{Rp}(a)) = \{g\}$;

(5.1.5) $\mathrm{firstops}(a) = g$, $\mathrm{ops}(a_1) = \{g\}$;

(5.1.6) $\mathrm{firstops}(a) = g$, $\mathrm{ops}(a_1) \neq \{g\}$.

Proposition 5.1. *Let $a = f(a_1, a_2)$, $\mathrm{op}(a) > 1$, $\mathrm{op}(b) = 0$, $\mathrm{var}(a) = \{x_1, x_2\}$, and $b = x_1$. Then, the following hold:*

(i) *if a satisfies* (5.1.1) *or* (5.1.3), *then the order of $\sigma_{a,b}$ is infinite;*

(ii) *if a satisfies* (5.1.2) *or* (5.1.4), *then the order of $\sigma_{a,b}$ is less than or equal to 2.*

Proof. (i) Assume a and b satisfy (5.1.1). Then, $\mathrm{firstops}(a) = f$, $\mathrm{var}(a) = \{x_1, x_2\}$, and $f \in \mathrm{ops}(\mathrm{Lp}(a))$. Then, $\mathrm{op}(a) < \mathrm{op}(\widehat{\sigma}_{a,b}[a])$. For any natural number n, $\mathrm{op}(\widehat{\sigma}^{n+1}_{a,b}[a]) > \mathrm{op}(\widehat{\sigma}^{n}_{a,b}[a])$ makes $\sigma_{a,b}$ have infinite order. A similar argument works for (5.1.3).

(ii) Assume a and b satisfy (5.1.2). Then, $\mathrm{firstops}(a) = f$, $\mathrm{var}(a) = \{x_1, x_2\}$, and $\mathrm{ops}(\mathrm{Lp}(a)) = \{g\}$. If $\mathrm{leftmost}(a_1) = x_1$, then $\widehat{\sigma}_{a,b}[a] = a$ and $\widehat{\sigma}_{a,b}[b] = b$. Hence, $\sigma_{a,b}$ is idempotent. If $\mathrm{leftmost}(a_1) = x_2$, then $\widehat{\sigma}^{2}_{a,b}[a] = \widehat{\sigma}_{a,b}[a]$ and $\widehat{\sigma}_{a,b}[b] = b$. Hence, the order of $\sigma_{a,b}$ is equal to 2.

Assume a and b satisfy (5.1.4). Then, $\mathrm{firstops}(a) = f$, $\mathrm{var}(a) = \{x_1, x_2\}$ and $\mathrm{ops}(\mathrm{Rp}(a)) = \{g\}$. If $\mathrm{rightmost}(a_2) = x_2$, then $\widehat{\sigma}_{a,b}[a] = a$ and $\widehat{\sigma}_{a,b}[b] = b$. Hence, $\sigma_{a,b}$ is idempotent. If $\mathrm{leftmost}(a_2) = x_1$, then $\widehat{\sigma}^{2}_{a,b}[a] = \widehat{\sigma}_{a,b}[a]$ and $\widehat{\sigma}_{a,b}[b] = b$. Hence, the order of $\sigma_{a,b}$ is equal to 2. □

Proposition 5.2. *Let $a = g(a_1)$, $\mathrm{op}(a) > 1$, $\mathrm{op}(b) = 0$, $\mathrm{var}(a) = \{x_1, x_2\}$, and $b = x_1$. If a satisfies* (5.1.5) *or* (5.1.6), *then, the order of $\sigma_{a,b}$ is less than or equal to 2.*

Proof. Assume that a and b satisfy (5.1.5). Then, $\mathrm{firstops}(a_1) = g$ and $\mathrm{ops}(a_1) = \{g\}$. If $\mathrm{leftmost}(a_1) = x_1$, then $\widehat{\sigma}_{a,b}[a] = b$ and $\widehat{\sigma}_{a,b}[b] = b$. Hence, the order of $\sigma_{a,b}$ is equal to 2. If $\mathrm{leftmost}(a_1) = x_2$, then $\widehat{\sigma}^{2}_{a,b}[a] = \widehat{\sigma}_{a,b}[a]$ and $\widehat{\sigma}_{a,b}[b] = b$. Again, the order of $\sigma_{a,b}$ is equal to 2.

Assume that a and b satisfy (5.1.6). Then $\mathrm{firstops}(a_1) = f$ and $f \in \mathrm{ops}(a_1)$. If $\mathrm{leftmost}(a_1) = x_1$, then $\widehat{\sigma}_{a,b}[a] = a$ and $\widehat{\sigma}_{a,b}[b] = b$. Hence, $\sigma_{a,b}$ is idempotent. If $\mathrm{leftmost}(a_1) = x_2$, then $\widehat{\sigma}^{2}_{a,b}[a] = \widehat{\sigma}_{a,b}[a]$ and $\widehat{\sigma}_{a,b}[b] = b$. Hence, the order of $\sigma_{a,b}$ is equal to 2. □

In Case (V-2), we have $\text{var}(a) = \{x_1\}$, $\text{var}(b) = \{x_1\}$. We separate this case into following subcases:

(5.2.1) $\text{firstops}(a) = f$, $a_1 = x_1$;

(5.2.2) $\text{firstops}(a) = f$, $a_1 \notin X_2$, $\text{ops}(\text{Lp}(a)) = \{g\}$;

(5.2.3) $\text{firstops}(a) = f$, $a_1 \notin X_2$, $\text{ops}(\text{Lp}(a)) \neq \{g\}$;

(5.2.4) $\text{firstops}(a) = g$, $\text{ops}(a_1) = \{g\}$;

(5.2.5) $\text{firstops}(a) = g$, $\text{ops}(a_1) \neq \{g\}$.

Proposition 5.3. *Let* $a = f(a_1, a_2)$, $\text{op}(a) > 1$, $\text{op}(b) = 0$, $\text{var}(a) = \{x_1\}$, *and* $b = x_1$. *Then, the following hold:*

(i) *if a satisfies* (5.2.1) *or* (5.2.2), *then the order of* $\sigma_{a,b}$ *is equal to 1;*

(ii) *if a satisfies* (5.2.3), *then the order of* $\sigma_{a,b}$ *is infinite.*

Proof. (i) Assume that a and b satisfy (5.2.1). Since $\text{firstops}(a_1) = f$ and $\text{var}(a) = \{x_1\}$, we have $\hat{\sigma}_{a,b}[a] = a$. Since $b = x_1$, $\hat{\sigma}_{a,b}[b] = b$. Hence, $\sigma_{a,b}$ is idempotent.

Assume a and b satisfy (5.2.2). Since $\text{ops}(a_1) = g$ and $\text{var}(a) = \{x_1\}$, we have $\hat{\sigma}_{a,b}[a] = a$ and $\hat{\sigma}_{a,b}[b] = b$. Hence, $\sigma_{a,b}$ is idempotent.

(ii) Assume a and b satisfy (5.2.3). Since $\text{firstops}(a_1) = f$ and $\text{var}(a) = \{x_1\}$, we have $\text{op}(\hat{\sigma}^{k+1}_{a,b}[a_1]) \geq \text{op}(\hat{\sigma}^{k}_{a,b}[a_1])$ for all $k \in \mathbb{N}$. For any natural number n, $\text{op}(\hat{\sigma}^{n+1}_{a,b}[a]) > \text{op}(\hat{\sigma}^{n}_{a,b}[a])$ makes $\sigma_{a,b}$ have infinite order. $\square$

Proposition 5.4. *Let* $a = g(a_1)$, $\text{op}(a) > 1$, $\text{op}(b) = 0$, $\text{var}(a) = \{x_1\}$, *and* $b = x_1$. *Then, the following hold:*

(i) *if a satisfies* (5.2.4), *then the order of* $\sigma_{a,b}$ *is equal to 2;*

(ii) *if a satisfies* (5.2.5), *then the order of* $\sigma_{a,b}$ *is equal to 1 or is infinite.*

Proof. (i) Assume a and b satisfy (5.2.4). Then, $\hat{\sigma}_{a,b}[a] = b = \hat{\sigma}_{a,b}[b]$. Hence, the order of $\sigma_{a,b}$ is equal to 2.

(ii) Assume a and b satisfy (5.2.5). If there is only one $f \in \text{ops}(a_1)$, then $\hat{\sigma}_{a,b}[a] = x_1(x_1(\cdots(a))) = a$ and $\hat{\sigma}_{a,b}[b] = b$. Hence, $\sigma_{a,b}$ is idempotent.

Assume that the symbol f occurs more than twice in term a_1. Then, $\text{op}(\hat{\sigma}^{k+1}_{a,b}[a_1]) \geq \text{op}(\hat{\sigma}^{k}_{a,b}[a_1])$ for all $k \in \mathbb{N}$. For any natural number n, $\text{op}(\hat{\sigma}^{n+1}_{a,b}[a]) > \text{op}(\hat{\sigma}^{n}_{a,b}[a])$ makes $\sigma_{a,b}$ have infinite order. $\square$

In Case (V-3), we have $\text{var}(a) = \{x_2\}$, $b = x_1$. We separate into the following subcases:

(5.3.1) $\text{firstops}(a) = f$, $a_2 = x_2$;

(5.3.2) $\text{firstops}(a) = f$, $a_2 \notin X_2$, $\text{ops}(a_2) = \{g\}$;

(5.3.3) $\text{firstops}(a) = f$, $a_2 \notin X_2$, $\text{ops}(a_2) \neq \{g\}$;

(5.3.4) $\text{firstops}(a) = g$, $\text{ops}(a_1) = \{g\}$;

(5.3.5) $\text{firstops}(a) = g$, $\text{ops}(a_1) \neq \{g\}$.

Proposition 5.5. *Let* $a = f(a_1, a_2)$, $\mathrm{op}(a) > 1$, $\mathrm{op}(b) = 0$, $\mathrm{var}(a) = \{x_2\}$, *and* $b = x_1$. *Then, the following hold:*

(i) *if a satisfies* (5.3.1) *or* (5.3.2), *then the order of $\sigma_{a,b}$ is equal to 1;*

(ii) *if a satisfies* (5.3.3), *then the order of $\sigma_{a,b}$ is infinite.*

Proof. This can be proved similarly to the proof of Proposition 5.3. □

Proposition 5.6. *Let* $a = g(a_1)$, $\mathrm{op}(a) > 1$, $\mathrm{op}(b) = 0$, $\mathrm{var}(a) = \{x_2\}$, *and* $b = x_1$. *Then, the following hold:*

(i) *if a satisfies* (5.3.4), *then the order of $\sigma_{a,b}$ is equal to 2;*

(ii) *if a satisfies* (5.3.5), *then the order of $\sigma_{a,b}$ is equal to 1 or is infinite.*

Proof. This can be proved similarly to the proof of Proposition 5.4. □

6. Case VI: $\mathrm{op}(a) = 0$ and $\mathrm{op}(b) > 1$

In this case, we consider the following cases:

(6.1) $a = x_1$;

(6.1.1) $\mathrm{firstops}(b) = f$, $b_1 = x_1$;
(6.1.2) $\mathrm{firstops}(b) = f$, $b_1 \notin X_1$, $\mathrm{ops}(\mathrm{Lp}(b)) = \{f\}$;
(6.1.3) $\mathrm{firstops}(b) = f$, $b_1 \notin X_1$, $\mathrm{ops}(\mathrm{Lp}(b)) \neq \{f\}$;
(6.1.4) $\mathrm{firstops}(b) = g$, $\mathrm{ops}(b_1) = \{f\}$;
(6.1.5) $\mathrm{firstops}(b) = g$, $\mathrm{ops}(b_1) \neq \{f\}$;

(6.2) $a = x_2$;

(6.2.1) $\mathrm{firstops}(b) = f$, $b_2 = x_1$;
(6.2.2) $\mathrm{firstops}(b) = f$, $b_2 \notin X_1$, $\mathrm{ops}(\mathrm{Rp}(b)) = \{f\}$;
(6.2.3) $\mathrm{firstops}(b) = f$, $b_2 \notin X_1$, $\mathrm{ops}(\mathrm{Rp}(b)) \neq \{f\}$;
(6.2.4) $\mathrm{firstops}(b) = g$, $\mathrm{ops}(b_1) = \{f\}$;
(6.2.5) $\mathrm{firstops}(b) = g$, $\mathrm{ops}(b_1) \neq \{f\}$.

Proposition 6.1. *Let* $a = x_1$, $b = f(b_1, b_2)$, $\mathrm{op}(b) > 1$, *and* $\mathrm{var}(b) = \{x_1\}$. *Then, the following hold:*

(i) *if b satisfies* (6.1.1) *or* (6.1.2), *then the order of $\sigma_{a,b}$ is equal to 2;*

(ii) *if b satisfies* (6.1.3), *then the order of $\sigma_{a,b}$ is equal to 1 or is infinite.*

Proof. (i) If a and b satisfy (6.1.1) or (6.1.2), then $\widehat{\sigma}_{a,b}[a] = a$ and $\widehat{\sigma}_{a,b}[b] = a$. Hence, the order of $\sigma_{a,b}$ is equal to 2.

(ii) Assume a and b satisfy (6.1.3). If there is only one occurrence of g in b_1, then $\widehat{\sigma}_{a,b}[a] = a$ and $\widehat{\sigma}_{a,b}[b] = b$. Hence, $\sigma_{a,b}$ is idempotent.

Assume that g occurs more than twice in b_1. Then, $\operatorname{op}(\widehat{\sigma}_{a,b}^{k+1}[b_1]) \geq \operatorname{op}(\widehat{\sigma}_{a,b}^{k}[b_1])$ for all $k \in \mathbb{N}$. Then,

$$\begin{aligned}
\operatorname{op}\left(\widehat{\sigma}_{a,b}^{k+1}[b]\right) &= \operatorname{op}\left(\widehat{\sigma}_{a,b}^{k}[\widehat{\sigma}_{a,b}[b]]\right) \\
&= \operatorname{op}\left(\widehat{\sigma}_{a,b}^{k}[\sigma_{a,b}(f)(\widehat{\sigma}_{a,b}[b_1],\widehat{\sigma}_{a,b}[b_2])]\right) \\
&= \operatorname{op}\left(\sigma_{a,b}^{k}(\sigma_{a,b}(f))\left(\widehat{\sigma}_{a,b}^{k+1}[b_1],\widehat{\sigma}_{a,b}^{k+1}[b_2]\right)\right) \\
&> \operatorname{op}\left(\sigma_{a,b}^{k}(f)\left(\widehat{\sigma}_{a,b}^{k}[b_1],\widehat{\sigma}_{a,b}^{k}[b_2]\right)\right) \\
&= \operatorname{op}\left(\widehat{\sigma}_{a,b}^{k}[b]\right).
\end{aligned} \tag{6.1}$$

This shows that the order of $\sigma_{a,b}$ is infinite. □

Proposition 6.2. *Let* $a = x_1$, $b = g(b_1)$, $\operatorname{op}(b) > 1$, *and* $\operatorname{var}(b) = \{x_1\}$. *Then, the following hold:*

(i) *if b satisfies* (6.1.4), *then the order of $\sigma_{a,b}$ is equal to 1;*

(ii) *if b satisfies* (6.1.5), *then the order of $\sigma_{a,b}$ is infinite.*

Proof. (i) If a and b satisfy (6.1.4), then $\widehat{\sigma}_{a,b}[a] = a$ and $\widehat{\sigma}_{a,b}[b] = b$. Hence, $\sigma_{a,b}$ is idempotent.

(ii) Assume a and b satisfy (6.1.5). Then, $\operatorname{op}(\widehat{\sigma}_{a,b}^{k+1}[b_1]) \geq \operatorname{op}(\widehat{\sigma}_{a,b}^{k}[b_1])$ for all $k \in \mathbb{N}$. Then,

$$\begin{aligned}
\operatorname{op}\left(\widehat{\sigma}_{a,b}^{k+1}[b]\right) &= \operatorname{op}\left(\widehat{\sigma}_{a,b}^{k}[\widehat{\sigma}_{a,b}[b]]\right) \\
&= \operatorname{op}\left(\widehat{\sigma}_{a,b}^{k}[\sigma_{a,b}(g)(\widehat{\sigma}_{a,b}[b_1])]\right) \\
&= \operatorname{op}\left(\sigma_{a,b}^{k}(\sigma_{a,b}(g))\left(\widehat{\sigma}_{a,b}^{k+1}[b_1]\right)\right) \\
&> \operatorname{op}\left(\sigma_{a,b}^{k}(g)\left(\widehat{\sigma}_{a,b}^{k}[b_1]\right)\right) \\
&= \operatorname{op}\left(\widehat{\sigma}_{a,b}^{k}[b]\right).
\end{aligned} \tag{6.2}$$

This shows that the order of $\sigma_{a,b}$ is infinite. □

Proposition 6.3. *Let* $a = x_2$, $b = f(b_1,b_2)$, $\operatorname{op}(b) > 1$, *and* $\operatorname{var}(b) = \{x_1\}$. *Then, the following hold:*

(i) *if b satisfies* (6.2.1) *or* (6.2.2), *then the order of $\sigma_{a,b}$ is equal to 2;*

(ii) *if b satisfies* (6.2.3), *then the order of $\sigma_{a,b}$ is equal to 1 or is infinite.*

Proof. (i) If a and b satisfy (6.2.1) or (6.2.2), then $\widehat{\sigma}_{a,b}[a] = a$ and $\widehat{\sigma}_{a,b}^{2}[b] = \widehat{\sigma}_{a,b}[b]$. Hence the order of $\sigma_{a,b}$ is equal to 2.

(ii) This case can be proved similarly to the proof of Proposition 6.1(ii). □

Proposition 6.4. *Let* $a = x_2$, $b = g(b_1)$, $\mathrm{op}(b) > 1$, *and* $\mathrm{var}(b) = \{x_1\}$. *Then, the following hold:*

(i) *if b satisfies* (6.2.4), *then the order of* $\sigma_{a,b}$ *is equal to 1;*

(ii) *if b satisfies* (6.2.5), *then the order of* $\sigma_{a,b}$ *is infinite.*

Proof. The proofs follow those of Proposition 6.2 (i) and (ii), respectively. □

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