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Research Article

Strong Convergence Theorems of the General Iterative Methods for Nonexpansive Semigroups in Banach Spaces

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Let E be a real reflexive Banach space which admits a weakly sequentially continuous duality mapping from E to E^*. Let $\mathcal{S} = \{T(s) : 0 \leq s < \infty\}$ be a nonexpansive semigroup on E such that $\mathrm{Fix}(\mathcal{S}) := \bigcap_{t\geq 0}\mathrm{Fix}(T(t)) \neq \emptyset$, and f is a contraction on E with coefficient $0 < \alpha < 1$. Let F be δ-strongly accretive and λ-strictly pseudocontractive with $\delta + \lambda > 1$ and γ a positive real number such that $\gamma < 1/\alpha(1 - \sqrt{1-\delta/\lambda})$. When the sequences of real numbers $\{\alpha_n\}$ and $\{t_n\}$ satisfy some appropriate conditions, the three iterative processes given as follows: $x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n F)T(t_n)x_n$, $n \geq 0$, $y_{n+1} = \alpha_n \gamma f(T(t_n)y_n) + (I - \alpha_n F)T(t_n)y_n$, $n \geq 0$, and $z_{n+1} = T(t_n)(\alpha_n \gamma f(z_n) + (I - \alpha_n F)z_n)$, $n \geq 0$ converge strongly to $\tilde{x}$, where $\tilde{x}$ is the unique solution in $\mathrm{Fix}(\mathcal{S})$ of the variational inequality $\langle (F - \gamma f)\tilde{x}, j(x - \tilde{x})\rangle \geq 0$, $x \in \mathrm{Fix}(\mathcal{S})$. Our results extend and improve corresponding ones of Li et al. (2009) Chen and He (2007), and many others.

1. Introduction

Let E be a real Banach space. A mapping T of E into itself is said to be *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for each $x, y \in E$. We denote by $\mathrm{Fix}(T)$ the set of fixed points of T. A mapping $f : E \to E$ is called α-contraction if there exists a constant $0 < \alpha < 1$ such that $\|f(x) - f(y)\| \leq \alpha\|x - y\|$ for all $x, y \in E$. A family $\mathcal{S} = \{T(t) : 0 \leq t < \infty\}$ of mappings of E into itself is called a *nonexpansive semigroup* on E if it satisfies the following conditions:

(i) $T(0)x = x$ for all $x \in E$;

(ii) $T(s + t) = T(s)T(t)$ for all $s, t \geq 0$;

(iii) $\|T(t)x - T(t)y\| \leq \|x - y\|$ for all $x, y \in E$ and $t \geq 0$;

(iv) for all $x \in E$, the mapping $t \mapsto T(t)x$ is continuous.

We denote by $\mathrm{Fix}(\mathcal{S})$ the set of all common fixed points of $\mathcal{S}$, that is,

$$\mathrm{Fix}(\mathcal{S}) := \{x \in E : T(t)x = x, 0 \leq t < \infty\} = \bigcap_{t \geq 0} \mathrm{Fix}(T(t)). \tag{1.1}$$

In [1], Shioji and Takahashi introduced the following implicit iteration in a Hilbert space

$$x_n = \alpha_n x + (1 - \alpha_n) \frac{1}{t_n} \int_0^{t_n} T(s) x_n ds, \quad \forall n \in \mathbb{N}, \tag{1.2}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{t_n\}$ is a sequence of positive real numbers which diverges to ∞. Under certain restrictions on the sequence $\{\alpha_n\}$, Shioji and Takahashi [1] proved strong convergence of the sequence $\{x_n\}$ to a member of $F(\mathcal{S})$. In [2], Shimizu and Takahashi studied the strong convergence of the sequence $\{x_n\}$ defined by

$$x_{n+1} = \alpha_n x + (1 - \alpha_n) \frac{1}{t_n} \int_0^{t_n} T(s) x_n ds, \quad \forall n \in \mathbb{N} \tag{1.3}$$

in a real Hilbert space where $\{T(t) : t \geq 0\}$ is a strongly continuous semigroup of nonexpansive mappings on a closed convex subset C of a Banach space E and $\lim_{n \to \infty} t_n = \infty$. Using viscosity method, Chen and Song [3] studied the strong convergence of the following iterative method for a nonexpansive semigroup $\{T(t) : t \geq 0\}$ with $\mathrm{Fix}(\mathcal{S}) \neq \emptyset$ in a Banach space:

$$x_{n+1} = \alpha_n f(x) + (1 - \alpha_n) \frac{1}{t_n} \int_0^{t_n} T(s) x_n ds, \quad \forall n \in \mathbb{N}, \tag{1.4}$$

where f is a contraction. Note however that their iterate x_n at step n is constructed through the average of the semigroup over the interval $(0, t)$. Suzuki [4] was the first to introduce again in a Hilbert space the following implicit iteration process:

$$x_n = \alpha_n u + (1 - \alpha_n) T(t_n) x_n, \quad \forall n \in \mathbb{N}, \tag{1.5}$$

for the nonexpansive semigroup case. In 2002, Benavides et al. [5], in a uniformly smooth Banach space, showed that if $\mathcal{S}$ satisfies an asymptotic regularity condition and $\{\alpha_n\}$ fulfills the control conditions $\lim_{n \to \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, and $\lim_{n \to \infty} \alpha_n / \alpha_{n+1} = 0$, then both the implicit iteration process (1.5) and the explicit iteration process (1.6),

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) T(t_n) x_n, \quad \forall n \in \mathbb{N}, \tag{1.6}$$

converge to a same point of $F(\mathcal{S})$. In 2005, Xu [6] studied the strong convergence of the implicit iteration process (1.2) and (1.5) in a uniformly convex Banach space which admits a weakly sequentially continuous duality mapping. Recently, Chen and He [7] introduced the viscosity approximation process:

$$x_{n+1} = \alpha_n f(x_n) + (1-\beta_n)T(t_n)x_n, \quad \forall n \in \mathbb{N}, \tag{1.7}$$

where f is a contraction and $\{\alpha_n\}$ is a sequence in $(0,1)$ and a nonexpansive semigroup $\{T(t) : t \geq 0\}$. The strong convergence theorem of $\{x_n\}$ is proved in a reflexive Banach space which admits a weakly sequentially continuous duality mapping. In [8], Chen et al. introduced and studied modified Mann iteration for nonexpansive mapping in a uniformly convex Banach space.

On the other hand, iterative approximation methods for nonexpansive mappings have recently been applied to solve convex minimization problems; see, for example, [9–11] and the references therein. Let H be a real Hilbert space, whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. Let A be a strongly positive bounded linear operator on H; that is, there is a constant $\overline{\gamma} > 0$ with property

$$\langle Ax, x \rangle \geq \overline{\gamma} \|x\|^2 \quad \forall x \in H. \tag{1.8}$$

A typical problem is to minimize a quadratic function over the set of the fixed points of a nonexpansive mapping on a real Hilbert space H:

$$\min_{x \in C} \frac{1}{2} \langle Ax, x \rangle - \langle x, b \rangle, \tag{1.9}$$

where C is the fixed point set of a nonexpansive mapping T on H and b is a given point in H. In 2003, Xu [10] proved that the sequence $\{x_n\}$ defined by the iterative method below, with the initial guess $x_0 \in H$ chosen arbitrarily,

$$x_{n+1} = (I - \alpha_n A)Tx_n + \alpha_n u, \quad n \geq 0, \tag{1.10}$$

converges strongly to the unique solution of the minimization problem (1.9) provided the sequence $\{\alpha_n\}$ satisfies certain conditions. Using the viscosity approximation method, Moudafi [12] introduced the following iterative process for nonexpansive mappings (see [13] for further developments in both Hilbert and Banach spaces). Let f be a contraction on H. Starting with an arbitrary initial $x_0 \in H$, define a sequence $\{x_n\}$ recursively by

$$x_{n+1} = (1 - \alpha_n)Tx_n + \alpha_n f(x_n), \quad n \geq 0, \tag{1.11}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$. It is proved [12, 13] that, under certain appropriate conditions imposed on $\{\alpha_n\}$, the sequence $\{x_n\}$ generated by (1.11) strongly converges to the unique solution x^* in C of the variational inequality

$$\langle (I - f)x^*, x - x^* \rangle \geq 0, \quad x \in H. \tag{1.12}$$

Recently, Marino and Xu [14] mixed the iterative method (1.10) and the viscosity approximation method (1.11) and considered the following general iterative method:

$$x_{n+1} = (I - \alpha_n A)Tx_n + \alpha_n \gamma f(x_n), \quad n \geq 0, \tag{1.13}$$

where A is a strongly positive bounded linear operator on H. They proved that if the sequence $\{\alpha_n\}$ of parameters satisfies the certain conditions, then the sequence $\{x_n\}$ generated by (1.13) converges strongly to the unique solution x^* in H of the variational inequality

$$\langle (A - \gamma f)x^*, x - x^* \rangle \geq 0, \quad x \in H \tag{1.14}$$

which is the optimality condition for the minimization problem, $\min_{x\in C}(1/2)\langle Ax, x\rangle - h(x)$, where h is a potential function for γf (i.e., $h'(x) = \gamma f(x)$ for $x \in H$).

Very recently, Li et al. [15] introduced the following iterative procedures for the approximation of common fixed points of a one-parameter nonexpansive semigroup on a Hilbert space H:

$$x_0 = x \in H, \quad x_{n+1} = (I - \alpha_n A)\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds + \alpha_n \gamma f(x_n), \quad n \geq 0, \tag{1.15}$$

where A is a strongly positive bounded linear operator on H.

Let δ and λ be two positive real numbers such that $\delta, \lambda < 1$. Recall that a mapping F with domain $D(F)$ and range $R(F)$ in E is called *δ-strongly accretive* if, for each $x, y \in D(F)$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Fx - Fy, j(x - y)\rangle \geq \delta \|x - y\|^2, \tag{1.16}$$

where J is the normalized duality mapping from E into the dual space E^*. Recall also that a mapping F is called *λ-strictly pseudocontractive* if, for each $x, y \in D(F)$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Fx - Fy, j(x - y)\rangle \leq \|x - y\|^2 - \lambda \|(x - y) - (Fx - Fy)\|^2. \tag{1.17}$$

It is easy to see that (1.17) can be rewritten as

$$\langle (I - F)x - (I - F)y, j(x - y)\rangle \geq \lambda \|(I - F)x - (I - F)y\|^2, \tag{1.18}$$

see [16].

In this paper, motivated by the above results, we introduce and study the strong convergence theorems of the general iterative scheme $\{x_n\}$ defined by (1.19) in the framework of a reflexive Banach space E which admits a weakly sequentially continuous duality mapping:

$$x_0 = x \in E, \quad x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n F)T(t_n)x_n, \quad n \geq 0, \tag{1.19}$$

where F is δ-strongly accretive and λ-strictly pseudocontractive with $\delta + \lambda > 1$, f is a contraction on E with coefficient $0 < \alpha < 1$, γ is a positive real number such that $\gamma < (1/\alpha)(1 - \sqrt{(1-\delta)/\lambda})$, and $\mathcal{S} = \{T(t) : 0 \le t < \infty\}$ is a nonexpansive semigroup on E. The strong convergence theorems are proved under some appropriate control conditions on parameters $\{\alpha_n\}$ and $\{t_n\}$. Furthermore, by using these results, we obtain strong convergence theorems of the following new general iterative schemes $\{y_n\}$ and $\{z_n\}$ defined by

$$y_0 = y \in E, \quad y_{n+1} = \alpha_n \gamma f\big(T(t_n)y_n\big) + (I - \alpha_n F)T(t_n)y_n, \quad n \ge 0, \tag{1.20}$$

$$z_0 = z \in E, \quad z_{n+1} = T(t_n)\big(\alpha_n \gamma f(z_n) + (I - \alpha_n F)z_n\big), \quad n \ge 0. \tag{1.21}$$

The results presented in this paper extend and improve the main results in Li et al. [15], Chen and He [7], and many others.

2. Preliminaries

Throughout this paper, it is assumed that E is a real Banach space with norm $\|\cdot\|$ and let J denote the normalized duality mapping from E into E^* given by

$$J(x) = \left\{ f \in E^* : \langle x, f \rangle = \|x\|^2 = \|f\|^2 \right\} \tag{2.1}$$

for each $x \in E$, where E^* denotes the dual space of E, $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing, and $\mathbb{N}$ denotes the set of all positive integers. In the sequel, we will denote the single-valued duality mapping by j, and consider $F(T) = \{x \in C : Tx = x\}$. When $\{x_n\}$ is a sequence in E, then $x_n \to x$ (resp., $x_n \rightharpoonup x$, $x_n \overset{*}{\rightharpoonup} x$) will denote strong (resp., weak, weak*) convergence of the sequence $\{x_n\}$ to x. In a Banach space E, the following result (*the subdifferential inequality*) is well known [17, Theorem 4.2.1]: for all $x, y \in E$, for all $j(x+y) \in J(x+y)$, for all $j(x) \in J(x)$,

$$\|x\|^2 + 2\langle y, j(x) \rangle \le \|x + y\|^2 \le \|x\|^2 + \langle y, j(x+y) \rangle. \tag{2.2}$$

A real Banach space E is said to be *strictly convex* if $\|x + y\|/2 < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \ne y$. It is said to be *uniformly convex* if, for all $\epsilon \in [0, 2]$, there exits $\delta_\epsilon > 0$ such that

$$\|x\| = \|y\| = 1 \quad \text{with } \|x - y\| \ge \epsilon \quad \text{implies } \frac{\|x + y\|}{2} < 1 - \delta_\epsilon. \tag{2.3}$$

The following results are well known and can be founded in [17]:

(i) a uniformly convex Banach space E is reflexive and strictly convex [17, Theorems 4.2.1 and 4.1.6],

(ii) if E is a strictly convex Banach space and $T : E \to E$ is a nonexpansive mapping, then fixed point set $F(T)$ of T is a closed convex subset of E [17, Theorem 4.5.3].

If a Banach space E admits a sequentially continuous duality mapping J from weak topology to weak star topology, then from Lemma 1 of [18], it follows that the duality mapping J is single-valued and also E is smooth. In this case, duality mapping J is also said to be *weakly sequentially continuous*, that is, for each $\{x_n\} \subset E$ with $x_n \rightharpoonup x$, then $J(x_n) \stackrel{*}{\rightharpoonup} J(x)$ (see [18, 19]).

In the sequel, we will denote the single-valued duality mapping by j. A Banach space E is said to satisfy *Opial's condition* if, for any sequence $\{x_n\}$ in E, $x_n \rightharpoonup x$ as $n \to \infty$ implies

$$\limsup_{n \to \infty} \|x_n - x\| < \limsup_{n \to \infty} \|x_n - y\| \quad \forall y \in E \text{ with } x \neq y. \tag{2.4}$$

By Theorem 1 of [18], we know that if E admits a weakly sequentially continuous duality mapping, then E satisfies Opial's condition and E is smooth; for the details, see [18].

Now, we present the concept of uniformly asymptotically regular semigroup (also see [20, 21]). Let C be a nonempty closed convex subset of a Banach space E, $\mathcal{S} = \{T(t) : 0 \leq t < \infty\}$ a continuous operator semigroup on C. Then, $\mathcal{S}$ is said to be *uniformly asymptotically regular* (in short, u.a.r.) on C if, for all $h \geq 0$ and any bounded subset D of C,

$$\lim_{t \to \infty} \sup_{x \in D} \|T(h)(T(t)x) - T(t)x\| = 0. \tag{2.5}$$

The nonexpansive semigroup $\{\sigma_t : t > 0\}$ defined by the following lemma is an example of u.a.r. operator semigroup. Other examples of u.a.r. operator semigroup can be found in [20, Examples 17 and 18].

Lemma 2.1 (see [3, Lemma 2.7]). *Let C be a nonempty closed convex subset of a uniformly convex Banach space E, D a bounded closed convex subset of C, and $\mathcal{S} = \{T(s) : 0 \leq s < \infty\}$ a nonexpansive semigroup on C such that $F(\mathcal{S}) \neq \emptyset$. For each $h > 0$, set $\sigma_t(x) = (1/t)\int_0^t T(s)x\,ds$, then*

$$\lim_{t \to \infty} \sup_{x \in D} \|\sigma_t(x) - T(h)\sigma_t(x)\| = 0. \tag{2.6}$$

Example 2.2. The set $\{\sigma_t : t > 0\}$ defined by **Lemma 2.1** is u.a.r. nonexpansive semigroup. In fact, it is obvious that $\{\sigma_t : t > 0\}$ is a nonexpansive semigroup. For each $h > 0$, we have

$$\begin{aligned} \|\sigma_t(x) - \sigma_h\sigma_t(x)\| &= \left\|\sigma_t(x) - \frac{1}{h}\int_0^h T(s)\sigma_t(x)\,ds\right\| \\ &= \left\|\frac{1}{h}\int_0^h (\sigma_t(x) - T(s)\sigma_t(x))\,ds\right\| \\ &\leq \frac{1}{h}\int_0^h \|\sigma_t(x) - T(s)\sigma_t(x)\|\,ds. \end{aligned} \tag{2.7}$$

Applying Lemma 2.1, we have

$$\lim_{t\to\infty}\sup_{x\in D}\|\sigma_t(x)-\sigma_h\sigma_t(x)\|\le\frac{1}{h}\int_0^h\lim_{t\to\infty}\sup_{x\in D}\|\sigma_t(x)-T(s)\sigma_t(x)\|ds=0. \tag{2.8}$$

Let C be a nonempty closed and convex subset of a Banach space E and D a nonempty subset of C. A mapping $Q: C\to D$ is said to be sunny if

$$Q(Qx+t(x-Qx))=Qx, \tag{2.9}$$

whenever $Qx+t(x-Qx)\in C$ for $x\in C$ and $t=0$. A mapping $Q: C\to D$ is called a retraction if $Qx=x$ for all $x\in D$. Furthermore, Q is a sunny nonexpansive retraction from C onto D if Q is a retraction from C onto D which is also sunny and nonexpansive. A subset D of C is called a sunny nonexpansive retraction of C if there exists a sunny nonexpansive retraction from C onto D. The following lemma concerns the sunny nonexpansive retraction.

Lemma 2.3 (see [22, 23]). *Let C be a closed convex subset of a smooth Banach space E. Let D be a nonempty subset of C and $Q: C\to D$ be a retraction. Then, Q is sunny and nonexpansive if and only if*

$$\langle u-Qu, j(y-Qu)\rangle\le 0 \tag{2.10}$$

for all $u\in C$ and $y\in D$.

Lemma 2.4 (see [24, Lemma 2.3]). *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the property*

$$a_{n+1}\le(1-t_n)a_n+t_nc_n+b_n,\quad \forall n\ge 0, \tag{2.11}$$

where $\{t_n\},\{b_n\}$, and $\{c_n\}$ satisfy the restrictions

(i) $\sum_{n=1}^{\infty}t_n=\infty$;

(ii) $\sum_{n=1}^{\infty}b_n<\infty$;

(iii) $\limsup_{n\to\infty}c_n\le 0$.

Then, $\lim_{n\to\infty}a_n=0$.

The following lemma will be frequently used throughout the paper and can be found in [25].

Lemma 2.5 (see [25, Lemma 2.7]). *Let E be a real smooth Banach space and $F: E\to E$ a mapping.*

(i) *If F is δ-strongly accretive and λ-strictly pseudocontractive with $\delta+\lambda>1$, then $I-F$ is contractive with constant $\sqrt{(1-\delta)/\lambda}$.*

(i) *If F is δ-strongly accretive and λ-strictly pseudocontractive with $\delta+\lambda>1$, then, for any fixed number $\tau\in(0,1)$, $I-\tau F$ is contractive with constant $1-\tau(1-\sqrt{(1-\delta)/\lambda})$.*

3. Main Results

Now, we are in a position to state and prove our main results.

Theorem 3.1. *Let E be a reflexive Banach space which admits a weakly sequentially continuous duality mapping J. Let $\mathcal{S} = \{T(t) : 0 \le t < \infty\}$ be a u.a.r. nonexpansive semigroup on E such that* $\mathrm{Fix}(\mathcal{S}) \neq \emptyset$. *Suppose that the real sequences $\{\alpha_n\} \subset [0,1]$, $\{t_n\} \subset (0,\infty)$ satisfy the conditions*

$$\lim_{n\to\infty} \alpha_n = 0, \qquad \sum_{n=0}^{\infty} \alpha_n = \infty, \qquad \lim_{n\to\infty} t_n = \infty. \tag{3.1}$$

Let F be δ-strongly accretive and λ-strictly pseudocontractive with $\delta + \lambda > 1$, $f : E \to E$ a contraction mapping with coefficient $\alpha \in (0,1)$, and γ a positive real number such that $\gamma < (1/\alpha)(1 - \sqrt{(1-\delta)/\lambda})$. Then, the sequence $\{x_n\}$ defined by (1.19) *converges strongly to $\tilde{x}$, where $\tilde{x}$ is the unique solution in* $\mathrm{Fix}(\mathcal{S})$ *of the variational inequality*

$$\left\langle (F - \gamma f)\tilde{x}, j(x - \tilde{x}) \right\rangle \ge 0, \quad x \in \mathrm{Fix}(\mathcal{S}) \tag{3.2}$$

or equivalently $\tilde{x} = Q_{\mathrm{Fix}(\mathcal{S})}(I - F + \gamma f)\tilde{x}$, where $Q_{\mathrm{Fix}(\mathcal{S})}$ is the sunny nonexpansive retraction of E onto $\mathrm{Fix}(\mathcal{S})$.

Proof. Note that $\mathrm{Fix}(\mathcal{S})$ is a nonempty closed convex set. We first show that $\{x_n\}$ is bounded. Let $q \in \mathrm{Fix}(\mathcal{S})$. Thus, by Lemma 2.5, we have

$$\begin{aligned}
\|x_{n+1} - q\| &= \|\alpha_n \gamma f(x_n) + (I - \alpha_n F)T(t_n)x_n - (I - \alpha_n F)q - \alpha_n Fq\| \\
&\le \alpha_n \|\gamma f(x_n) - Fq\| + \|I - \alpha_n F\| \|T(t_n)x_n - q\| \\
&\le \alpha_n \gamma \|f(x_n) - f(q)\| + \alpha_n \|\gamma f(q) - Fq\| + \|I - \alpha_n F\| \|x_n - q\| \\
&\le \alpha_n \alpha\gamma \|x_n - q\| + \alpha_n \|\gamma f(q) - Fq\| \\
&\quad + \left(1 - \alpha_n\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)\right)\|x_n - q\| \\
&= \left(1 - \alpha_n\left(1 - \sqrt{\frac{1-\delta}{\lambda}} - \alpha\gamma\right)\right)\|x_n - q\| \\
&\quad + \alpha_n\left(1 - \sqrt{\frac{1-\delta}{\lambda}} - \alpha\gamma\right)\frac{\|\gamma f(q) - Fq\|}{1 - \sqrt{(1-\delta)/\lambda} - \alpha\gamma} \\
&\le \max\left\{\|x_n - q\|, \frac{1}{1 - \sqrt{(1-\delta)/\lambda} - \alpha\gamma}\|\gamma f(q) - Fq\|\right\}, \quad \forall n \ge 0.
\end{aligned} \tag{3.3}$$

By induction, we get

$$\|x_n - q\| \le \max\left\{\|x_0 - q\|, \frac{1}{1-\sqrt{(1-\delta)/\lambda} - \alpha\gamma}\|\gamma f(q) - Fq\|\right\}, \quad n \ge 0. \tag{3.4}$$

This implies that $\{x_n\}$ is bounded and, hence, so are $\{f(x_n)\}$ and $\{FT(t_n)x_n\}$. This implies that

$$\lim_{n\to\infty}\|x_{n+1} - T(t_n)x_n\| = \lim_{n\to\infty}\alpha_n\|\gamma f(x_n) - FT(t_n)x_n\| = 0. \tag{3.5}$$

Since $\{T(t)\}$ is a u.a.r. nonexpansive semigroup and $\lim_{n\to\infty} t_n = \infty$, we have, for all $h > 0$,

$$\lim_{n\to\infty}\|T(h)(T(t_n)x_n) - T(t_n)x_n\| \le \lim_{n\to\infty}\sup_{x\in\{x_n\}}\|T(h)(T(t_n)x) - T(t_n)x\| = 0. \tag{3.6}$$

Hence, for all $h > 0$,

$$\begin{aligned}\|x_{n+1} - T(h)x_{n+1}\| &\le \|x_{n+1} - T(t_n)x_n\| + \|T(t_n)x_n - T(h)T(t_n)x_n\| + \|T(h)T(t_n)x_n - T(h)x_{n+1}\| \\ &\le 2\|x_{n+1} - T(t_n)x_n\| + \|T(t_n)x_n - T(h)T(t_n)x_n\| \longrightarrow 0.\end{aligned} \tag{3.7}$$

That is, for all $h > 0$,

$$\lim_{n\to\infty}\|x_n - T(h)x_n\| = 0. \tag{3.8}$$

Let $\Phi = Q_{\mathrm{Fix}(\mathcal{S})}$. Then, $\Phi(I - F - \gamma f)$ is a contraction on E. In fact, from Lemma 2.5(i), we have

$$\begin{aligned}\|\Phi(I-F-\gamma f)x - \Phi(I-F-\gamma f)y\| &\le \|(I-F-\gamma f)x - (I-F-\gamma f)y\| \\ &\le \|(I-F)x - (I-F)y\| + \gamma\|f(x) - f(y)\| \\ &\le \sqrt{\frac{1-\delta}{\lambda}}\|x-y\| + \alpha\gamma\|x-y\| \\ &= \left(\sqrt{\frac{1-\delta}{\lambda}} + \alpha\gamma\right)\|x-y\|, \quad \forall x, y \in E.\end{aligned} \tag{3.9}$$

Therefore, $\Phi(I-F-\gamma f)$ is a contraction on E due to $(\sqrt{(1-\delta)/\lambda}+\alpha\gamma) \in (0,1)$. Thus, by Banach contraction principle, $Q_{\mathrm{Fix}(\mathcal{S})}(I - F - \gamma f)$ has a unique fixed point $\tilde{x}$. Then, using Lemma 2.3, $\tilde{x}$ is the unique solution in Fix(S) of the variational inequality (3.2). Next, we show that

$$\limsup_{n\to\infty}\langle\gamma f(\tilde{x}) - F\tilde{x}, j(x_n - \tilde{x})\rangle \le 0. \tag{3.10}$$

Indeed, we can take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\left\langle \gamma f(\tilde{x}) - F\tilde{x}, j(x_n - \tilde{x})\right\rangle = \lim_{k\to\infty}\left\langle \gamma f(\tilde{x}) - F\tilde{x}, j(x_{n_k} - \tilde{x})\right\rangle. \tag{3.11}$$

We may assume that $x_{n_k} \rightharpoonup p \in E$ as $k \to \infty$, since a Banach space E has a weakly sequentially continuous duality mapping J satisfying Opial's condition [13]. We will prove that $p \in \mathrm{Fix}(\mathcal{S})$. Suppose the contrary, $p \notin \mathrm{Fix}(\mathcal{S})$, that is, $T(h_0)p \neq p$ for some $h_0 > 0$. It follows from (3.8) and Opial's condition that

$$\begin{aligned}
\liminf_{k\to\infty}\left\|x_{n_k} - p\right\| &< \liminf_{k\to\infty}\left\|x_{n_k} - T(h_0)p\right\| \\
&\le \liminf_{k\to\infty}\left\{\|x_{n_k} - T(h_0)x_{n_k}\| + \left\|T(h_0)x_{n_k} - T(h_0)p\right\|\right\} \\
&\le \liminf_{k\to\infty}\left\{\|x_{n_k} - T(h_0)x_{n_k}\| + \left\|x_{n_k} - p\right\|\right\} \\
&= \liminf_{k\to\infty}\left\|x_{n_k} - p\right\|.
\end{aligned} \tag{3.12}$$

This is a contradiction, which shows that $p \in F(T(h))$ for all $h > 0$, that is, $p \in \mathrm{Fix}(\mathcal{S})$. In view of the variational inequality (3.2) and the assumption that duality mapping J is weakly sequentially continuous, we conclude

$$\begin{aligned}
\limsup_{n\to\infty}\left\langle \gamma f(\tilde{x}) - F\tilde{x}, j(x_n - \tilde{x})\right\rangle &= \lim_{k\to\infty}\left\langle \gamma f(\tilde{x}) - F\tilde{x}, j(x_{n_k} - \tilde{x})\right\rangle \\
&\le \left\langle \gamma f(\tilde{x}) - F\tilde{x}, j(p - \tilde{x})\right\rangle \le 0.
\end{aligned} \tag{3.13}$$

Finally, we will show that $x_n \to \tilde{x}$. For each $n \ge 0$, we have

$$\begin{aligned}
\|x_{n+1} - \tilde{x}\|^2 &= \left\|\alpha_n\gamma f(x_n) + (I - \alpha_n F)T(t_n)x_n - (I - \alpha_n F)\tilde{x} - \alpha_n F\tilde{x}\right\|^2 \\
&\le \left\|\alpha_n\gamma f(x_n) - \alpha_n F\tilde{x} + (I - \alpha_n F)T(t_n)x_n - (I - \alpha_n F)\tilde{x}\right\|^2 \\
&= \|(I - \alpha_n F)T(t_n)x_n - (I - \alpha_n F)\tilde{x}\|^2 + 2\alpha_n\left\langle \gamma f(x_n) - F\tilde{x}, j(x_{n+1} - \tilde{x})\right\rangle \\
&\le \left(1 - \alpha_n\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)\right)^2 \|x_n - \tilde{x}\|^2 + 2\alpha_n\left\langle \gamma f(x_n) - \gamma f(\tilde{x}), j(x_{n+1} - \tilde{x})\right\rangle \\
&\quad + 2\alpha_n\left\langle \gamma f(\tilde{x}) - F\tilde{x}, j(x_{n+1} - \tilde{x})\right\rangle.
\end{aligned} \tag{3.14}$$

On the other hand,

$$
\begin{aligned}
&\langle \gamma f(x_n) - \gamma f(\tilde{x}), j(x_{n+1} - \tilde{x}) \rangle \\
&\quad \leq \gamma\alpha \|x_n - \tilde{x}\| \|x_{n+1} - \tilde{x}\| \\
&\quad \leq \gamma\alpha \|x_n - \tilde{x}\| \left[\sqrt{\left(1 - \alpha_n\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)\right)^2 \|x_n - \tilde{x}\|^2 + 2\alpha_n \left| \langle \gamma f(x_n) - F\tilde{x}, j(x_{n+1} - \tilde{x}) \rangle \right|} \right] \\
&\quad \leq \gamma\alpha \left(1 - \alpha_n\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)\right) \|x_n - \tilde{x}\|^2 \\
&\qquad + \gamma\alpha \|x_n - \tilde{x}\| \sqrt{2\left| \langle \gamma f(x_n) - F\tilde{x}, j(x_{n+1} - \tilde{x}) \rangle \right|} \sqrt{\alpha_n} \\
&\quad \leq \gamma\alpha \left(1 - \alpha_n\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)\right) \|x_n - \tilde{x}\|^2 + \sqrt{\alpha_n} M_0,
\end{aligned}
\tag{3.15}
$$

where M_0 is a constant satisfying $M_0 \geq \gamma\alpha \|x_n - \tilde{x}\| \sqrt{2|\langle \gamma f(x_n) - F\tilde{x}, j(x_{n+1} - \tilde{x}) \rangle|}$. Substituting (3.15) in (3.14), we obtain

$$
\begin{aligned}
\|x_{n+1} - \tilde{x}\|^2 \leq{}& \left(1 - \alpha_n\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)\right)^2 \|x_n - \tilde{x}\|^2 + 2\alpha_n\gamma\alpha\left(1 - \alpha_n\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)\right) \\
&\times \|x_n - \tilde{x}\|^2 + 2\alpha_n\sqrt{\alpha_n} M_0 + 2\alpha_n \langle \gamma f(\tilde{x}) - F\tilde{x}, j(x_{n+1} - \tilde{x}) \rangle \\
={}& \left(1 - 2\alpha_n\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right) + \alpha_n^2\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)^2\right) \|x_n - \tilde{x}\|^2 \\
&+ 2\alpha_n\gamma\alpha\left(1 - \alpha_n\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)\right) \|x_n - \tilde{x}\|^2 \\
&+ 2\alpha_n\sqrt{\alpha_n} M_0 + 2\alpha_n \langle \gamma f(\tilde{x}) - F\tilde{x}, j(x_{n+1} - \tilde{x}) \rangle \\
={}& \left(1 - 2\alpha_n\left[\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right) - \alpha\gamma + \alpha_n\gamma\alpha\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)\right]\right) \|x_n - \tilde{x}\|^2 \\
&+ \alpha_n\left[\alpha_n\left(1 - \sqrt{\frac{1-\delta}{\lambda}}\right)^2 \|x_n - \tilde{x}\|^2 + 2M_0\sqrt{\alpha_n} + 2\langle \gamma f(\tilde{x}) - F\tilde{x}, j(x_{n+1} - \tilde{x}) \rangle\right] \\
={}& (1 - \alpha_n\gamma_n) \|x_n - \tilde{x}\|^2 + \alpha_n\gamma_n \frac{\beta_n}{\gamma_n},
\end{aligned}
\tag{3.16}
$$

where

$$\gamma_n = 2\left[\left(1-\sqrt{\frac{1-\delta}{\lambda}}\right) - \alpha\gamma + \alpha_n\gamma\alpha\left(1-\sqrt{\frac{1-\delta}{\lambda}}\right)\right],$$
$$\beta_n = \left[\alpha_n\left(1-\sqrt{\frac{1-\delta}{\lambda}}\right)^2 \|x_n-\tilde{x}\|^2 + 2M_0\sqrt{\alpha_n} + 2\langle \gamma f(\tilde{x}) - F\tilde{x}, j(x_{n+1}-\tilde{x})\rangle\right]. \tag{3.17}$$

It is easily seen that $\sum_{n=1}^{\infty}\alpha_n\gamma_n = \infty$. Since $\{x_n\}$ is bounded and $\lim_{n\to\infty}\alpha_n = 0$, by (3.46), we obtain $\limsup_{n\to\infty}\beta_n/\gamma_n \le 0$, applying Lemma 2.4 to (3.16) to conclude $x_n \to \tilde{x}$ as $n \to \infty$. This completes the proof. □

Using Theorem 3.1, we obtain the following two strong convergence theorems of new iterative approximation methods for a nonexpansive semigroup $\{T(t) : 0 \le t < \infty\}$.

Corollary 3.2. *Let E be a reflexive Banach space which admits a weakly sequentially continuous duality mapping J. Let $\mathcal{S} = \{T(t) : 0 \le t < \infty\}$ be a u.a.r. nonexpansive semigroup on E such that* $\mathrm{Fix}(\mathcal{S}) \ne \emptyset$. *Suppose that the real sequences $\{\alpha_n\} \subset [0,1]$, $\{t_n\} \subset (0,\infty)$ satisfy the conditions*

$$\lim_{n\to\infty}\alpha_n = 0, \qquad \sum_{n=0}^{\infty}\alpha_n = \infty, \qquad \lim_{n\to\infty}t_n = \infty. \tag{3.18}$$

Let F be δ-strongly accretive and λ-strictly pseudocontractive with $\delta+\lambda > 1$, $f : E \to E$ a contraction mapping with coefficient $\alpha \in (0,1)$, and γ a positive real number such that $\gamma < (1/\alpha)(1-\sqrt{(1-\delta)/\lambda})$. Then, the sequence $\{y_n\}$ defined by (1.20) *converges strongly to $\tilde{x}$, where $\tilde{x}$ is the unique solution in* $\mathrm{Fix}(\mathcal{S})$ *of the variational inequality*

$$\langle (F-\gamma f)\tilde{x}, j(x-\tilde{x})\rangle \ge 0, \quad x \in \mathrm{Fix}(\mathcal{S}) \tag{3.19}$$

or equivalently $\tilde{x} = Q_{\mathrm{Fix}(\mathcal{S})}(I-F+\gamma f)\tilde{x}$, where $Q_{\mathrm{Fix}(\mathcal{S})}$ is the sunny nonexpansive retraction of E onto $\mathrm{Fix}(\mathcal{S})$.

Proof. Let $\{x_n\}$ be the sequence given by $x_0 = y_0$ and

$$x_{n+1} = \alpha_n\gamma f(x_n) + (I-\alpha_n F)T(t_n)x_n, \quad \forall n \ge 0. \tag{3.20}$$

Form Theorem 3.1, $x_n \to \tilde{x}$. We claim that $y_n \to \tilde{x}$. Indeed, we estimate

$$\begin{aligned}\|x_{n+1}-y_{n+1}\| &\le \alpha_n\gamma\|f(T(t_n)y_n) - f(x_n)\| + \|I-\alpha_n F\|\|T(t_n)x_n - T(t_n)y_n\| \\ &\le \alpha_n\gamma\alpha\|T(t_n)y_n - x_n\| + \left(1-\alpha_n\left(1-\sqrt{\frac{1-\delta}{\lambda}}\right)\right)\|x_n-y_n\|\end{aligned}$$

$$
\begin{aligned}
&\leq \alpha_n\gamma\alpha\left\|T(t_n)y_n - T(t_n)\tilde{x}\right\| + \alpha_n\gamma\alpha\|T(t_n)\tilde{x} - x_n\| + \left(1-\alpha_n\left(1-\sqrt{\frac{1-\delta}{\lambda}}\right)\right)\|x_n - y_n\| \\
&\leq \alpha_n\gamma\alpha\left\|y_n - \tilde{x}\right\| + \alpha_n\gamma\alpha\|\tilde{x} - x_n\| + \left(1-\alpha_n\left(1-\sqrt{\frac{1-\delta}{\lambda}}\right)\right)\|x_n - y_n\| \\
&\leq \alpha_n\gamma\alpha\left\|y_n - x_n\right\| + \alpha_n\gamma\alpha\|x_n - \tilde{x}\| + \alpha_n\gamma\alpha\|\tilde{x} - x_n\| + \left(1-\alpha_n\left(1-\sqrt{\frac{1-\delta}{\lambda}}\right)\right)\|x_n - y_n\| \\
&= \left(1-\alpha_n\left(1-\sqrt{\frac{1-\delta}{\lambda}}-\gamma\alpha\right)\right)\|x_n - y_n\| \\
&\quad + \alpha_n\left(1-\sqrt{\frac{1-\delta}{\lambda}}-\gamma\alpha\right)\frac{2\alpha\gamma}{\left(1-\sqrt{(1-\delta)/\lambda}-\gamma\alpha\right)}\|\tilde{x} - x_n\|.
\end{aligned}
\tag{3.21}
$$

It follows from $\sum_{n=1}^{\infty}\alpha_n = \infty$, $\lim_{n\to\infty}\|x_n - \tilde{x}\| = 0$, and Lemma 2.4 that $\|x_n - y_n\| \to 0$. Consequently, $y_n \to \tilde{x}$ as required. □

Corollary 3.3. *Let E be a reflexive Banach space which admits a weakly sequentially continuous duality mapping J. Let $\mathcal{S} = \{T(t) : 0 \leq t < \infty\}$ be a u.a.r. nonexpansive semigroup on E such that $\mathrm{Fix}(\mathcal{S}) \neq \emptyset$. Suppose that the real sequences $\{\alpha_n\} \subset [0,1]$, $\{t_n\} \subset (0,\infty)$ satisfy the conditions*

$$
\lim_{n\to\infty}\alpha_n = 0, \qquad \sum_{n=0}^{\infty}\alpha_n = \infty, \qquad \lim_{n\to\infty}t_n = \infty. \tag{3.22}
$$

Let F be δ-strongly accretive and λ-strictly pseudocontractive with $\delta + \lambda > 1$, $f : E \to E$ acontraction mapping with coefficient $\alpha \in (0,1)$, and γ a positive real number such that $\gamma < (1/\alpha)(1 - \sqrt{(1-\delta)/\lambda})$. Then, the sequence $\{z_n\}$ defined by (1.21) *converges strongly to $\tilde{x}$, where $\tilde{x}$ is the unique solution in* $\mathrm{Fix}(\mathcal{S})$ *of the variational inequality*

$$
\left\langle (F-\gamma f)\tilde{x}, j(x-\tilde{x})\right\rangle \geq 0, \qquad x \in \mathrm{Fix}(\mathcal{S}) \tag{3.23}
$$

or equivalently $\tilde{x} = Q_{\mathrm{Fix}(\mathcal{S})}(I - F + \gamma f)\tilde{x}$, where $Q_{\mathrm{Fix}(\mathcal{S})}$ is the sunny nonexpansive retraction of E onto $\mathrm{Fix}(\mathcal{S})$.

Proof. Define the sequences $\{y_n\}$ and $\{\beta_n\}$ by

$$
y_n = \alpha_n\gamma f(z_n) + (I - \alpha_n F)z_n, \quad \beta_n = \alpha_{n+1} \quad \forall n \in \mathbb{N}. \tag{3.24}
$$

Taking $p \in \mathrm{Fix}(\mathcal{S})$, we have

$$\begin{aligned}\|z_{n+1}-p\| &= \|T(t_n)y_n - T(t_n)p\| \le \|y_n - p\| \\ &= \|\alpha_n\gamma f(z_n) + (I-\alpha_n F)z_n - (I-\alpha_n F)p - \alpha_n Fp\| \\ &\le \left(1-\alpha_n\left(1-\sqrt{\frac{1-\delta}{\lambda}}\right)\right)\|z_n - p\| + \alpha_n\|\gamma f(z_n) - F(p)\| \\ &= \left(1-\alpha_n\left(1-\sqrt{\frac{1-\delta}{\lambda}}\right)\right)\|z_n - p\| + \alpha_n\left(1-\sqrt{\frac{1-\delta}{\lambda}}\right)\frac{\|\gamma f(z_n) - F(p)\|}{\left(1-\sqrt{(1-\delta)/\lambda}\right)}.\end{aligned} \tag{3.25}$$

It follows from induction that

$$\|z_{n+1}-p\| \le \max\left\{\|z_0 - p\|, \frac{\|\gamma f(z_0) - F(p)\|}{1-\sqrt{(1-\delta)/\lambda}}\right\}, \quad n \ge 0. \tag{3.26}$$

Thus, both $\{z_n\}$ and $\{y_n\}$ are bounded. We observe that

$$y_{n+1} = \alpha_{n+1}\gamma f(z_{n+1}) + (I-\alpha_{n+1}F)z_{n+1} = \beta_n\gamma f\left(T(t_n)y_n\right) + \left(I-\beta_n F\right)T(t_n)y_n. \tag{3.27}$$

Thus, Corollary 3.2 implies that $\{y_n\}$ converges strongly to some point $\tilde{x}$. In this case, we also have

$$\|z_n - \tilde{x}\| \le \|z_n - y_n\| + \|y_n - \tilde{x}\| = \alpha_n\|\gamma f(z_n) - Fz_n\| + \|y_n - \tilde{x}\| \longrightarrow 0. \tag{3.28}$$

Hence, the sequence $\{z_n\}$ converges strongly to some point $\tilde{x}$. This complete the proof. □

Using Theorem 3.1, Lemma 2.1, and Example 2.2, we have the following result.

Corollary 3.4. *Let E be a uniformly convex Banach space which admits a weakly sequentially continuous duality mapping J. Let $\mathcal{S} = \{T(t) : 0 \le t < \infty\}$ be a nonexpansive semigroup on E such that $\mathrm{Fix}(\mathcal{S}) \ne \emptyset$. Suppose that the real sequences $\{\alpha_n\} \subset [0,1]$, $\{t_n\} \subset (0,\infty)$ satisfy the conditions*

$$\lim_{n\to\infty}\alpha_n = 0, \qquad \sum_{n=0}^{\infty}\alpha_n = \infty, \qquad \lim_{n\to\infty}t_n = \infty. \tag{3.29}$$

Let F be δ-strongly accretive and λ-strictly pseudocontractive with $\delta + \lambda > 1$, $f : E \to E$ a contraction mapping with coefficient $\alpha \in (0,1)$, and γ a positive real number such that $\gamma < (1/\alpha)(1 - \sqrt{(1-\delta)/\lambda})$. Then, the sequence $\{x_n\}$ defined by

$$\begin{aligned} x_0 &= x \in E, \\ x_{n+1} &= \alpha_n\gamma f(x_n) + (I-\alpha_n F)\frac{1}{t_n}\int_0^{t_n} T(t)x_n\,ds, \quad n \ge 0 \end{aligned} \tag{3.30}$$

converges strongly to $\tilde{x}$, *where* $\tilde{x}$ *is the unique solution in* $\operatorname{Fix}(\mathcal{S})$ *of the variational inequality*

$$\langle (F-\gamma f)\tilde{x}, j(x-\tilde{x})\rangle \geq 0, \quad x \in \operatorname{Fix}(\mathcal{S}) \tag{3.31}$$

or equivalently $\tilde{x} = Q_{\operatorname{Fix}(\mathcal{S})}((I-F+\gamma f)\tilde{x})$, *where* $Q_{\operatorname{Fix}(\mathcal{S})}$ *is the sunny nonexpansive retraction of* E *onto* $\operatorname{Fix}(\mathcal{S})$.

Corollary 3.5. *Let* H *be a real Hilbert space. Let* $\mathcal{S} = \{T(t) : 0 \leq t < \infty\}$ *be a nonexpansive semigroup on* H *such that* $\operatorname{Fix}(\mathcal{S}) \neq \emptyset$. *Suppose that the real sequences* $\{\alpha_n\} \subset [0,1]$, $\{t_n\} \subset (0,\infty)$ *satisfy the conditions*

$$\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=0}^{\infty} \alpha_n = \infty, \quad \lim_{n\to\infty} t_n = \infty. \tag{3.32}$$

Let $f : E \to E$ *be a contraction mapping with coefficient* $\alpha \in (0,1)$ *and* A *a strongly positive bounded linear operator with coefficient* $\overline{\gamma} > 1/2$ *and* $0 < \gamma < (1-\sqrt{2-2\overline{\gamma}})/\alpha$. *Then, the sequence* $\{x_n\}$ *defined by*

$$\begin{gathered} x_0 = x \in E, \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I-\alpha_n A)\frac{1}{t_n}\int_0^{t_n} T(t)x_n ds, \quad n \geq 0 \end{gathered} \tag{3.33}$$

converges strongly to $\tilde{x}$, *where* $\tilde{x}$ *is the unique solution in* $\operatorname{Fix}(\mathcal{S})$ *of the variational inequality*

$$\langle (A-\gamma f)\tilde{x}, j(x-\tilde{x})\rangle \geq 0, \quad x \in \operatorname{Fix}(\mathcal{S}) \tag{3.34}$$

or equivalently $\tilde{x} = Q_{\operatorname{Fix}(\mathcal{S})}((I-A+\gamma f)\tilde{x})$, *where* $Q_{\operatorname{Fix}(\mathcal{S})}$ *is the sunny nonexpansive retraction of* E *onto* $\operatorname{Fix}(\mathcal{S})$.

Proof. Since A is a strongly positive bounded linear operator with coefficient $\overline{\gamma}$, we have

$$\langle Ax - Ay, x-y\rangle \geq \overline{\gamma}\|x-y\|^2. \tag{3.35}$$

Therefore, A is $\overline{\gamma}$-strongly accretive. On the other hand,

$$\begin{aligned} \|(I-A)x-(I-A)y\|^2 &= \langle (x-y)-(Ax-Ay), (x-y)-(Ax-Ay)\rangle \\ &= \langle x-y, x-y\rangle - 2\langle Ax-Ay, x-y\rangle + \langle Ax-Ay, Ax-Ay\rangle \\ &= \|x-y\|^2 - 2\langle Ax-Ay, x-y\rangle + \|Ax-Ay\|^2 \\ &\leq \|x-y\|^2 - 2\langle Ax-Ay, x-y\rangle + \|A\|^2\|x-y\|^2. \end{aligned} \tag{3.36}$$

Since A is strongly positive if and only if $(1/\|A\|)A$ is strongly positive, we may assume, without loss of generality, that $\|A\| = 1$, so that

$$\begin{aligned}\langle Ax - Ay, x - y\rangle &\leq \|x - y\|^2 - \frac{1}{2}\|(I - A)x - (I - A)y\|^2 \\ &= \|x - y\|^2 - \frac{1}{2}\|(x - y) - (Ax - Ay)\|^2.\end{aligned} \tag{3.37}$$

Hence, A is 12-strongly pseudocontractive. Applying Corollary 3.4, we conclude the result. □

Theorem 3.6. *Let E be a reflexive Banach space which admits a weakly sequentially continuous duality mapping J. Let $\mathcal{S} = \{T(t) : 0 < t < \infty\}$ be a u.a.r. nonexpansive semigroup on E such that $\mathrm{Fix}(\mathcal{S}) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{t_n\}$ be sequences of real number satisfying*

$$0 < \alpha_n < 1, \quad \sum_{n=0}^{\infty} \alpha_n = \infty, \quad t_n > 0, \quad \lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \frac{\alpha_n}{t_n} = 0. \tag{3.38}$$

Let F be δ-strongly accretive and λ-strictly pseudocontractive with $\delta + \lambda > 1$, $f : E \to E$ a contraction mapping with coefficient $\alpha \in (0,1)$, and γ a positive real number such that $\gamma < (1/\alpha)(1 - \sqrt{(1-\delta)/\lambda})$. Then, the sequence $\{x_n\}$ defined by

$$\begin{aligned} x_0 &= x \in E, \\ x_{n+1} &= \alpha_n \gamma f(x_n) + (I - \alpha_n F)T(t_n)x_n, \quad n \geq 0 \end{aligned} \tag{3.39}$$

converges strongly to $\tilde{x}$, where $\tilde{x}$ is the unique solution in $\mathrm{Fix}(\mathcal{S})$ of the variational inequality

$$\langle (F - \gamma f)\tilde{x}, j(x - \tilde{x})\rangle \geq 0, \qquad x \in \mathrm{Fix}(\mathcal{S}) \tag{3.40}$$

or equivalently $\tilde{x} = Q_{\mathrm{Fix}(\mathcal{S})}(I - F + \gamma f)\tilde{x}$, where $Q_{\mathrm{Fix}(\mathcal{S})}$ is the sunny nonexpansive retraction of E onto $\mathrm{Fix}(\mathcal{S})$.

Proof. By the same argument as in the proof of Theorem 3.1, we can obtain that $\{x_n\}$, $\{f(x_n)\}$, and $\{FT(t_n)x_n\}$ are bounded and $Q_{\mathrm{Fix}(\mathcal{S})}(I - F - \gamma f)$ is a contraction on E. Thus, by Banach contraction principle, $Q_{\mathrm{Fix}(\mathcal{S})}(I - F - \gamma f)$ has a unique fixed point $\tilde{x}$. Then, using Lemma 2.3, $\tilde{x}$ is the unique solution in $\mathrm{Fix}(\mathcal{S})$ of the variational inequality (3.40). Next, we show that

$$\limsup_{n\to\infty} \langle \gamma f(\tilde{x}) - F\tilde{x}, j(x_n - \tilde{x})\rangle \leq 0. \tag{3.41}$$

Indeed, we can take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty} \langle \gamma f(\tilde{x}) - F\tilde{x}, j(x_n - \tilde{x})\rangle = \lim_{k\to\infty} \langle \gamma f(\tilde{x}) - F\tilde{x}, j(x_{n_k} - \tilde{x})\rangle. \tag{3.42}$$

We may assume that $x_{n_k} \rightharpoonup p \in E$ as $k \to \infty$. Now, we show that $p \in \mathrm{Fix}(\mathcal{S})$. Put

$$x_k = x_{n_k}, \quad \alpha_k = \alpha_{n_k} \quad s_k = t_{n_k} \quad \forall k \in \mathbb{N}. \tag{3.43}$$

Fix $t > 0$, then we have

$$\begin{aligned}
\|x_k - T(t)p\| &= \sum_{i=0}^{[t/s_i]-1} \|T((i+1)s_k)x_k - T(is_k)x_k\| \\
&\quad + \left\|T\left(\left[\frac{t}{s_k}\right]s_k\right)x_k - T\left(\left[\frac{t}{s_k}\right]s_k\right)p\right\| + \left\|T\left(\left[\frac{t}{s_k}\right]s_k\right)p - T(t)p\right\| \\
&\leq \left[\frac{t}{s_k}\right]\|T(s_k)x_k - x_{k+1}\| + \|x_{k+1} - p\| + \left\|T\left(t - \left[\frac{t}{s_k}\right]s_k\right)p - p\right\| \\
&\leq \left[\frac{t}{s_k}\right]\alpha_k\|FT(s_k)x_k - f(x_k)\| + \|x_{k+1} - p\| + \left\|T\left(t - \left[\frac{t}{s_k}\right]s_k\right)p - p\right\| \\
&\leq \left(\frac{t\alpha_k}{s_k}\right)\|FT(s_k)x_k - f(x_k)\| + \|x_{k+1} - p\| + \max\{\|T(s)p - p\| : 0 \leq s \leq s_k\}.
\end{aligned} \tag{3.44}$$

Thus, for all $k \in \mathbb{N}$, we obtain

$$\limsup_{k\to\infty} \|x_k - T(t)p\| \leq \limsup_{k\to\infty}\|x_{k+1} - p\| = \limsup_{k\to\infty}\|x_k - p\|. \tag{3.45}$$

Since Banach space E has a weakly sequentially continuous duality mapping satisfying Opial's condition [13], we can conclude that $T(t)p = p$ for all $t > 0$, that is, $p \in \mathrm{Fix}(\mathcal{S})$. In view of the variational inequality (3.2) and the assumption that duality mapping J is weakly sequentially continuous, we conclude

$$\begin{aligned}
\limsup_{n\to\infty} \langle \gamma f(\tilde{x}) - F\tilde{x}, j(x_n - \tilde{x})\rangle &= \lim_{k\to\infty} \langle \gamma f(\tilde{x}) - F\tilde{x}, j(x_{n_k} - \tilde{x})\rangle \\
&\leq \langle \gamma f(\tilde{x}) - F\tilde{x}, J(p - \tilde{x})\rangle \leq 0.
\end{aligned} \tag{3.46}$$

By the same argument as in the proof of Theorem 3.1, we conclude that $x_n \to \tilde{x}$ as $n \to \infty$. This completes the proof. □

Using Theorem 3.6 and the method as in the proof of Corollary 3.7, we have the following result.

Corollary 3.7. *Let E be a reflexive Banach space which admits a weakly sequentially continuous duality mapping J. Let $\mathcal{S} = \{T(t) : 0 < t < \infty\}$ be a u.a.r. nonexpansive semigroup on E such that* $\mathrm{Fix}(\mathcal{S}) \neq \emptyset$. *Let $\{\alpha_n\}$ and $\{t_n\}$ be sequences of real number satisfying*

$$0 < \alpha_n < 1, \quad \sum_{n=0}^{\infty} \alpha_n = \infty, \qquad t_n > 0, \qquad \lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \frac{\alpha_n}{t_n} = 0. \tag{3.47}$$

Let F be a δ-strongly accretive and λ-strictly pseudocontractive with $\delta + \lambda > 1$, $f : E \to E$ a contraction mapping with coefficient $\alpha \in (0,1)$, and γ is a positive real number such that $\gamma < 1/\alpha(1 - \sqrt{(1-\delta)/\lambda})$. Then, the sequence $\{y_n\}$ defined by

$$\begin{gathered} y_0 = y \in E, \\ y_{n+1} = \alpha_n \gamma f\big(T(t_n)y_n\big) + (I - \alpha_n F)T(t_n)y_n, \quad n \geq 0 \end{gathered} \tag{3.48}$$

converges strongly to $\tilde{x}$, where $\tilde{x}$ is the unique solution in $\operatorname{Fix}(\mathcal{S})$ *of the variational inequality*

$$\big\langle (F - \gamma f)\tilde{x}, j(x - \tilde{x})\big\rangle \geq 0, \quad x \in \operatorname{Fix}(\mathcal{S}) \tag{3.49}$$

or equivalently $\tilde{x} = Q_{\operatorname{Fix}(\mathcal{S})}(I - F + \gamma f)\tilde{x}$, where $Q_{\operatorname{Fix}(\mathcal{S})}$ is the sunny nonexpansive retraction of E onto $\operatorname{Fix}(\mathcal{S})$.

Using Theorem 3.6 and the method as in the proof of Corollary 3.8, we have the following result.

Corollary 3.8. *Let E be a reflexive Banach space which admits a weakly sequentially continuous duality mapping J. Let $\mathcal{S} = \{T(t) : 0 < t < \infty\}$ be a u.a.r. nonexpansive semigroup on E such that* $\operatorname{Fix}(\mathcal{S}) \neq \emptyset$. *Let $\{\alpha_n\}$ and $\{t_n\}$ be sequences of real number satisfying*

$$0 < \alpha_n < 1, \quad \sum_{n=0}^{\infty} \alpha_n = \infty, \qquad t_n > 0, \qquad \lim_{n \to \infty} \alpha_n = \lim_{n \to \infty} \frac{\alpha_n}{t_n} = 0. \tag{3.50}$$

Let F be a δ-strongly accretive and λ-strictly pseudocontractive with $\delta + \lambda > 1$, $f : E \to E$ a contraction mapping with coefficient $\alpha \in (0,1)$, and γ is a positive real number such that $\gamma < (1/\alpha)(1 - \sqrt{(1-\delta)/\lambda})$. Then, the sequence $\{z_n\}$ defined by

$$\begin{gathered} z_0 = z \in E, \\ z_{n+1} = T(t_n)\big(\alpha_n \gamma f(z_n) + (I - \alpha_n F)z_n\big), \quad n \geq 0 \end{gathered} \tag{3.51}$$

converges strongly to $\tilde{x}$, where $\tilde{x}$ is the unique solution in $\operatorname{Fix}(\mathcal{S})$ *of the variational inequality*

$$\big\langle (F - \gamma f)\tilde{x}, j(x - \tilde{x})\big\rangle \geq 0, \quad x \in \operatorname{Fix}(\mathcal{S}) \tag{3.52}$$

or equivalently $\tilde{x} = Q_{\operatorname{Fix}(\mathcal{S})}(I - F + \gamma f)\tilde{x}$, where $Q_{\operatorname{Fix}(\mathcal{S})}$ is the sunny nonexpansive retraction of E onto $\operatorname{Fix}(\mathcal{S})$.

Using Theorem 3.6, Lemma 2.1, and Example 2.2, we have the following result.

Corollary 3.9. *Let E be a uniformly convex Banach space which admits a weakly sequentially continuous duality mapping J. Let $\mathcal{S} = \{T(t) : 0 < t < \infty\}$ be a nonexpansive semigroup on E such that* $\operatorname{Fix}(\mathcal{S}) \neq \emptyset$*. Let $\{\alpha_n\}$ and $\{t_n\}$ be sequences of real numbers satisfying*

$$0 < \alpha_n < 1, \quad \sum_{n=0}^{\infty} \alpha_n = \infty, \qquad t_n > 0, \qquad \lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \frac{\alpha_n}{t_n} = 0. \tag{3.53}$$

Let F be δ-strongly accretive and λ-strictly pseudocontractive with $\delta + \lambda > 1$, $f : E \to E$ a contraction mapping with coefficient $\alpha \in (0,1)$, and γ a positive real number such that $\gamma < (1/\alpha)(1 - \sqrt{(1-\delta)/\lambda})$. Then, the sequence $\{x_n\}$ defined by

$$\begin{gathered} x_0 = x \in E, \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n F)\frac{1}{t_n}\int_0^{t_n} T(t)x_n ds, \quad n \geq 0 \end{gathered} \tag{3.54}$$

converges strongly to $\widetilde{x}$, where $\widetilde{x}$ is the unique solution in $\operatorname{Fix}(\mathcal{S})$ *of the variational inequality*

$$\langle (F - \gamma f)\widetilde{x}, j(x - \widetilde{x}) \rangle \geq 0, \quad x \in \operatorname{Fix}(\mathcal{S}) \tag{3.55}$$

or equivalently $\widetilde{x} = Q_{\operatorname{Fix}(\mathcal{S})}(I - F + \gamma f)\widetilde{x}$, where $Q_{\operatorname{Fix}(\mathcal{S})}$ is the sunny nonexpansive retraction of E onto $\operatorname{Fix}(\mathcal{S})$.

Corollary 3.10. *Let H be a real Hilbert space. Let $\mathcal{S} = \{T(t) : 0 \leq t < \infty\}$ be a nonexpansive semigroup on H such that* $\operatorname{Fix}(\mathcal{S}) \neq \emptyset$*. Suppose that the real sequences $\{\alpha_n\} \subset [0,1]$, $\{t_n\} \subset (0,\infty)$ satisfy the conditions*

$$0 < \alpha_n < 1, \quad \sum_{n=0}^{\infty} \alpha_n = \infty, \qquad t_n > 0, \qquad \lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \frac{\alpha_n}{t_n} = 0. \tag{3.56}$$

Let $f : E \to E$ be a contraction mapping with coefficient $\alpha \in (0,1)$ and A a strongly positive bounded linear operator with coefficient $\overline{\gamma} > 1/2$ and $0 < \gamma < (1 - \sqrt{2 - 2\overline{\gamma}})/\alpha$. Then, the sequence $\{x_n\}$ defined by

$$\begin{gathered} x_0 = x \in E, \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)\frac{1}{t_n}\int_0^{t_n} T(t)x_n ds, \quad n \geq 0 \end{gathered} \tag{3.57}$$

converges strongly to $\widetilde{x}$, where $\widetilde{x}$ is the unique solution in $\operatorname{Fix}(\mathcal{S})$ *of the variational inequality*

$$\langle (A - \gamma f)\widetilde{x}, j(x - \widetilde{x}) \rangle \geq 0, \quad x \in \operatorname{Fix}(\mathcal{S}) \tag{3.58}$$

or equivalently $\widetilde{x} = Q_{\operatorname{Fix}(\mathcal{S})}((I - A + \gamma f)\widetilde{x})$, where $Q_{\operatorname{Fix}(\mathcal{S})}$ is the sunny nonexpansive retraction of E onto $\operatorname{Fix}(\mathcal{S})$.

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