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Research Article

On BE-Semigroups

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The notion of a BE-semigroup is introduced, and related properties are investigated. The concept of left (resp., right) deductive systems of a BE-semigroup is also introduced.

1. Introduction

Hu and Li, Iséki and Tanaka, respectively, introduced two classes of abstract algebras: BCK-algebras and BCI-algebras [1–3]. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. In [1, 4] Hu and Li introduced a wide class of abstract algebras: BCH-algebras. They have shown that the class of BCI-algebras is a proper subclass of the class of BCH-algebras. We refer to [5] for general information on BCK-algebras. Neggers and Kim [6] introduced the notion of a d-algebra which is a generalization of BCK-algebras, and also they introduced the notion of a B-algebra [7, 8], that is, (I) $x * x = 0$, (II) $x * 0 = x$, (III) $(x * y) * z = x * (z * (0 * y))$, for any $x, y, z \in X$, which is equivalent to the idea of groups. Moreover, Jun et al. [9] introduced a new notion, called an *BH- algebra*, which is another generalization of $BCH/BCI/BCK$-algebras, that is, (I), (II), and (IV) $x * y = 0$ and $y * x = 0$ imply that $x = y$ for any $x, y \in X$. Walendziak obtained other equivalent set of axioms for a B-algebra [10]. Kim et al. [11] introduced the notion of a (pre-) Coxeter algebra and showed that a Coxeter algebra is equivalent to an abelian group all of whose elements have order 2, that is, a Boolean group. C. B. Kim and H. S. Kim [12] introduced the notion of a BM-algebra which is a specialization of B-algebras. They proved that the class of BM-algebras is a proper subclass of B-algebras and also showed that a BM-algebra is equivalent to a 0-commutative B-algebra. In [13], H. S. Kim and Y. H. Kim introduced the notion of a BE-algebra as a generalization of a BCK-algebra. Using the notion of upper sets, they gave an equivalent condition of the filter in BE-algebras. In [14, 15], Ahn and So introduced the notion of ideals in BE-algebras and proved several characterizations of such ideals.

In this paper, by combining BE-algebras and semigroups, we introduce the notion of BE-semigroups. We define left (resp., right) deductive systems (LDS (resp., RDS) for short) of a BE-semigroup, and then we describe LDS generated by a nonempty subset in a BE-semigroup as a simple form.

2. Preliminaries

We recall some definitions and results discussed in [13].

Definition 2.1 (see [13]). An algebra $(X;*,1)$ of type $(2,0)$ is called a *BE-algebra* if

(BE1) $x*x=1$ for all $x \in X$,

(BE2) $x*1=1$ for all $x \in X$,

(BE3) $1*x=x$ for all $x \in X$,

(BE4) $x*(y*z)=y*(x*z)$ for all $x,y,z \in X$ (*exchange*).

We introduce a relation "$\le$" on X by $x \le y$ if and only if $x*y=1$.

Proposition 2.2 (see [13]). *If $(X;*,1)$ is a BE-algebra, then $x*(y*x)=1$ for any $x,y \in X$.*

Example 2.3 (see [13]). Let $X := \{1,a,b,c,d,0\}$ be a set with the following table:

$*$	1	a	b	c	d	0
1	1	a	b	c	d	0
a	1	1	a	c	c	d
b	1	1	1	c	c	c
c	1	a	b	1	a	b
d	1	1	a	1	1	a
0	1	1	1	1	1	1

(2.1)

Then $(X;*,1)$ is a BE-algebra.

Definition 2.4 (see [13]). A BE-algebra $(X;*,1)$ is said to be *self-distributive* if $x*(y*z)=(x*y)*(x*z)$ for all $x,y,z \in X$.

Example 2.5 (see [13]). Let $X := \{1,a,b,c,d\}$ be a set with the following table:

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	d
b	1	a	1	c	c
c	1	1	b	1	b
d	1	1	1	1	1

(2.2)

Then it is easy to see that X is a self-distributive BE-algebra.

Note that the BE-algebra in Example 2.3 is not self-distributive, since $d*(a*0) = d*d = 1$, while $(d * a) * (d * 0) = 1 * a = a$.

Proposition 2.6. *Let X be a self-distributive BE-algebra. If* $x \leq y$*, then* $z*x \leq z*y$ *and* $y*z \leq x*z$ *for any* $x, y, z \in X$.

Proof. The proof is straightforward. □

3. BE-Semigroups

Definition 3.1. An algebraic system $(X; \odot, *, 1)$ is called a *BE-semigroup* if it satisfies the following:

(i) $(X; \odot)$ is a semigroup,

(ii) $(X; *, 1)$ is a BE-algebra,

(iii) the operation "$\odot$" is distributive (on both sides) over the operation "$*$".

Example 3.2. (1) Define two operations "$\odot$" and "$*$" on a set $X := \{1, a, b, c\}$ as follows:

$\odot$	1	a	b	c
1	1	1	1	1
a	1	1	1	1
b	1	1	1	1
c	1	a	b	c

$*$	1	a	b	c
1	1	a	b	c
a	1	1	b	c
b	1	a	1	c
c	1	1	1	1

(3.1)

It is easy to see that $(X; \odot, *, 1)$ is a BE-semigroup.

(2) Define two binary operations "$\odot$" and "$*$" on a set $A := \{1, a, b, c\}$ as follows:

$\odot$	1	a	b	c
1	1	1	1	1
a	1	1	1	1
b	1	1	1	b
c	1	1	b	c

$*$	1	a	b	c
1	1	a	b	c
a	1	1	b	c
b	1	a	1	c
c	1	1	1	1

(3.2)

It is easy to show that $(A; \odot, *, 1)$ is a BE-semigroup.

Proposition 3.3. *Let* $(X; \odot, *, 1)$ *be a BE-semigroup. Then*

(i) $(\forall x \in X)\ (1 \odot x = x \odot 1 = 1)$,

(ii) $(\forall x, y, z \in X)\ (x \leq y \Rightarrow x \odot z \leq y \odot z,\ z \odot x \leq z \odot y)$.

Proof. (i) For all $x \in X$, we have that $1 \odot x = (1 * 1) \odot x = (1 \odot x) * (1 \odot x) = 1$ and $x \odot 1 = x \odot (1 * 1) = (x \odot 1) * (x \odot 1) = 1$.

(ii) Let $x, y, z \in X$ be such that $x \leq y$. Then

$$\begin{aligned} (x \odot z) * (y \odot z) &= (x * y) \odot z = 1 \odot z = 1, \\ (z \odot x) * (z \odot y) &= z \odot (x * y) = z \odot 1 = 1. \end{aligned} \tag{3.3}$$

Hence $x \odot z \leq y \odot z$ and $z \odot x \leq z \odot y$. □

Definition 3.4. An element $a(\neq 1)$ in a *BE*-semigroup $(X; \odot, *, 1)$ is said to be a *left* (resp., *right*) *unit divisor* if

$$(\exists b(\neq 1) \in X) \quad (a \odot b = 1 \ (\text{resp.}, b \odot a = 1)). \tag{3.4}$$

A *unit divisor* is an element of X which is both a left and a right unit divisors.

Theorem 3.5. *Let $(X; \odot, *, 1)$ be a BE-semigroup. If it satisfies the left (resp., right) cancellation law for the operation $\odot$, that is,*

$$(\forall x(\neq 1), y, z \in A) \quad (x \odot y = y \odot z \ (\textit{resp.}, y \odot x = z \odot x) \Longrightarrow y = z), \tag{3.5}$$

then X contains no left (resp., right) unit divisors.

Proof. Let $(X; \odot, *, 1)$ satisfy the left cancellation law for the operation $\odot$ and assume that $x \odot y = 1$ where $x \neq 1$. Then $x \odot y = 1 = x \odot 1$ by Proposition 3.3(i), which implies $y = 1$. Similarly it holds for the right case. Hence there is no left (resp., right) unit divisors in X. □

Now we consider the converse of Theorem 3.5.

Theorem 3.6. *Let $(X; \odot, *, 1)$ be a BE-semigroup in which there are no left (resp., right) unit divisors. Then it satisfies the left (resp., right) cancellation law for the operation $\odot$.*

Proof. Let $x, y, z \in X$ be such that $x \odot y = x \odot z$ and $x \neq 1$. Then

$$\begin{aligned} x \odot (y * z) &= (x \odot y) * (x \odot z) = 1, \\ x \odot (z * y) &= (x \odot z) * (x \odot y) = 1. \end{aligned} \tag{3.6}$$

Since X has no left unit divisor, it follows that $y * z = 1 = z * y$ so that $y = z$. The argument is the same for the right case. □

Definition 3.7. Let $(X; \odot, *, 1)$ be a *BE*-semigroup. A nonempty subset D of X is called a *left* (resp., *right*) *deductive system* (LDS (resp., RDS), for short) if it satisfies

(ds1) $X \odot D \subseteq D$ (resp., $(D \odot X \subseteq D)$),

(ds2) $(\forall a \in D)$ $((\forall x \in X)$ $(a * x \in D \Rightarrow x \in D)$.

Example 3.8. Let $X := \{x, y, z, 1\}$ be a set with the following Cayley tables:

$$\begin{array}{c|cccc} \odot & 1 & x & y & z \\ \hline 1 & 1 & 1 & 1 & 1 \\ x & 1 & x & 1 & 1 \\ y & 1 & 1 & y & z \\ z & 1 & 1 & z & y \end{array} \qquad \begin{array}{c|cccc} * & 1 & x & y & z \\ \hline 1 & 1 & x & y & z \\ x & 1 & 1 & y & z \\ y & 1 & 1 & 1 & z \\ z & 1 & 1 & 1 & 1 \end{array} \tag{3.7}$$

It is easy to show that $(X; \odot, *, 1)$ is a *BE*-semigroup. We know that $D := \{1, x\}$ is an LDS of X, but $E := \{1, y\}$ is not an LDS of X, since $z \odot y = z \notin E$ and/or $y * x = 1 \in E$, $y \in E$ but $x \notin E$.

Let $(X; *, 1)$ be a *BE*-algebra, and let $a, b \in X$. Then the set

$$A(a, b) := \{x \in X \mid a * (b * x) = 1\} \tag{3.8}$$

is nonempty, since $1, a, b \in A(a, b)$.

Proposition 3.9. *If D is an LDS of a BE-semigroup* $(X; \odot, *, 1)$, *then*

$$(\forall a, b \in D) \quad (A(a, b) \subseteq D). \tag{3.9}$$

Proof. Let $x \in A(a, b)$ where $a, b \in D$. Then $a*(b*x) = 1 \in D$ and so $x \in D$ by (ds2). Therefore $A(a, b) \subseteq D$. □

Theorem 3.10. *Let* $\{D_i\}$ *be an arbitrary collection of LDSs of a BE-semigroup* $(X; \odot, *, 1)$, *where i ranges over some index set I. Then* $\cap_{i \in I} D_i$ *is also an LDS of A.*

Proof. The proof is straightforward. □

Let $(X; \odot, *, 1)$ be a *BE*-semigroup. For any subset D of X, the intersection of all LDSs (resp., RDSs) of X containing D is called the LDSs (resp., RDSs) *generated by D*, and is denoted by $\langle D \rangle_l$ (resp., $\langle D \rangle_r$). It is clear that if D and E are subsets of a *BE*-semigroup $(X; \odot, *, 1)$ satisfying $D \subseteq E$, then $\langle D \rangle_l \subseteq \langle E \rangle_l$ (resp., $\langle D \rangle_r \subseteq \langle E \rangle_r$), and if D is an LDS (resp., RDS) of X, then $\langle D \rangle_l = D$ (resp., $\langle D \rangle_r = D$).

A *BE*-semigroup $(X; \odot, *, 1)$ is said to be *self-distributive* if $(X; *, 1)$ is a self-distributive *BE*-algebra.

Theorem 3.11. *Let* $(X; \odot, *, 1)$ *be a self-distributive BE-semigroup and let D be a nonempty subset of X such that* $A \odot D \subseteq D$. *Then* $\langle D \rangle_l := \{a \in X \mid y_n * (\cdots * (y_1 * a) \cdots) = 1$ *for some* $y_1, \ldots, y_n \in D\}$.

Proof. Denote

$$B := \{a \in X \mid y_n * (\cdots * (y_1 * a) \cdots) = 1 \text{ for some } y_1, \ldots, y_n \in D\}. \tag{3.10}$$

Let $a \in X$ and $b \in B$. Then there exist $y_1, \ldots, y_n \in D$ such that $y_n * (\cdots * (y_1 * b) \cdots) = 1$. It follows that

$$\begin{aligned} 1 &= x \odot 1 \\ &= x \odot (y_n * (\cdots * (y_1 * b) \cdots)) \\ &= (x \odot y_n) * (\cdots * ((x \odot y_1) * (x \odot b)) \cdots). \end{aligned} \tag{3.11}$$

Since $x \odot y_i \in D$ for $i = 1, \ldots, n$, we have that $x \odot b \in B$. Let $x, a \in X$ be such that $a * x \in B$ and $a \in B$. Then there exist $y_1, \ldots, y_n, z_1, \ldots, z_m \in D$ such that

$$y_n * (\cdots * (y_1 * (a * x)) \cdots) = 1, \tag{3.12}$$

$$z_m * (\cdots * (z_1 * a) \cdots) = 1. \tag{3.13}$$

Using (BE4), it follows from (3.12) that $a * (y_n * (\cdots * (y_1 * x) \cdots)) = 1$, that is, $a \leq y_n * (\cdots * (y_1 * x) \cdots)$, and so from (3.13) and Proposition 2.6 it follows that

$$\begin{aligned} 1 &= z_m * (\cdots * (z_1 * a) \cdots) \\ &\leq z_m * (\cdots * (z_1 * (y_n * (\cdots * (y_1 * x) \cdots))) \cdots). \end{aligned} \tag{3.14}$$

Thus $z_m * (\cdots * (z_1 * (y_n * (\cdots * (y_1 * x) \cdots))) \cdots) = 1$, which implies $x \in B$. Therefore B is an LDS of X. Obviously $D \subseteq B$. Let G be an LDS containing D. To show $B \subseteq G$, let a be any element of B. Then there exist $y_1, \ldots, y_n \in D$ such that $y_n * (\cdots * (y_1 * a) \cdots) = 1$. It follows from (ds2) that $a \in G$ so that $B \subseteq G$. Consequently, we have that $\langle D \rangle_l = B$. □

In the following example, we know that the union of any LDSs (resp., RDSs) D and E may not be an LDS (resp., RDS) of a self-distributive BE-semigroup $(X; \cdot, *, 1)$.

Example 3.12. Let $X := \{1, a, b, c, d\}$ be a set with the following Cayley tables:

⊙	1	a	b	c	d
1	1	1	1	1	1
a	1	1	1	1	1
b	1	1	1	1	1
c	1	1	1	1	1
d	1	1	1	1	d

*	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	b	d
b	1	a	1	a	d
c	1	1	1	1	d
d	1	1	b	b	1

(3.15)

It is easy to check that $(X; \odot, *, 1)$ is a self-distributive BE-semigroup. We know that $D := \{1, a\}$ and $E := \{1, b\}$ are LDSs of X, but $D \cup E = \{1, a, b\}$ is not an LDS of X, since $b * c = a \in D \cup E$, $c \notin D \cup E$.

Theorem 3.13. *Let D and E be LDSs of a self-distributive BE-semigroup $(X; \cdot, *, 1)$. Then*

$$\langle D \cup E \rangle_l := \{a \in X \mid x * (y * a) = 1 \textit{ for some } x \in D,\ y \in E\}. \tag{3.16}$$

Proof. Denote

$$K := \{a \in X \mid x * (y * a) = 1 \text{ for some } x \in D,\ y \in E\}. \tag{3.17}$$

Obviously, $K \subseteq \langle D \cup E\rangle_l$. Let $b \in \langle D \cup E\rangle_l$. Then there exist $y_1, \ldots, y_n \in D \cup E$ such that $y_n * (\cdots * (y_1 * b) \cdots) = 1$ by Theorem 3.11. If $y_i \in D$ (resp., E) for all $i = 1, \ldots, n$, then $b \in D$ (resp., E). Hence $b \in K$ since $b * (1 * b) = 1$ (resp., $1 * (b * b) = 1$). If some of $y_1, \ldots, y_n$ belong to D and others belong to E, then we may assume that $y_1, \ldots, y_k \in D$ and $y_{k+1}, \ldots, y_n \in E$ for $1 \leq k < n$, without loss of generality. Let $p = y_k * (\cdots * (y_1 * b) \cdots)$. Then

$$\begin{aligned} &y_n * (\cdots * (y_{k+1} * p) \cdots) \\ &\quad = y_n * (\cdots * (y_{k+1} * (y_k * (\cdots * (y_1 * b) \cdots))) \cdots) \\ &\quad = 1, \end{aligned} \tag{3.18}$$

and so $p \in E$. Now let $q = p * b = (y_k * (\cdots * (y_1 * b) \cdots)) * b$. Then

$$\begin{aligned} &y_k * (\cdots * (y_1 * q) \cdots) \\ &\quad = y_k * (\cdots * (y_1 * ((y_k * (\cdots * (y_1 * b) \cdots)) * b)) \cdots) \\ &\quad = (y_k * (\cdots * (y_1 * b) \cdots)) * (y_k * (\cdots * (y_1 * b) \cdots)) \\ &\quad = 1, \end{aligned} \tag{3.19}$$

which implies that $q \in D$. Since $p * (q * b) = q * (p * b) = q * q = 1$, it follows that $b \in K$ so that $\langle D \cup E\rangle_l \subseteq K$. This completes the proof. □

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