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Research Article

Fixed Point and Common Fixed Point Theorems for Generalized Weak Contraction Mappings of Integral Type in Modular Spaces

Chirasak Mongkolkeha and Poom Kumam

Department of Mathematics, Faculty of Science, King Mongkut's University of Technology Thonburi (KMUTT), Bangmod, Bangkok 10140, Thailand

Correspondence should be addressed to Poom Kumam, poom.kum@kmutt.ac.th

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We prove new fixed point and common fixed point theorems for generalized weak contractive mappings of integral type in modular spaces. Our results extend and generalize the results of A. Razani and R. Moradi (2009) and M. Beygmohammadi and A. Razani (2010).

1. Introduction

Let (X, d) be a metric space. A mapping $T : X \to X$ is a contraction if

$$d(T(x), T(y)) \leq k d(x, y), \tag{1.1}$$

where $0 < k < 1$. The Banach Contraction Mapping Principle appeared in explicit form in Banach's thesis in 1922 [1]. For its simplicity and usefulness, it has become a very popular tool in solving existence problems in many branches of mathematical analysis. Banach contraction principle has been extended in many different directions; see [2–6]. In 1997 Alber and Guerre-Delabriere [7] introduced the concept of weak contraction in Hilbert spaces, and Rhoades [8] has showed that the result by Akber et al. is also valid in complete metric spaces A mapping $T : X \to X$ is said to be weakly contractive if

$$d(T(x), T(y)) \leq d(x, y) - \phi(d(x, y)), \tag{1.2}$$

where $\phi : [0,\infty) \to [0,\infty)$ is continuous and nondecreasing function such that $\phi(t) = 0$ if and only if $t = 0$. If one takes $\phi(t) = (1-k)t$ where $0 < k < 1$, then (1.2) reduces to (1.1). In 2002, Branciari [9] gave a fixed point result for a single mapping an analogue of Banach's contraction principle for an integral-type inequality, which is stated as follow.

Theorem 1.1. *Let (X,d) be a complete metric space, $\alpha \in [0,1)$, $f : X \to X$ a mapping such that for each $x, y \in X$,*

$$\int_0^{d(f(x),f(y))} \varphi(t)dt \le \alpha \int_0^{d(x,y)} \varphi(t)dt, \tag{1.3}$$

where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a Lebesgue integrable which is summable, nonnegative, and for all $\varepsilon > 0$, $\int_0^{\varepsilon} \varphi(t)dt > 0$. Then, f has a unique fixed point $z \in X$ such that for each $x \in X$, $lim_{n\to\infty} f^n x = z$.

Afterward, many authors extended this work to more general contractive conditions. The works noted in [10–12] are some examples from this line of research.

The notion of modular spaces, as a generalize of metric spaces, was introduced by Nakano [13] and redefined by Musielak and Orlicz [14]. A lot of mathematicians are interested, fixed points of Modular spaces, for example [15–22]. In 2009, Razani and Moradi [23] studied fixed point theorems for ρ-compatible maps of integral type in modular spaces.

Recently, Beygmohammadi and Razani [24] proved the existence for mapping defined on a complete modular space satisfying contractive inequality of integral type.

In this paper, we study the existence of fixed point and common fixed point theorems for ρ-compatible mapping satisfying a generalize weak contraction of integral type in modular spaces.

First, we start with a brief recollection of basic concepts and facts in modular spaces.

Definition 1.2. Let X be a vector space over $\mathbb{R}$(or $\mathbb{C}$). A functional $\rho : X \to [0,\infty]$ is called a modular if for arbitrary f and g, elements of X satisfy the following conditions:

(1) $\rho(f) = 0$ if and only if $f = 0$;

(2) $\rho(\alpha f) = \rho(f)$ for all scalar α with $|\alpha| = 1$;

(3) $\rho(\alpha f + \beta g) \le \rho(f) + \rho(g)$, whenever $\alpha, \beta \ge 0$ and $\alpha + \beta = 1$.If we replace (3) by

(4) $\rho(\alpha f + \beta g) \le \alpha^s \rho(f) + \beta^s \rho(g)$, for $\alpha, \beta \ge 0$, $\alpha^s + \beta^s = 1$ with an s $\in (0,1]$, then the modular ρ is called s-convex modular, and if $s = 1$, ρ is called convex modular.

If ρ is modular in X, then the set defined by

$$X_\rho = \left\{x \in X : \rho(\lambda x) \longrightarrow 0 \text{ as } \lambda \longrightarrow 0\right\} \tag{1.4}$$

is called a modular space. X_ρ is a vector subspace of X.

Definition 1.3. A modular ρ is said to satisfy the Δ_2-condition if $\rho(2f_n) \to 0$ as $n \to \infty$, whenever $\rho(f_n) \to 0$ as $n \to \infty$.

Definition 1.4. Let X_ρ be a modular space. Then,

(1) the sequence $(f_n)_{n\in\mathbb{N}}$ in X_ρ is said to be ρ-convergent to $f \in X_\rho$ if $\rho(f_n - f) \to 0$, as $n \to \infty$,

(2) the sequence $(f_n)_{n\in\mathbb{N}}$ in X_ρ is said to be ρ-Cauchy if $\rho(f_n - f_m) \to 0$, as $n, m \to \infty$,

(3) a subset C of X_ρ is said to be ρ-closed if the ρ-limit of a ρ-convergent sequence of C always belong to C,

(4) a subset C of X_ρ is said to be ρ-complete if any ρ-Cauchy sequence in C is ρ-convergent sequence and its is in C,

(5) a subset C of X_ρ is said to be ρ-bounded if $\delta_\rho(C) = \sup\{\rho(f-g); f, g \in C\} < \infty$.

Definition 1.5. Let C be a subset of X_ρ and $T : C \to C$ an arbitrary mapping. T is called a ρ-contraction if for each $f, g \in X_\rho$ there exists $k < 1$ such that

$$\rho(T(f) - T(g)) \leq k\rho(f-g). \tag{1.5}$$

Definition 1.6. Let X_ρ be a modular space, where ρ satisfies the Δ_2-condition. Two self-mappings T and f of X_ρ are called ρ-compatible if $\rho(Tfx_n - fTx_n) \to 0$ as $n \to \infty$, whenever $\{x_n\}_{n\in\mathbb{N}}$ is a sequence in X_ρ such that $fx_n \to z$ and $Tx_n \to z$ for some point $z \in X_\rho$.

2. A Common Fixed Point Theorem for ρ-Compatible Generalized Weak Contraction Maps of Integral Type

Theorem 2.1. *Let X_ρ be a ρ-complete modular space, where ρ satisfies the Δ_2-condition. Let $c, l \in \mathbb{R}^+$, $c > l$ and $T, f : X_\rho \to X_\rho$ are two ρ-compatible mappings such that $T(X_\rho) \subseteq f(X_\rho)$ and*

$$\int_0^{\rho(c(Tx-Ty))} \varphi(t)dt \leq \int_0^{\rho(l(fx-fy))} \varphi(t)dt - \phi\left(\int_0^{\rho(l(fx-fy))} \varphi(t)dt\right), \tag{2.1}$$

for all $x, y \in X_\rho$, where $\varphi : [0,\infty) \to [0,\infty)$ is a Lebesgue integrable which is summable, nonnegative, and for all $\varepsilon > 0$, $\int_0^\varepsilon \varphi(t)dt > 0$ and $\phi : [0,\infty) \to [0,\infty)$ is lower semicontinuous function with $\phi(t) > 0$ for all $t > 0$ and $\phi(t) = 0$ if and only if $t = 0$. If one of T or f is continuous, then there exists a unique common fixed point of T and f.

Proof. Let $x \in X_\rho$ and generate inductively the sequence $\{Tx_n\}_{n\in\mathbb{N}}$ as follow: $Tx_n = fx_{n+1}$. First, we prove that the sequence $\{\rho(c(Tx_n - Tx_{n-1}))\}$ converges to 0. Since,

$$\begin{aligned}
\int_0^{\rho(c(Tx_n-Tx_{n-1}))} \varphi(t)dt &\leq \int_0^{\rho(l(fx_n-fx_{n-1}))} \varphi(t)dt - \phi\left(\int_0^{\rho(l(fx_n-fx_{n-1}))} \varphi(t)dt\right) \\
&\leq \int_0^{\rho(l(fx_n-fx_{n-1}))} \varphi(t)dt \\
&\leq \int_0^{\rho(l(Tx_{n-1}-Tx_{n-2}))} \varphi(t)dt \\
&< \int_0^{\rho(c(Tx_{n-1}-Tx_{n-2}))} \varphi(t)dt.
\end{aligned} \tag{2.2}$$

This means that the sequence $\{\int_0^{\rho(c(Tx_n - Tx_{n-1}))}\}$ is decreasing and bounded below. Hence, there exists $r \geq 0$ such that

$$\lim_{n\to\infty}\int_0^{\rho(c(Tx_n - Tx_{n-1}))}\varphi(t)dt = r. \tag{2.3}$$

If $r > 0$, then $\lim_{n\to\infty}\int_0^{\rho(c(Tx_n - Tx_{n-1}))}\varphi(t)dt = r > 0$. Taking $n \longrightarrow \infty$ in the inequality (2.2) which is a contradiction, thus $r = 0$. This implies that

$$\rho(c(Tx_n - Tx_{n-1})) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{2.4}$$

Next, we prove that the sequence $\{Tx_n\}_{n\in\mathbb{N}}$ is ρ-*Cauchy*. Suppose $\{cTx_n\}_{n\in\mathbb{N}}$ is not ρ-*Cauchy*, then there exists $\varepsilon > 0$ and sequence of integers $\{m_k\}, \{n_k\}$ with $m_k > n_k \geq k$ such that

$$\rho(c(Tx_{m_k} - Tx_{n_k})) \geq \varepsilon \quad \text{for } k = 1, 2, 3, \ldots \tag{2.5}$$

We can assume that

$$\rho(c(Tx_{m_k-1} - Tx_{n_k})) < \varepsilon. \tag{2.6}$$

Let m_k be the smallest number exceeding n_k for which (2.5) holds, and

$$\theta_k = \{m \in \mathbb{N} \mid \exists n_k \in \mathbb{N};\ \rho(c(Tx_m - Tx_{n_k})) \geq \varepsilon,\ m > n_k \geq k\}. \tag{2.7}$$

Since $\theta_k \subset \mathbb{N}$ and clearly $\theta_k \neq \emptyset$, by well ordering principle, the minimum element of θ_k is denoted by m_k and obviously (2.6) holds. Now, let $\alpha \in \mathbb{R}^+$ be such that $l/c + 1/\alpha = 1$, then we get

$$\begin{aligned}
\int_0^{\varepsilon}\varphi(t)dt &\leq \int_0^{\rho(c(Tx_{m_k} - Tx_{n_k}))}\varphi(t)dt \\
&\leq \int_0^{\rho(l(fx_{m_k} - fx_{n_k}))}\varphi(t)dt - \phi\left(\int_0^{\rho(l(fx_{m_k} - fx_{n_k}))}\varphi(t)dt\right) \\
&\leq \int_0^{\rho(l(fx_{m_k} - fx_{n_k}))}\varphi(t)dt \\
&\leq \int_0^{\rho(l(Tx_{m_k-1} - Tx_{n_k-1}))}\varphi(t)dt,
\end{aligned} \tag{2.8}$$

$$\begin{aligned}\rho(l(Tx_{m_k-1}-Tx_{n_k-1})) &= \rho(l(Tx_{m_k-1}-Tx_{n_k}+Tx_{n_k}-Tx_{n_k-1})) \\ &= \rho\left(\frac{l}{c}c(Tx_{m_k-1}-Tx_{n_k})+\frac{1}{\alpha}\alpha l(Tx_{n_k}-Tx_{n_k-1})\right) \\ &\le \rho(c(Tx_{m_k-1}-Tx_{n_k}))+\rho(\alpha l(Tx_{n_k}-Tx_{n_k-1})) \\ &< \varepsilon+\rho(\alpha l(Tx_{n_k}-Tx_{n_k-1})).\end{aligned} \tag{2.9}$$

Using the Δ_2-*condition* and (2.4), we obtain

$$\lim_{n\to\infty}\rho(\alpha l(Tx_{n_k}-Tx_{n_k-1}))=0. \tag{2.10}$$

It follows that

$$\lim_{k\to\infty}\int_0^{\rho(l(Tx_{m_k-1}-Tx_{n_k-1}))}\varphi(t)dt<\int_0^{\varepsilon}\varphi(t)dt. \tag{2.11}$$

From (2.8) and (2.11), we also have

$$\begin{aligned}\int_0^{\varepsilon}\varphi(t)dt &\le \int_0^{\rho(l(Tx_{m_k-1}-Tx_{n_k-1}))}\varphi(t)dt \\ &< \int_0^{\varepsilon}\varphi(t)dt,\end{aligned} \tag{2.12}$$

which is a contradiction. Hence, $\{cTx_n\}_{n\in\mathbb{N}}$ is ρ-*Cauchy* and by the Δ_2-*condition*, $\{Tx_n\}_{n\in\mathbb{N}}$ is ρ-*Cauchy*. Since X_ρ is ρ-*complete*, there exists a point $u\in X_\rho$ such that $\rho(Tx_n-u)\to 0$ as $n\to\infty$. If T is continuous, then $T^2x_n\to Tu$ and $Tfx_n\to Tu$ as $n\to\infty$. Since $\rho(c(fTx_n-Tfx_n))\to 0$ as $n\to\infty$, by ρ-*compatible*, $fTx_n\to Tu$ as $n\to\infty$. Next, we prove that u is a unique fixed point of T. Indeed,

$$\begin{aligned}\int_0^{\rho(c(T^2x_n-Tx_n))}\varphi(t)dt &= \int_0^{\rho(c(T(Tx_n)-Tx_n))}\varphi(t)dt \\ &\le \int_0^{\rho(l(fTx_n-fx_n))}\varphi(t)dt-\phi\left(\int_0^{\rho(l(fTx_n-fx_n))}\varphi(t)dt\right) \\ &\le \int_0^{\rho(l(fTx_n-fx_n))}\varphi(t)dt.\end{aligned} \tag{2.13}$$

Taking $n\to\infty$ in the inequality (2.13), we have

$$\int_0^{\rho(c(Tu-u))}\varphi(t)dt\le\int_0^{\rho(l(Tu-u))}\varphi(t)dt, \tag{2.14}$$

which implies that $\rho(c(Tu-u)) = 0$ and $Tu = u$. Since $T(X_\rho) \subseteq f(X_\rho)$, there exists u_1 such that $u = Tu = fu_1$. The inequality,

$$\begin{aligned}\int_0^{\rho(c(T^2x_n-Tu_1))}\varphi(t)dt &\leq \int_0^{\rho(l(fTx_n-fu_1))}\varphi(t)dt - \phi\left(\int_0^{\rho(l(fTx_n-fu_1))}\varphi(t)dt\right)\\ &\leq \int_0^{\rho(l(fTx_n-fu_1))}\varphi(t)dt\end{aligned} \tag{2.15}$$

as $n \to \infty$, yields

$$\int_0^{\rho(c(Tu-Tu_1))}\varphi(t)dt \leq \int_0^{\rho(l(Tu-fu_1))}\varphi(t)dt \tag{2.16}$$

and, thus,

$$\begin{aligned}\int_0^{\rho(c(u-Tu_1))}\varphi(t)dt &\leq \int_0^{\rho(l(u-fu_1))}\varphi(t)dt\\ &\leq \int_0^{\rho(l(u-u))}\varphi(t)dt\\ &= 0,\end{aligned} \tag{2.17}$$

which implies that, $u = Tu_1 = fu_1$ and also $fu = fTu_1 = Tfu_1 = Tu = u$ (see [25]). Hence, $fu = Tu = u$. Suppose that there exists $w \in X_\rho$ such that $w = Tw = fw$ and $w \neq u$, we have $\int_0^{\rho(c(w-u))}\varphi(t)dt > 0$ and

$$\begin{aligned}\int_0^{\rho(c(w-u))}\varphi(t)dt &= \int_0^{\rho(c(Tw-Tu))}\varphi(t)dt\\ &\leq \int_0^{\rho(l(fw-fu))}\varphi(t)dt - \phi\left(\int_0^{\rho(l(fw-fu))}\varphi(t)dt\right)\\ &< \int_0^{\rho(l(fw-fu))}\varphi(t)dt\\ &< \int_0^{\rho(c(w-u))}\varphi(t)dt,\end{aligned} \tag{2.18}$$

which is a contradiction. Hence, $u = w$ and the proof is complete. □

In fact, if take $\phi(t) = (1-k)t$ where $0 < k < 1$ and take $\phi(t) = t - \psi(t)$, respectively, where $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is a nondecreasing and right continuous function with $\psi(t) < t$ for all $t > 0$, we obtain following corollaries.

Corollary 2.2 (see [23]). *Let X_ρ be a ρ-complete modular space, where ρ satisfies the Δ_2-condition. Suppose $c, l \in \mathbb{R}^+$, $c > l$ and $T, h : X_\rho \to X_\rho$ are two ρ-compatible mappings such that $T(X_\rho) \subseteq h(X_\rho)$ and*

$$\int_0^{\rho(c(Tx-Ty))} \varphi(t)dt \le k \int_0^{\rho(l(hx-hy))} \varphi(t)dt, \tag{2.19}$$

for some $k \in (0,1)$, where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a Lebesgue integrable which is summable, nonnegative, and for all $\varepsilon > 0$, $\int_0^\varepsilon \varphi(t)dt > 0$. If one of h or T is continuous, then there exists a unique common fixed point of h and T.

Corollary 2.3 (see [23]). *Let X_ρ be a ρ-complete modular space, where ρ satisfies the Δ_2-condition. Suppose $c, l \in \mathbb{R}^+$, $c > l$ and $T, h : X_\rho \to X_\rho$ are two ρ-compatible mappings such that $T(X_\rho) \subseteq h(X_\rho)$ and*

$$\int_0^{\rho(c(Tx-Ty))} \varphi(t)dt \le \psi\left(\int_0^{\rho(l(hx-hy))} \varphi(t)dt\right), \tag{2.20}$$

where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a Lebesgue integrable which is summable, nonnegative, and for all $\varepsilon > 0$, $\int_0^\varepsilon \varphi(t)dt > 0$ and $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is a nondecreasing and right continuous function with $\psi(t) < t$ for all $t > 0$. If one of h or T is continuous, then there exists a unique common fixed point of h and T.

3. A Fixed Point Theorem for Generalized Weak Contraction Mapping of Integral Type

Theorem 3.1. *Let X_ρ be a ρ-complete modular space, where ρ satisfies the Δ_2-condition. Let $c, l \in \mathbb{R}^+$, $c > l$ and $T : X_\rho \to X_\rho$ be a mapping such that for each $x, y \in X_\rho$,*

$$\int_0^{\rho(c(Tx-Ty))} \varphi(t)dt \le \int_0^{\rho(l(x-y))} \varphi(t)dt - \phi\left(\int_0^{\rho(l(x-y))} \varphi(t)dt\right), \tag{3.1}$$

where $\varphi : [0,\infty) \to [0,\infty)$ is a Lebesgue integrable which is summable, nonnegative, and for all $\varepsilon > 0$, $\int_0^\varepsilon \varphi(t)dt > 0$ and $\phi : [0,\infty) \to [0,\infty)$ is lower semicontinuous function with $\phi(t) > 0$ for all $t > 0$ and $\phi(t) = 0$ if and only if $t = 0$. Then, T has a unique fixed point.

Proof. First, we prove that the sequence $\{\rho(c(T^n x - T^{n-1}x))\}$ converges to 0. Since,

$$\begin{aligned}\int_0^{\rho(c(T^n x-T^{n-1}x))} \varphi(t)dt &\le \int_0^{\rho(l(T^{n-1}x-T^{n-2}x))} \varphi(t)dt - \phi\left(\int_0^{\rho(l(T^{n-1}x-T^{n-2}x))} \varphi(t)dt\right) \\ &\le \int_0^{\rho(l(T^{n-1}x-T^{n-2}x))} \varphi(t)dt \\ &< \int_0^{\rho(c(T^{n-1}x-T^{n-2}x))} \varphi(t)dt,\end{aligned} \tag{3.2}$$

it follows that the sequence $\{\int_0^{\rho(c(T^n x - T^{n-1}x))}\}$ is decreasing and bounded below. Hence, there exists $r \geq 0$ such that

$$\lim_{n\to\infty}\int_0^{\rho(c(T^n x - T^{n-1}x))}\varphi(t)dt = r. \tag{3.3}$$

If $r > 0$, then $\lim_{n\to\infty}\int_0^{\rho(c(T^n x - T^{n-1}x))}\varphi(t)dt = r > 0$, taking $n \to \infty$ in the inequality (3.2) which is a contradiction, thus $r = 0$. So, we have

$$\rho\Big(c\Big(T^n x - T^{n-1}x\Big)\Big) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.4}$$

Next, we prove that the sequence $\{T^n(x)\}_{n\in\mathbb{N}}$ is ρ-*Cauchy*. Suppose $\{cT^n(x)\}_{n\in\mathbb{N}}$ is not ρ-*Cauchy*, there exists $\varepsilon > 0$ and sequence of integers $\{m_k\}, \{n_k\}$ with $m_k > n_k \geq k$ such that

$$\rho(c(T^{m_k}x - T^{n_k}x)) \geq \varepsilon \quad \text{for } k = 1, 2, 3, \ldots. \tag{3.5}$$

We can assume that

$$\rho\Big(c\Big(T^{m_k-1}x - T^{n_k}x\Big)\Big) < \varepsilon. \tag{3.6}$$

Let m_k be the smallest number exceeding n_k for which (3.5) holds, and

$$\theta_k = \big\{m \in \mathbb{N} \mid \exists n_k \in \mathbb{N};\ \rho(c(T^m x - T^{n_k}x)) \geq \varepsilon,\ m > n_k \geq k\big\}. \tag{3.7}$$

Since $\theta_k \subset \mathbb{N}$ and clearly $\theta_k \neq \emptyset$, by well ordering principle, the minimum element of θ_k is denoted by m_k and obviously (3.6) holds. Now, let $\alpha \in \mathbb{R}^+$ be such that $l/c + 1/\alpha = 1$, then we get

$$\begin{aligned}
\int_0^{\varepsilon}\varphi(t)dt &\leq \int_0^{\rho(c(T^{m_k}x - T^{n_k}x))}\varphi(t)dt \\
&\leq \int_0^{\rho(l(T^{m_k-1}x - T^{n_k-1}x))}\varphi(t)dt - \phi\left(\int_0^{\rho(l(T^{m_k-1}x - T^{n_k-1}x))}\varphi(t)dt\right), \\
&\leq \int_0^{\rho(l(T^{m_k-1}x - T^{n_k-1}x))}\varphi(t)dt,
\end{aligned} \tag{3.8}$$

$$\begin{aligned}
\rho\Big(l\Big(T^{m_k-1}x - T^{n_k-1}x\Big)\Big) &= \rho\Big(l\Big(T^{m_k-1}x - T^{n_k}x + T^{n_k}x - T^{n_k-1}x\Big)\Big) \\
&= \rho\left(\frac{l}{c}c\Big(T^{m_k-1}x - T^{n_k}x\Big) + \frac{1}{\alpha}\alpha l\Big(T^{n_k}x - T^{n_k-1}x\Big)\right) \\
&\leq \rho\Big(c\Big(T^{m_k-1}x - T^{n_k}x\Big)\Big) + \rho\Big(\alpha l\Big(T^{n_k}x - T^{n_k-1}x\Big)\Big) \\
&< \varepsilon + \rho\Big(\alpha l\Big(T^{n_k}x - T^{n_k-1}x\Big)\Big).
\end{aligned} \tag{3.9}$$

Using the Δ_2-*condition* and (3.4), we obtain

$$\lim_{k\to\infty} \rho\left(\alpha l\left(T^{n_k}x - T^{n_k-1}x\right)\right) = 0, \tag{3.10}$$

$$\lim_{k\to\infty} \int_0^{\rho(l(T^{m_k-1}x - T^{n_k-1}x))} \varphi(t)dt < \int_0^{\varepsilon} \varphi(t)dt. \tag{3.11}$$

From (3.8) and (3.11), we have

$$\begin{aligned} \int_0^{\varepsilon} \varphi(t)dt &\le \int_0^{\rho(l(T^{m_k-1}x - T^{n_k-1}x))} \varphi(t)dt \\ &< \int_0^{\varepsilon} \varphi(t)dt, \end{aligned} \tag{3.12}$$

which is a contradiction. Hence, $\{cT^n(x)\}_{n\in\mathbb{N}}$ is ρ-*Cauchy* and again by the Δ_2-*condition*, $\{T^n(x)\}_{n\in\mathbb{N}}$ is ρ-*Cauchy*. Since X_ρ is ρ-*complete*, there exists a point $u \in X_\rho$ such that $\rho(T^n x - u) \to 0$ as $n \to \infty$. Next, we prove that u is a unique fixed point of T. Indeed,

$$\begin{aligned} \rho\left(\frac{c}{2}(u - Tu)\right) &= \rho\left(\frac{c}{2}\left(u - T^{n+1}x + T^{n+1}x - Tu\right)\right) \\ &\le \rho\left(c\left(u - T^{n+1}x\right)\right) + \rho\left(c\left(T^{n+1}x - Tu\right)\right), \end{aligned} \tag{3.13}$$

$$\begin{aligned} \int_0^{\rho(c(T^{n+1}x - Tu))} \varphi(t)dt &\le \int_0^{\rho(l(T^n x - u))} \varphi(t)dt - \phi\left(\int_0^{\rho(l(T^n x - u))} \varphi(t)dt\right) \\ &\le \int_0^{\rho(l(T^n x - u))} \varphi(t)dt. \end{aligned} \tag{3.14}$$

Since $\rho(T^n x - u) \to 0$ as $n \to \infty$, we obtain

$$\lim_{n\to\infty} \int_0^{\rho(c(T^{n+1}x - Tu))} \varphi(t)dt \le 0, \tag{3.15}$$

which implies that

$$\rho\left(c\left(T^{n+1}x - Tu\right)\right) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.16}$$

So, we have

$$\rho\left(c\left(u - T^{n+1}x\right)\right) + \rho\left(c\left(T^{n+1}x - Tu\right)\right) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.17}$$

Thus $\rho(c/2(u-Tu))=0$ and $Tu=u$. Suppose that there exists $w\in X_\rho$ such that $Tw=w$ and $w\neq u$, we have $\int_0^{\rho(c(w-u))}\varphi(t)dt>0$ and

$$\begin{aligned}\int_0^{\rho(c(w-u))}\varphi(t)dt&=\int_0^{\rho(c(Tw-Tu))}\varphi(t)dt\\&\leq\int_0^{\rho(l(w-u))}\varphi(t)dt-\phi\left(\int_0^{\rho(l(w-u))}\varphi(t)dt\right)\\&<\int_0^{\rho(l(w-u))}\varphi(t)dt\\&<\int_0^{\rho(c(w-u))}\varphi(t)dt,\end{aligned}\tag{3.18}$$

which is a contradiction. Hence, $u=w$ and the proof is complete. □

Corollary 3.2. *Let X_ρ be a ρ-complete modular space, where ρ satisfies the Δ_2-condition. Let $f:X_\rho\to X_\rho$ be a mapping such that there exists an $\lambda\in(0,1)$ and $c,l\in\mathbb{R}^+$ where $l<c$ and for each $x,y\in X_\rho$,*

$$\int_0^{\rho(c(fx-fy))}\varphi(t)dt\leq\lambda\int_0^{\rho(l(x-y))}\varphi(t)dt\tag{3.19}$$

where $\varphi:[0,\infty)\to[0,\infty)$ is a Lebesgue integrable which is summable, nonnegative, and for all $\varepsilon>0$, $\int_o^{\varepsilon}\varphi(t)dt>o$. Then, T has a unique fixed point in X_ρ.

Corollary 3.3 (see [24]). *Let X_ρ be a ρ-complete modular space where ρ satisfies the Δ_2-condition. Assume that $\psi:\mathbb{R}^+\to[0,\infty)$ is an increasing and upper semicontinuous function satisfying $\psi(t)<t$ for all $t>0$. Let $\varphi:[0,\infty)\to[0,\infty)$ be a Lebesgue integrable which is summable, nonnegative, and for all $\varepsilon>0$, $\int_0^{\varepsilon}\varphi(t)dt>0$ and let $f:X_\rho\to X_\rho$ be a mapping such that there are $c,l\in\mathbb{R}^+$ where $l<c$,*

$$\int_0^{\rho(c(Tx-Ty))}\varphi(t)dt\leq\psi\left(\int_0^{\rho(l(x-y))}\varphi(t)dt\right),\tag{3.20}$$

for each $x,y\in X_\rho$. Then, T has a unique fixed point in X_ρ.

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References

[1] S. Banach, "Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales," *Fundamenta Mathematicae*, vol. 3, pp. 133–181, 1922.

[2] G. Jungck, "Compatible mappings and common fixed points," *International Journal of Mathematics and Mathematical Sciences*, vol. 9, no. 4, pp. 771–779, 1986.

[3] G. Jungck, "Common fixed points for noncontinuous nonself maps on nonmetric spaces," *Far East Journal of Mathematical Sciences*, vol. 4, no. 2, pp. 199–215, 1996.

[4] G. Jungck and B. E. Rhoades, "Fixed points for set valued functions without continuity," *Indian Journal of Pure and Applied Mathematics*, vol. 29, no. 3, pp. 227–238, 1998.

[5] W. Sintunavarat and P. Kumam, "Coincidence and common fixed points for hybrid strict contractions without the weakly commuting condition," *Applied Mathematics Letters*, vol. 22, no. 12, pp. 1877–1881, 2009.

[6] W. Sintunavarat and P. Kumam, "Weak condition for generalized multi-valued (f, α, β)-weak contraction mappings," *Applied Mathematics Letters*, vol. 24, no. 4, pp. 460–465, 2011.

[7] Y. I. Alber and S. Guerre-Delabriere, "Principle of weakly contractive maps in Hilbert spaces," in *New Results in Operator Theory and Its Applications*, vol. 98 of *Operator Theory: Advances and Applications*, pp. 7–22, Birkhäuser, Basel, Switzerland, 1997.

[8] B. E. Rhoades, "Some theorems on weakly contractive maps," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 47, no. 4, pp. 2683–2693, 2001.

[9] A. Branciari, "A fixed point theorem for mappings satisfying a general contractive condition of integral type," *International Journal of Mathematics and Mathematical Sciences*, vol. 29, no. 9, pp. 531–536, 2002.

[10] B. E. Rhoades, "Two fixed-point theorems for mappings satisfying a general contractive condition of integral type," *International Journal of Mathematics and Mathematical Sciences*, vol. 2003, no. 63, pp. 4007–4013, 2003.

[11] W. Sintunavart and P. Kumam, "Gregus-type common fixed point theorems for tangential multivalued mappings of integral type in metric spaces," *International Journal of Mathematics and Mathematical Sciences*, vol. 2011, Article ID 923458, 12 pages, 2011.

[12] W. Sintunavarat and P. Kumam, "Gregus type fixed points for a tangential multi-valued mappings satisfying contractive conditions of integral type," *Journal of Inequalities and Applications*. In Press.

[13] H. Nakano, *Modulared Semi-Ordered Linear Spaces*, Tokyo Mathematical Book Series, Maruzen Co. Ltd, Tokyo, Japan, 1950.

[14] J. Musielak and W. Orlicz, "On modular spaces," *Studia Mathematica*, vol. 18, pp. 49–65, 1959.

[15] T. Dominguez Benavides, M. A. Khamsi, and S. Samadi, "Uniformly Lipschitzian mappings in modular function spaces," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 46, no. 2, pp. 267–278, 2001.

[16] M. A. Khamsi, "Quasicontraction mappings in modular spaces without Δ_2-condition," *Fixed Point Theory and Applications*, vol. 2008, Article ID 916187, 6 pages, 2008.

[17] M. A. Khamsi, W. M. Kozłowski, and S. Reich, "Fixed point theory in modular function spaces," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 14, no. 11, pp. 935–953, 1990.

[18] P. Kumam, "On nonsquare and Jordan-von Neumann constants of modular spaces," *Southeast Asian Bulletin of Mathematics*, vol. 30, no. 1, pp. 69–77, 2006.

[19] P. Kumam, "On uniform Opial condition, uniform Kadec-Klee property in modular spaces and application to fixed point theory," *Journal of Interdisciplinary Mathematics*, vol. 8, no. 3, pp. 377–385, 2005.

[20] P. Kumam, "Fixed point theorems for nonexpansive mappings in modular spaces," *Archivum Mathematicum*, vol. 40, no. 4, pp. 345–353, 2004.

[21] P. Kumam, "Some geometrical properties and fixed point theorems in modular spaces," in *International Conference on Fixed Point Theory and Applications*, pp. 173–188, Yokohama Publishers, Yokohama, Japan, 2004.

[22] A. Razani, E. Nabizadeh, M. B. Mohamadi, and S. H. Pour, "Fixed points of nonlinear and asymptotic contractions in the modular space," *Abstract and Applied Analysis*, vol. 2007, Article ID 40575, 10 pages, 2007.

[23] A. Razani and R. Moradi, "Common fixed point theorems of integral type in modular spaces," *Bulletin of the Iranian Mathematical Society*, vol. 35, no. 2, pp. 11–24, 2009.

[24] M. Beygmohammadi and A. Razani, “Two fixed-point theorems for mappings satisfying a general contractive condition of integral type in the modular space,” *International Journal of Mathematics and Mathematical Sciences*, vol. 2010, Article ID 317107, 10 pages, 2010.

[25] H. Kaneko and S. Sessa, “Fixed point theorems for compatible multi-valued and single-valued mappings,” *International Journal of Mathematics and Mathematical Sciences*, vol. 12, no. 2, pp. 257–262, 1989.