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Research Article

Generalized Derivations and Bilocal Jordan Derivations of Nest Algebras

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Let H be a complex Hilbert space and $B(H)$ the collection of all linear bounded operators, $\mathfrak{A}$ is the closed subspace lattice including 0 an H, then $\mathfrak{A}$ is a nest, accordingly alg $\mathfrak{A} = \{T \in B(H) : TN \subseteq N, \forall N \in \mathfrak{A}\}$ is a nest algebra. It will be shown that of nest algebra, generalized derivations are generalized inner derivations, and bilocal Jordan derivations are inner derivations.

1. Introduction

The concept of local derivations was introduced by Kadison [1] who showed that on a Von-nemann algebra all norm-continuous local derivations are derivations. Larson and Sourour [2] proved that on the algebra $B(X)$ local derivations are derivations. M. Brešar and P. Šemrl [3, 4] generalized the results of the three authors above under a weaker condition. Shulman [5] showed that all local derivations on C^*-algebra are derivations.

Based on a great deal of research works of many mathematicians, some scholars paid more interests in similar kind of problems under more generalized conditions, such as considering local derivations on nest algebras and generalized derivation. Zhu and Xiong [6, 7] proved that local derivations of nest algebra and standard operator algebra are derivations, Zhang [8] considered the Jordan derivations of nest algebras, Lee [9] discussed generalized derivations of left faithful rings. Recently, some scholars discussed some new types of derivations, as Li and Zhou [10] and Majieed and Zhou [11] investigated some new types of generalized derivations associated with Hochschild 2 cocycles, other examples are in [12–15]. In fact, under appropriate conditions, local derivations are derivations.

In this paper, we will show that of nest algebra, a generalized derivation is a generalized inner derivation, and bilocal Jordan derivations are inner derivations.

2. Some Notations and Definitions

In what follows, some notations and basic definitions are introduced.

Let H be a complex Hilbert space and $B(H)$ the collection of all linear bounded operators on H, $\mathfrak{A}$ is the closed subspace lattice including 0 an H, then $\mathfrak{A}$ is a nest, correspondingly the Nest algebra is $\operatorname{alg}\mathfrak{A} = \{T \in B(H) : TN \subseteq N,\ \forall N \in \mathfrak{A}\}$.

If $N \neq 0$, we denote $N_- = \vee\{M \in \mathfrak{A} : M \subsetneq N\}$ and if $N \neq H$, denote $N_+ = \wedge\{M \in \mathfrak{A} : M \supsetneq N\}$, where $\subsetneq$ is real inclusion, and we define $0_- = 0$, $H_+ = H$.

For all $N \in \mathfrak{A}$, $P(N)$ represent the project operator from H to N, and $N^\perp = \{f \in H : \langle x, f\rangle = 0,\ \forall x \in N\}$.

Let $\mathfrak{A}$ be a Banach algebra and $\mathfrak{A}_1$ a subalgebra of $\mathfrak{A}$, we call the linear map $\varphi : \mathfrak{A}_1 \to \mathfrak{A}$ a generalized inner derivation if and only if for all $T \in \mathfrak{A}_1$, there exist operators A and B in $\mathfrak{A}$ such that $\varphi(T) = AT + TB$; if for all $T \in \mathfrak{A}_1$, we have $\varphi(T^2) = \varphi(T)T + T\varphi(T)$, then φ is called a Jordan derivation; if for all $T \in \mathfrak{A}_1$, there is a Jordan derivation $\varphi_T : \mathfrak{A}_1 \to \mathfrak{A}$, such that $\varphi(T) = \varphi_T(T)$, then φ is said to be a local Jordan derivation.

Definition 2.1. Let $\varphi : \operatorname{alg}\mathfrak{A} \to \operatorname{alg}\mathfrak{A}$ be an additive mapping, if there exists a derivation $\delta : \operatorname{alg}\mathfrak{A} \to \operatorname{alg}\mathfrak{A}$ that $\varphi(ST) = \varphi(S)T + S\delta(T)$, for all $S, T \in \operatorname{alg}\mathfrak{A}$, then φ is called a generalized derivation.

Definition 2.2. We call the linear mapping $\varphi : \operatorname{alg}\mathfrak{A} \to \operatorname{alg}\mathfrak{A}$ a bilocal Jordan derivation, if for every $u \in H$, there is a Jordan derivation $\delta_{T,u} : \operatorname{alg}\mathfrak{A} \to \operatorname{alg}\mathfrak{A}$, such that $\varphi(T)u = \delta_{T,u}(T)u$.

3. Main Results

Next to give out the main conclusions.

Theorem 3.1. *If $\varphi : \operatorname{alg}\mathfrak{A} \to \operatorname{alg}\mathfrak{A}$ is a generalized derivation, then there are operators A and B in $\operatorname{alg}\mathfrak{A}$, such that $\varphi(T) = AT + TB$, for all $T \in \operatorname{alg}\mathfrak{A}$.*

Proof. From the definition of generalized derivation, we can find a derivation: $\delta : \operatorname{alg}\mathfrak{A} \to \operatorname{alg}\mathfrak{A}$, such that $\varphi(ST) = \varphi(S)T + S\delta(T)$, for all $S, T \in \operatorname{alg}\mathfrak{A}$, so when $S = I$, we have $\varphi(T) = \varphi(I)T + \delta(T)$, for all $T \in \operatorname{alg}\mathfrak{A}$, denote $\varphi(I) = C$, apparently $C \in \operatorname{alg}\mathfrak{A}$ and $\varphi(T) = CT + \delta(T)$, for all $T \in \operatorname{alg}\mathfrak{A}$.

Since $\delta : \operatorname{alg}\mathfrak{A} \to \operatorname{alg}\mathfrak{A}$ is a derivation, by [6], it is an inner derivation, namely, there exists $D \in \operatorname{alg}\mathfrak{A}$, such that $\delta(T) = DT - TD$, consequently

$$\varphi(T) = CT + DT - TD = (C + D)T - TD. \tag{3.1}$$

Denote $A = C + D$, $B = -D$, then $\varphi(T) = AT + TB$, for all $T \in \operatorname{alg}\mathfrak{A}$. □

Theorem 3.2. *If $\varphi : \operatorname{alg}\mathfrak{A} \to \operatorname{alg}\mathfrak{A}$ is a local Jordan derivation, then φ is an inner derivation.*

Proof. Since φ is a local Jordan derivation, there exists a Jordan derivation $\varphi_T : \operatorname{alg}\mathfrak{A} \to \operatorname{alg}\mathfrak{A}$, such that $\varphi(T) = \varphi_T(T)$, from Theorem 2.12 in [8], we know that the Jordan derivation of nest algebra $\operatorname{alg}\mathfrak{A}$ is an inner derivation, so there exists $A_T \in \operatorname{alg}\mathfrak{A}$, such that $\varphi_T(T) = TA_T - A_TT$, by imitating the proof in [6], we can conclude that $\varphi_T(T) = TA - AT$, so $\varphi(T) = TA - AT$, namely, φ is an inner derivation. □

The following is the main result.

Theorem 3.3. *If* $\varphi : \operatorname{alg}\mathfrak{A} \to \operatorname{alg}\mathfrak{A}$ *is a bilocal Jordan derivation, then it is an inner derivation.*

Proof. We will prove this proposition by the following three steps.

(1) $\varphi(T)(\ker T) \subseteq \operatorname{ran}T$, where $\ker T$ and $\operatorname{ran}T$ are the kernal of T and range of T, respectively. In fact, since $\varphi(T)u = \delta_{T,u}(T)u$ for all $T \in \operatorname{alg}\mathfrak{A}$, for all $u \in H$, and $\delta_{T,u}$ is a Jordan derivation, by Theorem 2.12 in [8], there is an $A_{T,u} \in \operatorname{alg}\mathfrak{A}$, such that $\varphi(T)u = (TA_{T,u} - A_{T,u}T)u$, so if $u \in \ker T$, we have $\varphi(T)u = TA_{T,u}u \in \operatorname{ran}T$.

(2) For all $N \in \mathfrak{A}$, $\{o\} \subset N \subset H$, there exists $C_N \in B(H)$ and $B_N \in B(H)$, such that $\varphi(x \otimes f) = x \otimes C_N f + B_N x \otimes f$, for all $x \in N$, $f \in N_-^{\perp}$.

For arbitrary fixed $f \in N_-^{\perp}$, $f \neq 0$, and for all $x \in N$, we know that $x \otimes y \in \operatorname{alg}\mathfrak{A}$, from step (1), we have $\varphi(x \otimes f)\{f\}^{\perp} \subseteq \operatorname{span}\{x\}$, so there exists a linear function $\lambda_{x,f}$over$\{f\}^{\perp}$, such that $\varphi(x \otimes f)(u) = \langle u, \lambda_{x,f}\rangle x$, for all $u \in \{f\}^{\perp}$, in succession we will prove that $\lambda_{x,f}$ is independent of x. Take a $z \in N$ which is linear independent of x, we have $z \otimes f \in \operatorname{alg}\mathfrak{A}$, then

$$\varphi((x+z) \otimes f)(u) = \varphi(x \otimes f)(u) + \varphi(z \otimes f)(u) = \langle u, \lambda_{x,f}\rangle x + \langle u, \lambda_{z,f}\rangle z. \tag{3.2}$$

On the other hand, $\varphi((x+z) \otimes f)(u) = \langle u, \lambda_{x+z,f}\rangle (x+z)$, so

$$\langle u, \lambda_{x+z,f} - \lambda_{x,f}\rangle x = \langle u, \lambda_{z,f} - \lambda_{x+z,f}\rangle z. \tag{3.3}$$

Since x is linear independent of z, we know that $\lambda_{x,f} = \lambda_{x+z,f}$, that is, $\lambda_{x,f}$ is independent of x, so $\lambda_{x,f}$ can be denoted by λ_f, and

$$\varphi(x \otimes f)(u) = \langle u, \lambda_f\rangle x, \quad \forall u \in \{f\}^{\perp}. \tag{3.4}$$

Let g_f be the linear continuous span on H of λ_f, we define $B_{u,f} : N \to N$ as follows:

$$B_{u,f}(x) = \frac{1}{\langle u, f\rangle}\{\varphi(x \otimes f)(u) - \langle u, g_f\rangle x\}, \quad \text{where } u \in \{f\}^{\perp}. \tag{3.5}$$

Obviously, $B_{u,f}$ is linear and $\varphi(x \otimes f)(u) = \langle u, g_f\rangle x + \langle u, f\rangle B_{u,f}x$, $x \in N$, $u \in H$.

Next $B_{u,f}$ is independent of u, which reduce to show (i) $B_{au,f} = B_{u,f}$, $a \in C$; (ii) $B_{u,f} = B_{v,f}$, where $v \neq u$, $v \in H$.

In fact, (i) is evident. For (ii), since for all $x \in N$, $\varphi(x \otimes f)(v) = \langle v, g_f\rangle x + \langle v, f\rangle B_{v,f}x$ and $\varphi(x \otimes f)(u+v) = \langle u+v, g_f\rangle x + \langle u+v, f\rangle B_{u+v,f}x = \langle u+v, g_f\rangle x + \langle v, f\rangle B_{v,f}x + \langle u, f\rangle B_{u,f}x$, we have $\langle u, f\rangle B_{u+v,f} + \langle v, f\rangle B_{u+v,f} = \langle u, f\rangle B_{u,f} + \langle v, f\rangle B_{v,f}$, namely, $\langle u, f\rangle (B_{u+v,f} - B_{u,f}) = \langle v, f\rangle (B_{v,f} - B_{u+v,f})$, on account of $u \neq v$, so $B_{u+v,f} = B_{u,f}$, that is, $B_{u,f}$ is independent of u, so we can mark $B_{u,f}$ by B_f, as a result, we have

$$\varphi(x \otimes f)(u) = \langle u, g_f\rangle x + \langle u, f\rangle B_f x = x \otimes g_f(u) + B_f x \otimes f(u), \quad \forall u \in H. \tag{3.6}$$

Consequently

$$\varphi(x \otimes f) = x \otimes g_f + B_f x \otimes f. \tag{3.7}$$

Define $C_N : N_-^\perp \to N_-^\perp : C_N f \to g_f$, now we will show that C_N is a linear bounded operator. Because g_f is a continuous linear function, so g_f is bounded, consequently C_N is bounded, according to (3.4), we know

$$\varphi(x \otimes af)(u) = \langle u, \lambda_{af} \rangle x = \overline{a}\varphi(x \otimes f)(u) = \overline{a}\langle u, \lambda_f \rangle x = \langle u, a\lambda_f \rangle x, \tag{3.8}$$

so $\lambda_{a,f} = a\lambda_f$; on the other hand,

$$\begin{aligned} &\varphi(x \otimes (f_1 + f_2))(u) \\ &\quad = \langle u, \lambda_{f_1+f_2} \rangle x = \varphi(x \otimes f_1)(u) + \varphi(x \otimes f_2)(u) = \langle u, \lambda_{f_1} \rangle x + \langle u, \lambda_{f_2} \rangle x = \langle u, \lambda_{f_1} + \lambda_{f_2} \rangle x, \end{aligned} \tag{3.9}$$

so $\lambda_{f_1+f_2} = \lambda_{f_1} + \lambda_{f_2}$, this is enough to show that $g_{a_f} = ag_f$, $g_{f_1+f_2} = g_{f_1} + g_{f_2}$, so when $af_1 + f_2 \in N_-^\perp$,

$$C_N(af_1 + f_2) = g_{af_1+f_2} = g_{af_1} + g_{f_2} = aC_N f_1 + C_N f_2. \tag{3.10}$$

That is to say C_N is linear, so (3.7) has the form of

$$\varphi(x \otimes f) = x \otimes C_N f + B_f x \otimes f. \tag{3.11}$$

In succession we will prove that B_f is independent of f, arbitrarily choose $y \in N_-^\perp$, where y is linear independent of f, then $y + f \in N_-^\perp$ and

$$\begin{aligned} \varphi(x \otimes (f + y)) &= x \otimes C_N(f + y) + B_{f+y} x \otimes (f + y) \\ &= x \otimes C_N f + x \otimes C_N y + B_{f+y} x \otimes f + B_{f+y} x \otimes y. \end{aligned} \tag{3.12}$$

On the other hand,

$$\varphi(x \otimes (f + y)) = \varphi(x \times \otimes f) + \varphi(x \otimes y) = x \otimes C_N f + B_f x \otimes f + x \otimes C_N y + B_y x \otimes y. \tag{3.13}$$

So $(B_{f+y} - B_f)x \otimes f = (B_y - B_{y+f})x \otimes f$, then $B_f = B_{f+y} = B_y$, that is, B_f is independent of f, which can be marked by B_N, so (3.11) has the form of

$$\varphi(x \otimes f) = x \otimes C_N f + B_N x \otimes f. \tag{3.14}$$

We proved that B_N is bounded, because $\|B_N x \otimes f\| \leq \|\varphi(x \otimes f)\| + \|x \otimes C_N f\| \leq M + \|C_N\|\|x\|\|f\|$.

On account of the boundary of C_N and $\varphi(x\otimes f)\in \operatorname{alg} N\subseteq B(H)$, we know that B_N is bounded, namely, $B_N\in B(N)$, $C_N\in B(N_-^{\perp})$.

(3) For arbitrary $x\otimes y\in \operatorname{alg}\mathfrak{A}$, there is $\varphi(x\otimes f)=x\otimes Cf+Bx\otimes f$.

For all $M,N\in\mathfrak{A}$, $\{0\}\subset N\subset M\subset H$, select $x\in N$, $f\in M_-^{\perp}$, then $x\in M$, $f\in N_-^{\perp}$ and $x\otimes f\in \operatorname{alg}\mathfrak{A}$, from the result of step (2), it is easy to know that

$$\varphi(x\otimes f)=x\otimes C_N f+B_N x\otimes f,\quad \varphi(x\otimes f)=x\otimes C_M f+B_M x\otimes f. \tag{3.15}$$

Consequently $x\otimes(C_N-C_M)f=(B_M-B_N)x\otimes f$, so there exists a scalar $\lambda(N,M)$, such that

$$(C_N-C_M)|_{M_-^{\perp}}=\lambda P\left(M_-^{\perp}\right),\qquad (B_N-B_M)|_N=-\lambda P(N). \tag{3.16}$$

By imitating Lemma 2 mentioned in [6], we can prove that

$$\varphi(x\otimes f)=x\otimes Cf+Bx\otimes f,\quad x\otimes y\in \operatorname{alg} N. \tag{3.17}$$

Since the collection of all rank one operators is dense in $\operatorname{alg}\mathfrak{A}$, so for every $T\in\operatorname{alg}\mathfrak{A}$, we have $\varphi(T)=TC^*+BT$, let $T=I$, then $\varphi(I)=B+C^*$, considering φ to be a bilocal Jordan derivation, namely, $\varphi(I)(u)=\delta_{I,u}(I^2)u=(I\delta_{I,u}(I)+\delta_{I,u}(I)I)u=2\delta_{I,u}(I)u$, we can conclude that $\delta_{I,u}(I)=0$, so $\varphi(I)=0$, thereby $B+C^*=0$ and $\varphi(T)=BT-TB$, which shows that φ is an inner derivation. □

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References

[1] R.l V. Kadison, "Local derivations," *Journal of Algebra*, vol. 130, no. 2, pp. 494–509, 1990.

[2] D. R. Larson and A. R. Sourour, "Local derivations and local automorphsims of B(X)," in *Operator Theory: Operator Algebras and Applications, Part 2*, vol. 51 of *Proceedings of Symposia in Pure Mathematics*, pp. 187–194, American Mathematical Society, Providence, RI, USA, 1990.

[3] M. Brešar, "Characterizations of derivations on some normed algebras with involution," *Journal of Algebra*, vol. 152, no. 2, pp. 454–462, 1992.

[4] M. Brešar and P. Šemrl, "Mappings which preserve idempotents, local automorphisms, and local derivations," *Canadian Journal of Mathematics*, vol. 45, no. 3, pp. 483–496, 1993.

[5] V. S. Shulman, "Operators preserving ideals in C*-algebras," *Studia Mathematica*, vol. 109, no. 1, pp. 67–72, 1994.

[6] J. Zhu, "Local derivation of nest algebras," *Proceedings of the American Mathematical Society*, vol. 123, no. 3, pp. 739–742, 1995.

[7] J. Zhu and C. Xiong, "Bilocal derivations of standard operator algebras," *Proceedings of the American Mathematical Society*, vol. 125, no. 5, pp. 1367–1370, 1997.

[8] J. H. Zhang, "Jordan derivations on nest algebras," *Acta Mathematica Sinica*, vol. 41, no. 1, pp. 205–212, 1998.

[9] T.-K. Lee, "Generalized derivations of left faithful rings," *Communications in Algebra*, vol. 27, no. 8, pp. 4057–4073, 1999.

[10] J. Li and J. Zhou, "Generalize derivations associated with hochschild 2-cocycles on some algebras," *Czechoslovak Mathematical Journal*, vol. 60, pp. 909–932, 2010.

[11] A. Majieed and J. Zhou, "Generalized Jordan derivations associated with Hochschild 2-cocycles of triangular algebras," *Czechoslovak Mathematical Journal*, vol. 60, no. 1, pp. 211–219, 2010.

[12] A. Nowicki and I. Nowosad, "Local derivations of subrings of matrix rings," *Acta Mathematica Hungarica*, vol. 105, no. 1-2, pp. 145–150, 2004.

[13] A. Fošner and M. Fošner, "On ε-derivations and local ε-derivations," *Acta Mathematica Sinica*, vol. 26, no. 8, pp. 1555–1566, 2010.

[14] F. Lu and B. Liu, "Lie derivations of reflexive algebras," *Integral Equations and Operator Theory*, vol. 64, no. 2, pp. 261–271, 2009.

[15] F. Lu, "Jordan derivations of reflexive algebras," *Integral Equations and Operator Theory*, vol. 67, no. 1, pp. 51–56, 2010.