Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2011, Article ID 783084, 10 pages
doi:10.1155/2011/783084

Research Article

On Some Integral Operators for Certain Classes of p-Valent Functions

Lokesh Vijayvargy,[1] Pranay Goswami,[2] and Bushra Malik[3]

[1] *Faculty of Management, Jaipuria Institute of Management, Jaipur 302033, India*
[2] *Departement of Mathematics, Amity University Rajasthan, Jaipur 302002, India*
[3] *Mathematics Department, COMSATS Institute of Information Technology, Islamabad 44000, Pakistan*

Correspondence should be addressed to Pranay Goswami, pranaygoswami83@gmail.com

Received 3 December 2010; Revised 9 February 2011; Accepted 3 March 2011

Academic Editor: H. S. V. De Snoo

We study some generalized integral operators for the classes of p-valent functions with bounded radius and boundary rotation. Our work generalizes many previously known results. Many of our results are best possible.

1. Introduction

Let A_p denote the class of functions of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n, \quad p \in N = \{1, 2, \ldots\}, \tag{1.1}$$

which are analytic in the open unit disc $U = \{z : |z| < 1\}$.

Let f and g be analytic functions in U we say that f is subordinate to g, written as

$$f \prec g; \tag{1.2}$$

if there exists a Schwarz function $w(z)$ in U, with $w(0) = 0$ and $|w(z)| < 1$ $(z \in U)$, such that

$$f(z) = g(w(z)). \tag{1.3}$$

In particular, when g is univalent, then the above subordination is equivalent to

$$f(0) = 0, \qquad f(U) \subseteq g(U). \tag{1.4}$$

For functions $f, g \in A_p$, given by

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n, \quad g(z) = z^p + \sum_{n=p+1}^{\infty} b_n z^n, \quad z \in U, \tag{1.5}$$

we define the Hadamard product (or convolution) of f and g by

$$(f * g)(z) = z^p + \sum_{n=p+1}^{\infty} a_n b_n z^n, \quad z \in U. \tag{1.6}$$

Janowski [1] defined the class $P[A, B]$ as follows.

Let h be a function, analytic in U, with $h(0) = 1$. Then h is said to belong to the class $P[A, B]$, $-1 \leq B < A \leq 1$, if and only if, for $z \in U$,

$$h(z) = \frac{1 + Aw(z)}{1 + Bw(z)}, \quad \text{where } w(z) \text{ is a Schwarz function.} \tag{1.7}$$

Or equivalently, we can say that $h \in P[A, B]$, $-1 \leq B < A \leq 1$, if and only if,

$$h(z) \prec \frac{1 + Az}{1 + Bz}, \quad z \in U. \tag{1.8}$$

Geometrically, $h(z)$ is in the class $P[A, B]$, if and only if, $h(0) = 1$ and the image of $h(U)$ lies inside the open disc centered on the real axis with diameter end points,

$$D_1 = h(-1) = \frac{1 - A}{1 - B}, \quad D_2 = h(1) = \frac{1 + A}{1 + B}, \quad 0 < D_1 < 1 < D_2. \tag{1.9}$$

Clearly $P[A, B] \subset P((1 - A)/(1 - B))$.

In the recent paper, Noor [2] introduced the class $P_k(\alpha)$. We define it as follows. Let $P_k(\alpha), 0 \leq \alpha < p$, be the class of functions $p(z)$ with $p(0) = 1$ and satisfying the property

$$p(z) = \frac{1}{2} \int_0^{2\pi} \frac{1 + (1 - 2\alpha) z e^{-it}}{1 - z e^{-it}} d\mu(t), \tag{1.10}$$

where $\mu(t)$ is a real-valued function of bounded variation on $[0, 2\pi]$ and $\int_0^{2\pi} d\mu(t) = 2$ and $\int_0^{2\pi} |d\mu(t)| \leq k$.

The classes $V_k(\alpha)$ and $R_k(\alpha)$ are related to the class $P_k(\alpha)$ and can be defined as

$$\begin{aligned} f \in V_k(\alpha), \quad \text{iff } \frac{(zf'(z))'}{pf'(z)} \in P_k(\alpha), \quad z \in U, \\ f \in R_k(\alpha), \quad \text{iff } \frac{zf'(z)}{pf(z)} \in P_k(\alpha), \quad z \in U. \end{aligned} \tag{1.11}$$

We define a class $P_k[A,B]$ as follows.

Let $P_k[A,B]$, $k \geq 2$, $-1 \leq B < A \leq 1$, denote the class of p-valent analytic functions $h(z)$ that are represented by

$$h(z) = \left(\frac{k}{4} + \frac{1}{2}\right)h_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right)h_2(z), \quad z \in U, \tag{1.12}$$

where $h_1, h_2 \in P[A,B]$. For $A = 1 - 2\alpha$ $(0 \leq \alpha < p)$ and $B = -1$, it reduces to the class $P_k(\alpha)$ and $P_2(\alpha) = P(\alpha)$ is the class of p-valent analytic functions $h(z)$ with $\operatorname{Re} h(z) > \alpha$, $z \in U$. Taking $A = 1$, $B = -1$, and $p = 1$, we have $P_k[1,-1] = P_k$ (see [3]), and $P_2[1,-1] = P$ is the class of functions with positive real part.

Definition 1.1. A function f, analytic in U, and given by (1.1) is said to be in the class $R_k[A,B]$; $-1 \leq B < A \leq 1$, $k \geq 2$, if and only if,

$$\frac{zf'(z)}{pf(z)} \in P_k[A,B], \quad z \in U. \tag{1.13}$$

For $p = 1$, $R_k[A,B]$ is introduced and studied by Noor [4]. We note that

$$R_k[A,B] \subset R_k\left(\frac{1-A}{1-B}\right) \subset R_k, \tag{1.14}$$

where R_k is the class of functions with bounded radius rotation (see [5]). For $k = 2$, we have

$$R_2[A,B] \equiv S_p^*[A,B] \subset S_p^*\left(\frac{1-A}{1-B}\right) \subset S_p^*, \tag{1.15}$$

where S_p^* is the class of p-valent starlike functions. Similarly, we can define the class $V_k[A,B]$ as follows.

Definition 1.2. A function f, analytic in U, and given by (1.1) is said to be in the class $V_k[A,B]$; $-1 \leq B < A \leq 1$, $k \geq 2$, if and only if,

$$\frac{(zf'(z))'}{pf'(z)} \in P_k[A,B], \quad z \in U. \tag{1.16}$$

It is clear that

$$f \in V_k[A,B], \quad \text{iff } \frac{zf'(z)}{p} \in R_k[A,B], \quad z \in U. \tag{1.17}$$

For $p = 1$, $V_k[A,B]$ is the class introduced and studied by Noor [4]. It is easy to see that,

$$V_k[A,B] \subset V_k\left(\frac{1-A}{1-B}\right) \subset V_k, \tag{1.18}$$

where V_k is the class of functions with bounded boundary rotation see [5]. Also

$$V_2[A,B] \equiv C_p[A,B] \subset C_p\left(\frac{1-A}{1-B}\right) \subset C_p, \tag{1.19}$$

where C_p is the class of p-valent convex functions.

Very recently, Frasin [6], introduced the following general integral operators for p-valent functions,

$$F_p(z) = \int_0^z pt^{p-1}\left(\frac{f_1(t)}{t^p}\right)^{\alpha_1} \cdots \left(\frac{f_n(t)}{t^p}\right)^{\alpha_n} dt, \tag{1.20}$$

$$G_p(z) = \int_0^z pt^{p-1}\left(\frac{f_1'(t)}{pt^{p-1}}\right)^{\alpha_1} \cdots \left(\frac{f_n'(t)}{pt^{p-1}}\right)^{\alpha_n} dt, \quad \text{where } \alpha_i \in \mathbb{C},\ z \in U. \tag{1.21}$$

Clearly, we may see that for $p = 1$, these operators become the general integral operators

$$F_1(z) = F_n(z), \qquad G_1(z) = F_{\alpha_1,\alpha_2,\ldots,\alpha_n}(z), \tag{1.22}$$

introduced and studied by Breaz and Breaz [7] and Breaz et al. [8], (see also [9, 10]).

For $p = n = 1$, $\alpha_1 = \alpha \in [0,1]$ in (1.20), we obtain the integral operator $\int_0^z (f(t)/t)^{\alpha} dt$ studied in [11] and for $p = n = 1$, $\alpha_1 = \delta \in \mathbb{C}$, $|\delta| < 1/4$ in (1.21), we obtain the integral operator $\int_0^z (f'(t))^{\delta} dt$, studied in [12].

2. Main Results

Lemma 2.1. *Let $\beta > 0$, $\beta + \gamma > 0$, $\alpha \in [\alpha_0, 1)$, with $\alpha_0 = \max\{(\beta - \gamma - 1)/2\beta, -\gamma/\beta\}$. If*

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec \frac{1 + (1-2\alpha)z}{1-z}, \tag{2.1}$$

then

$$p(z) \prec Q(z) \prec \frac{1 + (1-2\alpha)z}{1-z}, \tag{2.2}$$

where $Q(z) = 1/\beta G(z) - \gamma/\beta$,

$$G(z) = \int_0^1 \left(\frac{1-z}{1-tz}\right)^{2\beta(1-\alpha)} t^{\beta+\gamma-1}dt = {}_2F_1\left(2\beta(1-\alpha), 1, \beta+\gamma+1; \frac{z}{1-z}\right), \tag{2.3}$$

$$\rho = \rho(\alpha,\beta,\gamma) = \frac{\beta+\gamma}{\beta_2 F_1(2\beta(1-\alpha), 1, \beta+\gamma+1; 1/2)} - \frac{\gamma}{\beta}, \tag{2.4}$$

${}_2F_1$ *denotes the Gauss hypergeometric function. From* (2.2), *we can deduce the sharp result* $p \in P(\rho)$, *where* ρ *is defined in* (2.4). *This result is a special case of one given in [11].*

Proof. To prove this Lemma we use Theorem 3.2j of [11, page 97]. Take $h(z) = (1 + (1 - 2\alpha)z)/(1 - z)$, $0 \le \alpha < 1$ and

$$H(z) = \beta h(z) + \gamma = \frac{a + bz}{1 - z}, \tag{2.5}$$

where $a = \beta + \gamma$ and $b = \beta(1 - 2\alpha) + \gamma$.

Since H is convex to apply Theorem 3.2j of [11, page 97] we only need to determine condition $\operatorname{Re} H(z) > 0$.

The range of $|z| \le 1$ under $H(z)$ is a half plane. In order to satisfy the required condition this half plane needs to lie in the right half plane. This requirement will be satisfied if $\operatorname{Re} H(-1) = \operatorname{Re} H(i)$ and $\operatorname{Re} H(0) > \operatorname{Re} H(-1) \ge 0$. Or we can write it as

$$\beta(1-\alpha) > 0, \quad \beta\alpha + \gamma \ge 0. \tag{2.6}$$

When $\beta > 0$, $\beta + \gamma > 0$, these conditions imply that $\alpha \in [-\gamma/\beta, 1)$, and if $\beta + \gamma > 1$, then $\alpha \in [(\beta - \gamma - 1)/2\beta, 1)$. Hence all the conditions of Theorem 3.2j of [11, page 97] are satisfied for $\alpha \in [\alpha_0, 1)$, with $\alpha_0 = \max\{(\beta - \gamma - 1)/2\beta, -\gamma/\beta\}$, thus we have the required result. □

To show that the solution $Q(z)$ can be represented in terms of hypergeometric functions we take $A = 1 - 2\alpha$, $B = -1$, $n = 1$ in Theorem 3.3d of [11, page 109].

Lemma 2.2. *Let* $f \in V_k(\alpha)$, $0 \le \alpha < p$, $k \ge 2$. *Then* $f \in R_k(\rho)$ *in* U, *where*

$$\rho = \rho(\alpha, p) = \frac{1}{{}_2F_1(2p(1-\alpha), 1, p+1; 1/2)}. \tag{2.7}$$

This result is sharp.

Proof. Let for $k \ge 2$, $z \in U$, we have

$$\frac{zf'(z)}{pf(z)} = h(z) = \left(\frac{k}{4} + \frac{1}{2}\right)h_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right)h_2(z), \tag{2.8}$$

where h, h_i are analytic in U with $h(0) = 1$, $h_i(0) = 1$, $i = 1, 2$.

We define

$$\phi_p(z) = z + \sum_{n=1}^{\infty} \frac{1}{p(1+(n-1))} z^n, \qquad z \in U. \tag{2.9}$$

By using (2.8), with convolution technique, see [13], we have

$$\frac{\phi_p(z)}{z} * h(z) = \left(\frac{k}{4}+\frac{1}{2}\right)\left(\frac{\phi_p(z)}{z} * h_1(z)\right) - \left(\frac{k}{4}-\frac{1}{2}\right)\left(\frac{\phi_p(z)}{z} * h_2(z)\right). \tag{2.10}$$

This implies that,

$$h(z) + \frac{zh'(z)}{ph(z)} = \left(\frac{k}{4}+\frac{1}{2}\right)\left(h_1(z) + \frac{zh_1'(z)}{ph_1(z)}\right) - \left(\frac{k}{4}-\frac{1}{2}\right)\left(h_2(z) + \frac{zh_2'(z)}{ph_2(z)}\right). \tag{2.11}$$

Logarithmic differentiation of (2.8) yields,

$$\frac{(zf'(z))'}{pf'(z)} = h(z) + \frac{zh'(z)}{ph(z)} = \left(\frac{k}{4}+\frac{1}{2}\right)\left(h_1(z) + \frac{zh_1'(z)}{ph_1(z)}\right) - \left(\frac{k}{4}-\frac{1}{2}\right)\left(h_2(z) + \frac{zh_2'(z)}{ph_2(z)}\right). \tag{2.12}$$

Since $(zf'(z))'/pf'(z) \in P_k(\alpha)$, $0 \le \alpha < p$, thus

$$h_i(z) + \frac{zh_i'(z)}{ph_i(z)} \in P(\alpha), \qquad i = 1, 2. \tag{2.13}$$

By using Lemma 2.1 (for $\beta = p$ and $\gamma = 0$), we deduce that $h_i \in P(\rho)$, where ρ is given in (2.7). This estimate is best possible because of the best dominant property of function $Q(z)$, where

$$Q(z) = \frac{1}{{}_2F_1(2p(1-\alpha), 1, p+1; z/(1-z))}, \qquad z \in U. \tag{2.14}$$

□

For $p = 1$, we have the sharp result proved in [14].

We begin with the following theorem.

Theorem 2.3. (i) *Let $\alpha_i > 0$, $f_i \in R_k[A,B]$ for all $i = 1, 2, \ldots, n$, and, $\sum_{i=1}^{n} \alpha_i = 1$. Then the integral operator $F_p \in V_k[A,B]$ in U, where $-1 \le B < A \le 1$, $k \ge 2$.*

(ii) *Let $f_i \in R_k(\alpha)$, $\alpha_i > 0$ for all $i = 1, 2, \ldots, n$ with $\alpha = (1-A)/(1-B)$, $k \ge 2$. If $\sum_{i=1}^{n} \alpha_i = 1$, then the integral operator F_p defined by* (1.20) *also belongs to the class $R_k(\rho)$ in U, where $\rho = \rho(\alpha, p)$ is defined by* (2.7). *This result is sharp.*

Proof (i). From (1.20), we can see that $F_p \in A_p$ in U, and

$$F_p'(z) = pz^{p-1}\left[\left(\frac{f_1(z)}{z^p}\right)^{\alpha_1}\cdots\left(\frac{f_n(z)}{z^p}\right)^{\alpha_n}\right]. \tag{2.15}$$

Differentiating logarithmically and multiplying by z, we obtain,

$$\frac{zF_p''(z)}{F_p'(z)} = (p-1) + \sum_{i=1}^{n}\alpha_i\left(\frac{zf_i'(z)}{f_i'(z)} - p\right), \quad z \in U. \tag{2.16}$$

Thus, we have

$$1 + \frac{zF_p''(z)}{F_p'(z)} = \sum_{i=1}^{n}\alpha_i\left(\frac{zf_i'(z)}{f_i'(z)}\right), \tag{2.17}$$

or

$$\begin{aligned}\frac{\left(zF_p'(z)\right)'}{pF_p'(z)} &= \sum_{i=1}^{n}\alpha_i\left(\frac{zf_i'(z)}{pf_i'(z)}\right)\\ &= \left(\frac{k}{4}+\frac{1}{2}\right)\left(\sum_{i=1}^{n}\alpha_i p_i(z)\right) - \left(\frac{k}{4}-\frac{1}{2}\right)\left(\sum_{i=1}^{n}\alpha_i h_i(z)\right),\end{aligned} \tag{2.18}$$

where $h_i, p_i \in P[A,B]$, for all $i = 1,2,\ldots,n$.

Since $P[A,B]$ is a convex set, see [15], it follows that,

$$\frac{\left(zF_p'(z)\right)'}{pF_p'(z)} = \left(\frac{k}{4}+\frac{1}{2}\right)H_1(z) - \left(\frac{k}{4}-\frac{1}{2}\right)H_2(z), \tag{2.19}$$

where $H_1, H_2 \in P[A,B]$ and therefore,

$$\frac{\left(zF_p'(z)\right)'}{pF_p'(z)} \in P_k[A,B], \quad z \in U. \tag{2.20}$$

This proves the result. □

Substituting $p = 1$, in Theorem 2.3(i), we have the following corollary.

Corollary 2.4. *Let $\alpha_i > 0$, $f_i \in R_k[A,B]$ for all $i = 1,2,\ldots,n$, $-1 \leq B < A \leq 1$, $k \geq 2$. Then the integral operator $F_p \in V_k[A,B]$ in U.*

Remark 2.5. Letting $\alpha_1 = \alpha$, $\alpha_2 = \beta$, and $n = 2$ in Corollary 2.4, we obtain a result due to Noor [4].

For $n = 1$, $\alpha_1 = \alpha = 1$, and $f_1 = f$ in Theorem 2.3(i), we have the following.

Corollary 2.6. *Let $f \in R_k[A,B]$ in U, $-1 \le B < A \le 1$, $k \ge 2$. Then the integral operator $\int_0^z p(f(t)/t)dt \in V_k[A,B]$, $z \in U$.*

Proof (ii). Taking $A = 1-2\alpha$, $B = -1$, with $\alpha = (1-A)/(1-B)$, we have for all $i = 1,2,\ldots,n$,

$$f_i \in R_k[1-2\alpha,-1] = R_k(\alpha), \tag{2.21}$$

using part (i) of Theorem 2.3, we have

$$F_p \in V_k[1-2\alpha,-1] = V_k(\alpha) \quad \text{in } U. \tag{2.22}$$

Now using Lemma 2.2 for $F_p \in V_k(\alpha)$, $\alpha = (1-A)/(1-B)$ implies that

$$F_p \in R_k(\rho), \quad \text{where } \rho = \rho(\alpha,p) \text{ is defined in (2.7).} \tag{2.23}$$

The sharpness of the result is clear from the function $Q(z)$ defined by (2.14). □

For $p = 1$, we have the following corollary.

Corollary 2.7. *Let $\alpha_i > 0$, $f_i \in R_k(\alpha)$ for all $i = 1,2,\ldots,n$, with $\alpha = (1-A)/(1-B)$ and $A = 1-2\alpha$, $B = -1$. Then the integral operator F_p defined by* (1.20) *also belongs to the class $R_k(\rho)$ in U, where*

$$\rho = \rho(\alpha) = \begin{cases} \dfrac{2\alpha-1}{2-2^{2(1-\alpha)}}, & \text{if } \alpha \ne \dfrac{1}{2} \\[2ex] \dfrac{1}{2\ln 2}, & \text{if } \alpha = \dfrac{1}{2}. \end{cases} \tag{2.24}$$

Remark 2.8. Letting $\alpha_1 = \mu$, $\alpha_2 = \eta$, and $n = 2$ in Corollary 2.7, we have the sharp result proved in [14].

For $A = 1$, $B = -1$, and $p = 1$, we have

$$f_i \in R_k(0) \quad \text{implies that } F_p \in V_k\left(\frac{1}{2}\right) \text{ in } U. \tag{2.25}$$

Theorem 2.9. (i) *Let $\alpha_i > 0, f_i \in V_k[A,B]$ for all $i = 1,2,\ldots,n$. If $\sum_{i=1}^n \alpha_i = 1$, then the integral operator G_p defined by* (1.21), *also belongs to the class $V_k[A,B]$ in U, where $-1 \le B < A \le 1$, $k \ge 2$.*

(ii) *Let for $\alpha_i > 0$, $\sum_{i=1}^n \alpha_i = 1$ and $f_i \in V_k(\alpha)$, for all $i = 1,2,\ldots,n$ with $0 \le \alpha < p$, $\alpha = (1-A)/(1-B)$, $k \ge 2$. Then the integral operator $G_p \in R_k(\rho)$ in U, where $\rho = \rho(\alpha,p)$ is defined by* (2.7). *This result is sharp.*

Proof (i). From definition (1.20), we have

$$1+\frac{zG_p''(z)}{G_p'(z)} = p+\sum_{i=1}^{n}\alpha_i\left(\frac{zf_i''(z)}{f_i'(z)}-p+1\right) = \sum_{i=1}^{n}\alpha_i\frac{(zf_i'(z))'}{f_i'(z)}, \tag{2.26}$$

or

$$\frac{\left(zG_p'(z)\right)'}{pG_p'(z)} = \sum_{i=1}^{n}\alpha_i\frac{(zf_i'(z))'}{pf_i'(z)} = \left(\frac{k}{4}+\frac{1}{2}\right)\left(\sum_{i=1}^{n}\alpha_i p_i(z)\right)-\left(\frac{k}{4}-\frac{1}{2}\right)\left(\sum_{i=1}^{n}\alpha_i h_i(z)\right), \tag{2.27}$$

where $h_i, p_i \in P[A,B]$, for all $i = 1,2,\ldots,n$.

Since $P[A,B]$ is a convex set, see [15], it follows that,

$$\frac{\left(zG_p'(z)\right)'}{pG_p'(z)} = \left(\frac{k}{4}+\frac{1}{2}\right)H_1(z)-\left(\frac{k}{4}-\frac{1}{2}\right)H_2(z), \quad z\in U, \tag{2.28}$$

where $H_1, H_2 \in P[A,B]$ and therefore,

$$\frac{\left(zG_p'(z)\right)'}{pG_p'(z)} \in P_k[A,B] \quad \text{in } U. \tag{2.29}$$

This implies that $G_p \in V_k[A,B]$. □

Letting $p = 1$ in Theorem 2.9(i), we have the following corollary.

Corollary 2.10. *Let $\alpha_i > 0$, $f_i \in V_k[A,B]$ for all $i = 1,2,\ldots,n$ and $-1 \le B < A \le 1$, $k \ge 2$. If $\sum_{i=1}^{n}\alpha_i = 1$, then $G_p \in V_k[A,B]$ in U.*

Proof (ii). Taking $A = 1-2\alpha$, $B = -1$, we have for all $i = 1,2,\ldots,n$

$$f_i \in V_k[1-2\alpha,-1] = V_k(\alpha), \quad \text{where } \alpha = \frac{1-A}{1-B}. \tag{2.30}$$

Now using part (i) of Theorem 2.9, we have

$$G_p \in V_k[1-2\alpha,-1] = V_k(\alpha) \quad \text{in } U. \tag{2.31}$$

Now using Lemma 2.2, for $\alpha = (1-A)/(1-B)$, we have

$$G_p \in V_k(\alpha) \text{ implies that } G_p \in R_k(\rho), \text{ in } U, \quad \text{where } \rho = \rho(\alpha, p) \text{ is defined in (2.7).} \tag{2.32}$$

The sharpness of the result is clear from the function $Q(z)$ defined by (2.14). □

For $p = 1$, we have the following corollary.

Corollary 2.11. (i) *Let $\alpha_i > 0$, $f_i \in V_k(\alpha)$, $i = 1, 2, \ldots, n$, with $\alpha = (1-A)/(1-B)$ and $A = 1-2\alpha$, $B = -1$. Then $G_p \in R_k(\rho)$ in U, where $\rho = \rho(\alpha)$ and defined in* (2.24).

Also for $A = 1$, $B = -1$, we have.

(ii) *If $f_i \in V_k(0)$ for all $i = 1, 2, \ldots, n$, then $G_p \in R_k(1/2)$ in U.*

Acknowledgments

Authors are thankful to anonymous referees for their very constructive comments to improve this paper. Second author (P. Goswami) is also grateful to Professor S. P. Goyal for his guidance and constant encouragement.

References

[1] W. Janowski, "Some extremal problems for certain families of analytic functions. I," *Annales Polonici Mathematici*, vol. 28, pp. 297–326, 1973.

[2] K. I. Noor, "Genealized integral operator and multivalent functions," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 6, no. 2, Article 51, 2005.

[3] B. Pinchuk, "Functions of bounded boundary rotation," *Israel Journal of Mathematics*, vol. 10, pp. 6–16, 1971.

[4] K. Inayat Noor, "Some properties of analytic functions with bounded radius rotation," *Complex Variables and Elliptic Equations*, vol. 54, no. 9, pp. 865–877, 2009.

[5] A. W. Goodman, *Univalent Functions*, vol. 1 and 2, Mariner, Tampa, Fla, USA, 1983.

[6] B. A. Frasin, "Convexity of integral operators of p-valent functions," *Mathematical and Computer Modelling*, vol. 51, no. 5-6, pp. 601–605, 2010.

[7] D. Breaz and N. Breaz, "Some convexity properties for a general integral operator," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 7, no. 5, Article 177, 2006.

[8] D. Breaz, S. Owa, and N. Breaz, "A new integral univalent operator," *Acta Universitatis Apulensis*, no. 16, pp. 11–16, 2008.

[9] D. Breaz, "A convexity property for an integral operator on the class $S_p(\beta)$," *General Mathematics*, vol. 15, no. 2, pp. 177–183, 2007.

[10] D. Breaz and H. Güney, "The integral operator on the classes $S_\alpha(b)$ and $C_\alpha(b)$," *Journal of Mathematical Inequalities*, vol. 2, no. 1, pp. 97–100, 2008.

[11] S. S. Miller and P. T. Mocanu, *Differential Subordinations: Theory and Applications*, vol. 225 of *Monographs and Textbooks in Pure and Applied Mathematics*, Marcel Dekker, New York, NY, USA, 2000.

[12] N. N. Pascu and V. Pescar, "On the integral operators of Kim-Merkes and Pfaltzgraff," *Mathematica*, vol. 32(55), no. 2, pp. 185–192, 1990.

[13] K. I. Noor, "On some differential operators for certain classes of analytic functions," *Journal of Mathematical Inequalities*, vol. 2, no. 1, pp. 129–137, 2008.

[14] K. I. Noor, W. Ul-Haq, M. Arif, and S. Mustafa, "On bounded boundary and bounded radius rotations," *Journal of Inequalities and Applications*, vol. 2009, Article ID 813687, 12 pages, 2009.

[15] K. I. Noor, "On subclasses of close-to-convex functions of higher order," *International Journal of Mathematics and Mathematical Sciences*, vol. 15, no. 2, pp. 279–289, 1992.