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Research Article

Monomiality Principle and Eigenfunctions of Differential Operators

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We apply the so-called *monomiality principle* in order to construct eigenfunctions for a wide set of ordinary differential operators, relevant to special functions and polynomials, including Bessel functions and generalized Gould-Hopper polynomials.

1. Introduction

In many paper the so-called *monomiality principle*, introduced by Dattoli et al. [1], was used in order to study in a standard way the most important properties of special polynomials and functions [2].

In this paper, we show that the abstract framework of monomiality can be used even to find in a constructive way the eigenfunctions of a wide set of linear differential operators connected with the Laguerre-type exponentials introduced in [3].

In this paper, we limit ourselves to consider the first Laguerre derivative $D_L := DxD$, so that we substitute the derivative D and multiplication operator x. with the corresponding derivative and multiplication operators $\widehat{P}$ and $\widehat{M}$, relevant to a given set of special polynomials or functions.

The same procedure could be generalized by considering (for any integer n) the higher-order Laguerre derivatives $D_{nL} := Dx \cdots DxDxD$ (containing $n + 1$ ordinary derivatives), showing that this method can be used to obtain eigenfunctions for each one of the infinite many operators obtained by using the same substitutions described before.

It can be noticed that this gives a further proof of the power of the monomiality technique.

2. The Monomiality Principle

The idea of monomiality traces back to Steffensen [4], who suggested the concept of *poweroid*, but only recently this idea was systematically used by Dattoli [2].

It was shown in [5] that all polynomial families are essentially the same, since it is possible to obtain one of them by transforming each other by means of suitable operators, called *derivative* and *multiplication* operators. However, the derivative and multiplication operators, relevant to a general polynomial set, are expressed by formal series of the derivative operator, so that it is in general impossible to obtain sufficiently simple formulas to work with.

However, for particular polynomials sets, relevant to suitable generating functions, the above-mentioned formal series reduce to finite sums, so that their main properties can be easily derived. The leading set in this field is given by the Hermite-Kampé de Fériet (shortly H-KdF) also called Gould-Hopper polynomials [6, 7].

Following Dattoli [2], we start with the following definition.

Definition 2.1. The polynomial set $\{p_n(x)\}_{n\in\mathbf{N}}$ is a quasimonomial set if there exist two linear operators $\widehat{P}$ and $\widehat{M}$, called, respectively, derivative operator and multiplication operator, verifying ($\forall n \in \mathbf{N}$) the identities

$$\begin{aligned} \widehat{P}(p_n(x)) &= np_{n-1}(x), \\ \widehat{M}(p_n(x)) &= p_{n+1}(x). \end{aligned} \tag{2.1}$$

The $\widehat{P}$ and $\widehat{M}$ operators are shown to satisfy the commutation property

$$\left[\widehat{P},\widehat{M}\right] = \widehat{P}\widehat{M} - \widehat{M}\widehat{P} = \widehat{1}, \tag{2.2}$$

and thus display a Weyl group structure.

If the considered polynomial set $\{p_n(x)\}$ is a quasi-monomial set, then its properties can be easily derived from those of the $\widehat{P}$ and $\widehat{M}$ operators. In fact the following holds.

(i) If $\widehat{P}$ and $\widehat{M}$ have a differential realization, then the polynomial $p_n(x)$ satisfies the differential equation

$$\widehat{M}\widehat{P}(p_n(x)) = np_n(x). \tag{2.3}$$

(ii) Assuming here and in the following $p_0(x) = 1$, then $p_n(x)$ can be explicitly constructed as

$$p_n(x) = \widehat{M}^n(1). \tag{2.4}$$

(iii) The last identity implies that the exponential generating function of $p_n(x)$ is given by

$$e^{t\widehat{M}}(1) = \sum_{n=0}^{\infty} \frac{\left(t\widehat{M}\right)^n}{n!}(1) = \sum_{n=0}^{\infty} \frac{t^n}{n!}\ \widehat{M}^n(1), \tag{2.5}$$

and therefore

$$e^{t\widehat{M}}(1) = \sum_{n=0}^{\infty} \frac{t^n}{n!}\ p_n(x). \tag{2.6}$$

3. Laguerre-Type Exponentials

For every positive integer n, the nL-exponential function is defined by

$$e_n(x) := \sum_{k=0}^{\infty} \frac{x^k}{(k!)^{n+1}}. \tag{3.1}$$

This function reduces to the classical exponential when $n = 0$, so that we can put $e_0(x) := e^x$.

Consider the operator (containing $n+1$ derivatives)

$$D_{nL} : = Dx \cdots DxDxD = S(n+1,1)D + S(n+1,2)xD^2 + \cdots + S(n+1,n+1)x^n D^{n+1}, \tag{3.2}$$

where $S(n+1,1),\ S(n+1,2), \dots, S(n+1,n+1)$ denote Stirling numbers of the second kind. In [3] (see also [8, 9] for applications), the following theorem is proved.

Theorem 3.1. *Let a be an arbitrary real or complex number. The nth Laguerre-type exponential $e_n(ax)$ is an eigenfunction of the operator D_{nL}, that is,*

$$D_{nL} e_n(ax) = a e_n(ax). \tag{3.3}$$

For $n = 0$, we have $D_{0L} := D$, and therefore (3.3) reduces to the classical property of the exponential function

$$De^{ax} = ae^{ax}. \tag{3.4}$$

It is worth noting that for all n, the nL-exponential function satisfies $e_n(0) = 1$, and it is an increasing convex function whenever $x \geq 0$; furthermore,

$$e^x = e_0(x) > e_1(x) > e_2(x) > \cdots > e_n(x) > \cdots, \quad \forall x > 0. \tag{3.5}$$

According to [10], for all $s = 1, 2, 3, \ldots$, it follows that

$$(DxD)^s = D^s x^s D^s, \qquad (DxDxD)^s = D^s x^s D^s x^s D^s, \tag{3.6}$$

and so on for every D_{nL} $(n = 1, 2, 3, \ldots)$.

4. Eigenfunctions of Differential Operators

We start assuming $a = 1$, $n = 1$, in (3.3), so that

$$DxD\left(\sum_{k=0}^{\infty} \frac{x^k}{(k!)^2}\right) = \sum_{k=0}^{\infty} \frac{x^k}{(k!)^2}. \tag{4.1}$$

By applying the monomiality principle to (4.1), we find the following result.

Theorem 4.1. *Let $\{p_k(x)\}$ be a polynomial (or function) set, and denote by $\widehat{P}$ and $\widehat{M}$ the corresponding derivative and multiplication operators. Then*

$$\widehat{P}\widehat{M}\widehat{P}\left(\sum_{k=0}^{\infty} \frac{p_k(x)}{(k!)^2}\right) = \sum_{k=0}^{\infty} \frac{p_k(x)}{(k!)^2}. \tag{4.2}$$

Therefore, the operator $\widehat{P}\widehat{M}\widehat{P}$ admits the eigenfunction $\sum_{k=0}^{\infty} p_k(x)/(k!)^2$.

Proof. Searching for an eigenfunction of the form $\sum_{k=0}^{\infty} a_k p_k(x)$, normalized assuming $a_0 := 1$, we find that, by using properties (2.1),

$$\begin{aligned} \widehat{P}\widehat{M}\widehat{P}\left(\sum_{k=0}^{\infty} a_k p_k(x)\right) &= \sum_{k=0}^{\infty} a_k \widehat{P}\widehat{M}\widehat{P}(p_k(x)) \\ &= \sum_{k=1}^{\infty} a_k k^2 p_{k-1}(x) = \sum_{k=0}^{\infty} a_{k+1}(k+1)^2 p_k(x) \end{aligned} \tag{4.3}$$

and consequently, recalling $a_0 := 1$,

$$\frac{a_{k+1}}{a_k} = \frac{1}{(k+1)^2} \Longleftrightarrow a_k = \frac{1}{(k!)^2}. \tag{4.4}$$

□

By the same method, we find, for any integer n, the general result.

Theorem 4.2. *The operator $\widehat{P}\widehat{M}\widehat{P}\cdots\widehat{M}\widehat{P}$, including $n+1$ copies of the derivative operator $\widehat{P}$, admits the eigenfunction $\sum_{k=0}^{\infty} p_k(x)/(k!)^{n+1}$, that is,*

$$\widehat{P}\widehat{M}\widehat{P}\cdots\widehat{M}\widehat{P}\left(\sum_{k=0}^{\infty} \frac{p_k(x)}{(k!)^{n+1}}\right) = \sum_{k=0}^{\infty} \frac{p_k(x)}{(k!)^{n+1}}. \tag{4.5}$$

We want to show, in the following sections, several examples of this method, deriving explicit eigenfunctions for a large set of differential operators, connected with classical polynomial (or function) sets.

5. Hermite and Gould-Hopper Polynomials

5.1. Hermite Polynomials

Consider first the Hermite polynomials defined by the Rodrigues formula

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}. \tag{5.1}$$

Recalling

$$\begin{aligned} \left(\frac{1}{2}\frac{d}{dx}\right) H_n(x) &= nH_{n-1}(x), \\ \left(2x - \frac{d}{dx}\right) H_n(x) &= H_{n+1}(x), \end{aligned} \tag{5.2}$$

we have

$$\widehat{P} = \frac{1}{2}\frac{d}{dx}, \qquad \widehat{M} = 2x - \frac{d}{dx}, \tag{5.3}$$

so that we find the operator

$$\widehat{P}\widehat{M}\widehat{P} = \frac{1}{2}\frac{d}{dx} + \frac{1}{4}\left(2x - \frac{d}{dx}\right)\frac{d^2}{dx^2} \tag{5.4}$$

and the corresponding eigenfunction

$$\sum_{n=0}^{+\infty} \frac{H_n(x)}{(n!)^2} = \sum_{n=0}^{+\infty} \sum_{h=0}^{[n/2]} (-1)^h \frac{(2x)^{n-2h}}{h!(n-2h)!n!}. \tag{5.5}$$

5.2. Gould-Hopper Polynomials

They are defined by [11]

$$H_n^{(m)}(x,y) := e^{y\partial_x^m} x^n = n! \sum_{h=0}^{[n/m]} \frac{y^h x^{n-mh}}{h!(n-mh)!}. \tag{5.6}$$

We have in this case

$$\hat{P} = \frac{\partial}{\partial x}, \qquad \widehat{M} = x + my\frac{\partial^{m-1}}{\partial x^{m-1}}, \tag{5.7}$$

so that we find the operator

$$\hat{P}\widehat{M}\hat{P} = \frac{\partial}{\partial x} + x\frac{\partial^2}{\partial x^2} + my\frac{\partial^{m+1}}{\partial x^{m+1}} \tag{5.8}$$

and the corresponding eigenfunction

$$\sum_{n=0}^{\infty}\frac{H_n^{(m)}(x)}{(n!)^2} = \sum_{n=0}^{\infty}\sum_{h=0}^{[n/m]}\frac{y^h x^{n-mh}}{h!(n-mh)!n!}. \tag{5.9}$$

6. Generalized Gould-Hopper Polynomials

In [12] a general set of polynomials, generalizing the Gould-Hopper ones is introduced. For shortness we will call them GGHP.

They are defined by the operational rule

$$G_n(x,g) = e^{g(D)}x^n, \tag{6.1}$$

where $g(t)$ is an analytic function and $D := D_x = d/dx$.

Of course, if $g(t) = yt^m$, then the Gould-Hopper polynomials are recovered.

It is worth noting that if $g(t)$ is a polynomial vanishing at $t = 0$, assuming $g(t) = x_1t + x_2t^2 + \cdots + x_rt^r$, the GGHP give back the many-variable one-index Hermite polynomials (see, e.g., [13]). Extensions of the last ones to many indices are given in [14].

The $G_n(x,g)$ satisfy [12]

$$\begin{gathered} DG_n(x,g) = nG_{n-1}(x,g), \\ \left[x + g'(D)\right]G_n(x,g) = G_{n+1}(x,g). \end{gathered} \tag{6.2}$$

Therefore, they belong to the Appell class, and

$$\hat{P} = D_x = D, \qquad \widehat{M} = x + g'(D_x) = x + g'(D). \tag{6.3}$$

As a consequence,

$$\hat{P}\widehat{M}\hat{P} = D\left(x + g'(D)\right)D = D + xD^2 + g'(D)D^2 = D + \left[x + g'(D)\right]D^2, \tag{6.4}$$

and the corresponding eigenfunction will be

$$\sum_{n=0}^{\infty}\frac{G_n(x,g)}{(n!)^2}. \tag{6.5}$$

7. Bessel Functions of the First Kind

The Bessel function of the first kind,

$$J_k(z) = \sum_{h=0}^{\infty} (-1)^h \frac{(z/2)^{2h+k}}{h!(h+k)!}, \tag{7.1}$$

satisfies the recurrence relations

$$\begin{aligned} \frac{2k}{z} J_k(z) &= J_{k-1}(z) + J_{k+1}(z), \\ 2\frac{d}{dz} J_k(z) &= J_{k-1}(z) - J_{k+1}(z). \end{aligned} \tag{7.2}$$

Adding and subtracting these equalities, we get

$$\begin{aligned} \left(\frac{d}{dz} + \frac{k}{z}\right) J_k(z) &= J_{k-1}(z), \\ \left(-\frac{d}{dz} + \frac{k}{z}\right) J_k(z) &= J_{k+1}(z), \end{aligned} \tag{7.3}$$

and, therefore, we have

$$\widehat{P} = k\frac{d}{dz} + \frac{k^2}{z}, \qquad \widehat{M} = -\frac{d}{dz} + \frac{k}{z}, \tag{7.4}$$

so that we find the operator

$$\widehat{P}\widehat{M}\widehat{P} = -k^2\frac{d^3}{dz^3} + \frac{k^3(k-1)}{z^2}\frac{d}{dz} + \frac{k^3(k-2)(k+1)}{z^3}, \tag{7.5}$$

and the corresponding eigenfunction

$$\sum_{k=0}^{+\infty} \frac{J_k(z)}{(k!)^2} = \sum_{k=0}^{+\infty} \sum_{h=0}^{\infty} (-1)^h \frac{(z/2)^{2h+k}}{h!(h+k)!(k!)^2}. \tag{7.6}$$

Note that the negative integer values of the index k do not contribute to the series.

8. A Direct Proof for Bessel Functions

Considering the case of Bessel functions J_n, we could proceed as follows.

(a) We define the "number operator" $\widehat{k}$ associated to them by putting

$$\widehat{k} J_n(z) := n J_n(z). \tag{8.1}$$

(b) And we define the relevant shift-down operator by

$$\widehat{E}_- := \frac{d}{dz} + \frac{1}{z}\widehat{k} \tag{8.2}$$

(note that z and $\widehat{k}$ do not commute).

Applying $\widehat{E}_-$ to J_n yields

$$\widehat{E}_- J_n(z) = \left(\frac{d}{dz} + \frac{1}{z}\widehat{k}\right) J_n(z) = \left(\frac{d}{dz} + \frac{n}{z}\right) J_n(z) = J_{n-1}(z). \tag{8.3}$$

Furthermore, by iteration,

$$\left(\widehat{E}_-\right)^2 J_n(z) = \widehat{E}_-\left(\widehat{E}_- J_n(z)\right) = \left(\frac{d}{dz} + \frac{n-1}{z}\right)\left(\frac{d}{dz} + \frac{n}{z}\right) J_n(z) = J_{n-2}(z), \tag{8.4}$$

and in general

$$\left(\widehat{E}_-\right)^m J_n(z) = \left(\frac{d}{dz} + \frac{n-(m-1)}{z}\right) \cdots \left(\frac{d}{dz} + \frac{n}{z}\right) J_n(z) = J_{n-m}(z). \tag{8.5}$$

The *derivative operator* $\widehat{P}$ for the Bessel functions can be written, in terms of the *number operator*, as follows:

$$\widehat{P} = \left(\widehat{k} + \widehat{1}\right)\widehat{E}_-. \tag{8.6}$$

In fact, by using the above rules, we have

$$\widehat{P} J_n(z) = \left(\widehat{k} + \widehat{1}\right)\widehat{E}_- J_n(z) = \left(\widehat{k} + \widehat{1}\right) J_{n-1}(z) = n J_{n-1}(z) = J_{n-1} + \frac{z}{2}\left[J_n(z) + J_{(n-2)}(z)\right]. \tag{8.7}$$

Since the number operator does not commute with z and d/dz, it does not commute with $\widehat{P}$ too.

(c) We define the shift-up operator by

$$\widehat{M} := \widehat{E}_+ := -\frac{d}{dz} + \frac{1}{z}\widehat{k}. \tag{8.8}$$

The action of the Laguerre derivative on Bessel functions becomes

$$\left[\left(\hat{k}+\hat{1}\right)\hat{E}_-\right]\hat{E}_+\left[\left(\hat{k}+\hat{1}\right)\hat{E}_-\right]\frac{J_n(z)}{n!} = n\frac{J_{n-1}(z)}{(n-1)!}. \tag{8.9}$$

The Bessel equation follows by using the factorization method:

$$\hat{E}_+\hat{E}_-J_n(z) = \left(-\frac{d}{dz}+\frac{n-1}{z}\right)\left(\frac{d}{dz}+\frac{n}{z}\right)J_n(z) = J_n(z). \tag{8.10}$$

Note that the derivative operator can be iterated without problem, since

$$\hat{P}^2 J_n(z) = \left(\hat{k}+\hat{1}\right)\hat{E}_-\left(\hat{k}+\hat{1}\right)\hat{E}_-J_n(z) = n(n-1)J_{n-2}(z), \tag{8.11}$$

and in general

$$\hat{P}^m J_n(z) = \frac{n!}{(n-m)!}J_{n-m}(z). \tag{8.12}$$

Note that

$$\hat{E}_-J_n(\lambda z) = \lambda J_{n-1}(\lambda z). \tag{8.13}$$

By using the preceding equations, it is easy to see that the function

$$F(z) := \sum_{s=0}^{\infty}\frac{\lambda^s J_s(z)}{s!} \tag{8.14}$$

is an eigenfunction of the operator $\hat{P}$. Therefore, the operator $\hat{P}\widehat{M}\hat{P}$ admits the eigenfunction

$$\sum_{k=0}^{\infty}\frac{J_k(z)}{[k!]^2}, \tag{8.15}$$

as can be checked directly.

Of course it should be very complicated to write similar equations for each monomial set, but this is useless, since the abstract Theorem 4.1 guarantees the validity of our result.

9. Parabolic Cylinder Functions

We deal with the functions

$$D_n(z) := 2^{-(n/2)}e^{-(z^2/4)}H_n\left(\frac{z}{\sqrt{2}}\right), \tag{9.1}$$

where H_n denotes the ordinary Hermite polynomials (5.1). Taking into account the recurrence relations

$$\begin{aligned} zD_n(z) &= nD_{n-1}(z) + D_{n+1}(z), \\ \frac{d}{dz}D_n(z) + \frac{z}{2}D_n(z) &= nD_{n-1}(z), \end{aligned} \tag{9.2}$$

we have in this case [15]

$$\widehat{P} = \frac{z}{2} + \frac{d}{dz}, \qquad \widehat{M} = \frac{z}{2} - \frac{d}{dz}. \tag{9.3}$$

Consequently, we find the operator

$$\widehat{P}\widehat{M}\widehat{P} = -\frac{d^3}{dz^3} + \frac{z^2+2}{4}\frac{d}{dz} + \frac{z^3+2z}{8}, \tag{9.4}$$

and the corresponding eigenfunction

$$\sum_{n=0}^{+\infty} \frac{D_n(z)}{(n!)^2} = e^{-z^2/4} \sum_{n=0}^{+\infty} \frac{He_n(z)}{(n!)^2}. \tag{9.5}$$

10. Bessel-Clifford Functions of the First Kind

They are defined by

$$C_n(z) := \frac{1}{\Gamma(n+1)}\, {}_0F_1(-, n+1; z) \tag{10.1}$$

and are a particular case of Wright functions [2, 16], connected with the Bessel functions of first kind by

$$J_n(z) = \left(\frac{z}{2}\right)^n C_n\left(-\frac{z^2}{4}\right). \tag{10.2}$$

The relevant generating function is given by

$$e^{t+(z/t)} = \sum_{n=-\infty}^{+\infty} C_n(z)t^n. \tag{10.3}$$

From the recurrence relations

$$\begin{aligned} \frac{d}{dz}C_n(z) &= C_{n+1}(z), \\ zC_{n+2}(z) + (n+1)C_{n+1}(z) &= C_n(z), \end{aligned} \tag{10.4}$$

we have in this case

$$\widehat{P} = nz\frac{d}{dz} + n^2, \qquad \widehat{M} = \frac{d}{dz}. \tag{10.5}$$

Therefore, we find the operator

$$\widehat{P}\widehat{M}\widehat{P} = n^2 z^2 \frac{d^3}{dz^3} + n^2(n+2)z\,\frac{d^2}{dz^2} + n^3\frac{d}{dz}, \tag{10.6}$$

and the corresponding eigenfunction

$$\sum_{k=0}^{+\infty} \frac{C_k(z)}{(k!)^2} = \sum_{k=0}^{+\infty}\sum_{h=0}^{+\infty} \frac{z^h}{h!(h+k)!(k!)^2}. \tag{10.7}$$

11. Modified Laguerre Polynomials

They are defined by

$$f_k^{\alpha}(x) = \sum_{h=0}^{k} \frac{\Gamma(k-h+\alpha)}{h!(k-h)!\Gamma(\alpha)} x^h, \tag{11.1}$$

and are related (see [16]) to the classical Laguerre polynomials $L_n^{(\alpha)}(x)$ and the Poisson-Charlier polynomials $c_n(x;\alpha)$ by

$$f_k^{-\alpha}(x) = (-1)^k L_k^{(\alpha-k)}(x) = \frac{x^k}{k!}\, c_k(x;\alpha). \tag{11.2}$$

We have in this case

$$\widehat{P} = k\frac{d}{dx} \tag{11.3}$$

coming directly from the differentiation of (11.1). Using the recurrence relation

$$f_{k+1}^{\alpha}(x) = \frac{1}{k+1}[k+(x+\alpha)]f_k^{\alpha}(x) - x\, f_{k-1}^{\alpha}(x), \tag{11.4}$$

we easily obtain the operator

$$\widehat{M} = \frac{1}{k+1}\quad [k+(x+\alpha)] - x\frac{d}{dx}. \tag{11.5}$$

Consequently, we find

$$\widehat{P}\widehat{M}\widehat{P} = k^2\left\{\frac{1}{k+1}\frac{d}{dx} + \frac{x+\alpha-1}{k+1}\frac{d^2}{dx^2} - x\frac{d^3}{dx^3}\right\}, \tag{11.6}$$

and the corresponding eigenfunction

$$\sum_{k=0}^{+\infty}\frac{f_k^{\alpha}(x)}{(k!)^2} = \sum_{k=0}^{+\infty}\sum_{h=0}^{k}\frac{\Gamma(k-h+\alpha)}{h!(k-h)!\Gamma(\alpha)(k!)^2}x^h. \tag{11.7}$$

12. Confluent Hypergeometric Polynomials

The Confluent Hypergeometric function $\Phi(a,c;z) := {}_1F_1(a,c;z)$ reduces to a polynomial when $a = -m$ (where m denotes an integer number).

Recalling the recurrences [15]

$$\begin{gathered}\left(a + z\frac{d}{dz}\right)\Phi(a,c;z) = a\,\Phi(a-1,c;z),\\ \left(\frac{c-a-z}{c-a} + \frac{z}{c-a}\frac{d}{dz}\right)\Phi(a,c;z) = \Phi(a+1,c;z),\end{gathered} \tag{12.1}$$

we have in this case

$$\widehat{P} = a + z\,\frac{d}{dz}, \qquad \widehat{M} = \frac{c-a-z}{c-a} + \frac{z}{c-a}\frac{d}{dz}, \tag{12.2}$$

so that putting again $a = -m$ we find the operator

$$\begin{aligned}\widehat{P}\widehat{M}\widehat{P} = \frac{1}{c-a}\bigg\{ & z^3\frac{d^3}{dz^3} + z^2(c+3-z)\frac{d^2}{dz^2}\\ & + z[(c-a-z+1)(a+1)-z]\frac{d}{dz} + a[(c-a-z)a-z]\bigg\}\end{aligned} \tag{12.3}$$

and the corresponding eigenfunction

$$\sum_{m=0}^{+\infty}\frac{\Phi(-m,c;z)}{(m!)^2}. \tag{12.4}$$

13. Hypergeometric Polynomials

A similar result holds for the Hypergeometric polynomials $F(a,b,c;z) := {}_2F_1(a,b,c;z)$ when $a = -m$ (integer number).

Recalling the recurrences [15]

$$\left(a+z\frac{d}{dz}\right)F(a,b,c;z)=a\ F(a+1,b,c;z),$$
$$\left(\frac{(a-c)+bz}{a-c}-\frac{z(1-z)}{a-c}\frac{d}{dz}\right)F(a,b,c;z)=F(a-1,b,c;z), \tag{13.1}$$

we have in this case

$$\widehat{M}=1+\frac{z}{a}\frac{d}{dz},\qquad \widehat{P}=\frac{az(1-z)}{c-a}\frac{d}{dz}-\frac{baz}{c-a}+a, \tag{13.2}$$

so that putting again $a=-m$ we find the operator

$$\begin{aligned}\widehat{P}\widehat{M}\widehat{P}=\frac{a}{(c-a)^2}\Bigg\{&z^3(1-z)^2\frac{d^3}{dz^3}+z^2(1-z)[c+3-z(a+b+5)]\frac{d^2}{dz^2}\\&+z[(1-z)(a+1)(1-2z+c-a-bz)+z(a+b-c-2+2z)]\frac{d}{dz}\\&+[bz(a+1)(2z-c+a+1)+a(c-a)(c-a-bz)]\Bigg\}\end{aligned} \tag{13.3}$$

and the corresponding eigenfunction

$$\sum_{m=0}^{+\infty}\frac{F(-m,b,c;z)}{(m!)^2}. \tag{13.4}$$

14. Conclusion

The above consideration shows that, even in the most simple case of the first-order Laguerre derivative $D_L := DxD$, the use of monomiality gives us the possibility to construct explicitly eigenfunctions for a wide set of linear differential operators, by using a very simple and standard method.

The extension of this method to the higher-order Lagaerre derivatives $D_{nL} := Dx\cdots DxDxD$ (containing $n+1$ ordinary derivatives) could be obtained in a similar way; however, the manual computation becomes very hard when the order n increases. The use of symbolic computer algebra programs like Mathematica could be exploited in order to obtain the relevant formulas in an easy way. However, we think that this extension does not add further elements of novelty to the above-described methodology.

Similar results can be obtained by using the operator

$$D_L+mD=DxD+mD=D(xD+m) \tag{14.1}$$

(m real or complex constant), or more generally the operator

$$D(xD+m)^n = \sum_{k=0}^{n} \binom{n}{k} D_{kL} m^{n-k} \tag{14.2}$$

(n positive integral number), and the corresponding eigenfunctions

$$\sum_{k=0}^{\infty} \frac{x^k}{k!(k+m)!}, \tag{14.3}$$

or

$$\sum_{k=0}^{\infty} \frac{x^k}{(k!)^n (k+m)!}. \tag{14.4}$$

The monomiality principle ensures that, by considering a quasi-monomial system $\{p_k(x)\}$ and the relevant derivative $\widehat{P}$ and multiplication $\widehat{M}$, the operators $\widehat{P}\widehat{M}\widehat{P} + m\widehat{P}$, or more generally $\widehat{P}(\widehat{M}\widehat{P} + m\widehat{I})^n$, admit the eigenfunctions

$$\sum_{k=0}^{\infty} \frac{p_k(x)}{k!(k+m)!} \tag{14.5}$$

or

$$\sum_{k=0}^{\infty} \frac{p_k(x)}{(k!)^n (k+m)!}. \tag{14.6}$$

The explicit expression of these operators in the case of several quasi-monomial systems will be considered in a forthcoming paper.

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