Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2011, Article ID 976723, 15 pages
doi:10.1155/2011/976723

Research Article

A Rademacher-Type Formula for *pod*(n)

Vyacheslav Kiria-Kaiserberg

Department of Mathematical Sciences, Georgia Southern University, Statesboro, GA 30458, USA

Correspondence should be addressed to Vyacheslav Kiria-Kaiserberg, vslkrg@yahoo.com

Received 24 May 2011; Accepted 7 September 2011

Academic Editor: Marianna Shubov

A Rademacher-type formula for the Fourier coefficients of the generating function for the partitions of n where no odd part is repeated is presented.

1. Partitions

A *partition* of a positive integer n is a representation of n as a sum of positive integers where order of summands (parts) does not matter. Let $p(n)$ represent the number of partitions of n. In 1937, Rademacher [1, 2] was able to express $p(n)$ as a convergent series:

$$p(n) = \frac{1}{\pi\sqrt{2}}\sum_{k=1}^{\infty} A_k(n)\sqrt{k}\frac{d}{dn}\left(\frac{\sinh\left((\pi/k)\sqrt{(2/3)(n-1/24)}\right)}{\sqrt{n-1/24}}\right), \tag{1.1}$$

where

$$A_k(n) = \sum_{0\le h<k\gcd(h,k)=1} e^{\pi i(s(h,k)-2nh/k)} \tag{1.2}$$

is a Kloosterman sum and

$$s(h,k) = \sum_{r=1}^{k-1}\frac{r}{k}\left(\frac{hr}{k} - \left\lfloor\frac{hr}{k}\right\rfloor - \frac{1}{2}\right) \tag{1.3}$$

is a Dedekind sum.

In 2011, Bruinier and Ono [3] announced a new formula that expresses $p(n)$ as a *finite* sum.

1.1. Formula for $\tilde{p}(n)$

Let

$$f(q) := \sum_{n=0}^{\infty} p(n)q^n = \prod_{m=1}^{\infty} \frac{1}{1-q^m} \tag{1.4}$$

be Euler's generating function for $p(n)$. H. Rademacher used the classical circle method to find the coefficients of q^n. There are many other infinite products to which this method could be applied. We introduce one of these infinite products here and derive the formula for the coefficients of q^n. Define

$$G(q) := \prod_{m=1}^{\infty} \frac{(1-q^m)}{(1-q^{2m})^2} = \frac{\left(f(q^2)\right)^2}{f(q)}. \tag{1.5}$$

Let $\tilde{p}(j)$ denote the coefficient of q^j in the expansion of $G(q)$, that is,

$$G(q) = \sum_{j=0}^{\infty} \tilde{p}(j)q^j. \tag{1.6}$$

We will find a closed expression for $\tilde{p}(j)$. Note that

$$G(-q) = \prod_{m=1}^{\infty} \frac{1+q^{2m-1}}{1-q^{2m}} = \sum_{n=0}^{\infty} pod(n)q^n, \tag{1.7}$$

where $pod(n)$ equals the number of partitions of n where no odd part is repeated. Thus

$$\begin{aligned} pod(n) &= (-1)^n \tilde{p}(n) \\ &= (-1)^n \frac{2}{\pi} \sum_{k=1 (k,2)=2}^{\infty} \sqrt{k} \sum_{0 \le h < k (h,k)=1} \frac{\omega^2(h,k/2)}{\omega(h,k)} e^{-2\pi i n h/k} \frac{d}{dn} \\ &\quad \times \left(\frac{\sinh\left(\pi\sqrt{8n-1}/2k\right)}{\sqrt{8n-1}} \right) \end{aligned} \tag{1.8}$$

which is simpler than the one given by Sills [4, page 4, Equation (1.13)] in 2010:

$$pod(n) = \frac{2}{\pi}\sum_{k=1}^{\infty}\sqrt{k\left(1-(-1)^k+\left\lfloor\frac{(4,k)}{4}\right\rfloor\right)}\sum_{0\le h<k(h,k)=1}\frac{\omega(h,k)\omega(4h/(k,4),k/(k,4))}{\omega(2h/(k,2),k/(k,2))}$$
$$\times e^{-2\pi inh/k}\frac{d}{dn}\left(\frac{\sinh\left(\pi\sqrt{(k,4)(8n-1)}/4k\right)}{\sqrt{8n-1}}\right), \quad (1.9)$$

where $\omega(h,k)$ is defined as

$$\omega(h,k) = \exp\left(\pi i\sum_{r=1}^{k-1}\frac{r}{k}\left(\frac{hr}{k}-\left\lfloor\frac{hr}{k}\right\rfloor-\frac{1}{2}\right)\right). \quad (1.10)$$

2. Evaluation of the Path Integral

2.1. Convergence and Cauchy Residue Theorem

Considering q as a complex variable in

$$\prod_{m=1}^{\infty}\frac{(1-q^m)}{(1-q^{2m})^2} = \prod_{m=1}^{\infty}\frac{1}{(1+q^m)(1-q^{2m})} = \prod_{m=1}^{\infty}\frac{1}{(1-(-q^m))(1-q^{2m})}, \quad (2.1)$$

we see from the right-hand side that infinite product and thus also infinite series are convergent for $|q|<1$ since

$$\sum_{n=0}^{\infty}\left(q^k\right)^n = \frac{1}{(1-q^k)}, \quad (2.2)$$

is a geometric series which converges for $|q|<1$ for any fixed $k\ge 1$.

Next, we note that from

$$G(q) = \sum_{j=0}^{\infty}\tilde{p}(j)q^j, \quad (2.3)$$

we get that

$$\frac{G(q)}{q^{n+1}} = \sum_{j=0}^{\infty}\frac{\tilde{p}(j)q^j}{q^{n+1}} \quad \text{if } 0<|q|<1. \quad (2.4)$$

The series on the right side of (2.4) is a Laurent series of $G(q)/q^{(n+1)}$. It has a pole of order $n+1$ at $q=0$ with residue $\tilde{p}(n)$. Applying Cauchy's Residue Theorem we get that

$$\tilde{p}(n) = \frac{1}{2\pi i}\int_C \frac{G(q)}{q^{n+1}}\, dq = \frac{1}{2\pi i}\int_C \frac{\left(f\left(q^2\right)\right)^2}{f(q)q^{n+1}}\, dq, \tag{2.5}$$

where C is any positively oriented simple closed countour lying inside the unit circle.

2.2. *Change of the Variable*

The change of the variable $q = e^{2\pi i\tau}$ maps the unit disk $|q| < 1$ into an infinite vertical strip of width 1 in the τ-plane. To see this we note that from $q = e^{2\pi i\tau}$ we get $\log q = 2\pi i\tau$, so $\tau = \log q/2\pi i$. Choosing the branch cut to be $[0,1]$, we get

$$\tau = \frac{\log|q|}{2\pi i} + \frac{\operatorname{Arg}(q)}{2\pi}. \tag{2.6}$$

As q traverses a circle centered at $q = 0$ of radius $e^{-2\pi}$ in the positive direction, the point τ varies from i to $i+1$ along a horizontal segment as could be easly deduced from (2.6).

Replacing the segment by the Rademacher path composed of upper arcs of the Ford circles formed by the Farey series $\mathcal{F}_N$, (2.5) becomes

$$\tilde{p}(n) = \frac{1}{2\pi i}\int_i^{i+1} \frac{\left(f\left(e^{4\pi i\tau}\right)\right)^2 2\pi i e^{2\pi i\tau}}{f\left(e^{2\pi i\tau}\right)e^{2\pi i\tau(n+1)}}\, d\tau, \tag{2.7}$$

which simplifies to

$$\begin{aligned}\tilde{p}(n) &= \int_i^{i+1} \frac{\left(f\left(e^{4\pi i\tau}\right)\right)^2}{f\left(e^{2\pi i\tau}\right)} e^{-2\pi i\tau n}\, d\tau,\\ &= \int_{P(N)} \frac{\left(f\left(e^{4\pi i\tau}\right)\right)^2}{f\left(e^{2\pi i\tau}\right)} e^{-2\pi i\tau n}\, d\tau.\end{aligned} \tag{2.8}$$

The above can be written as

$$\int_{P(N)} \frac{\left(f\left(e^{4\pi i\tau}\right)\right)^2}{f\left(e^{2\pi i\tau}\right)} e^{-2\pi i\tau n}\, d\tau = \sum_{k=1}^{N}\sum_{0\le h<k\,(h,k)=1}\int_{\gamma(h,k)} \frac{\left(f\left(e^{4\pi i\tau}\right)\right)^2}{f\left(e^{2\pi i\tau}\right)} e^{-2\pi i\tau n}\, d\tau, \tag{2.9}$$

where $\gamma(h,k)$ is the upper arc of the Ford circle $C(h,k)$.

2.3. Another Change of the Variable

Consider another change of variable

$$\tau = \frac{h}{k} + \frac{iz}{k}, \tag{2.10}$$

so that

$$z = -ik\left(\tau - \frac{h}{k}\right) \tag{2.11}$$

$$dz = -ik\ d\tau. \tag{2.12}$$

Under this transformation the Ford circle $C(h,k)$ in the τ-plane with center at $h/k + i1/2k^2$ and radius $1/2k^2$ is mapped to a negatively oriented circle C_k in the z-plane with center at $1/2k$ and radius $1/2k$. This follows from the fact that any point on the Ford circle $C(h,k)$ is given by

$$\tau = \left(\frac{h}{k} + i\frac{1}{2k^2}\right) + \frac{1}{2k^2}e^{i\theta}, \quad 0 \le \theta < 2\pi. \tag{2.13}$$

Substitution of (2.13) into (2.11) gives

$$z = \frac{1}{2k} + \frac{1}{2k}\left(-ie^{i\theta}\right), \tag{2.14}$$

which is a circle centered at $1/2k$ with radius $1/2k$. Now we make change of variable in (2.9). This gives

$$\widetilde{p}(n) = i\sum_{k=1}^{N} k^{-1} \sum_{0\le h<k\ (h,k)=1} e^{-2\pi i n h/k} \int_{s_{h,k}}^{t_{h,k}} \frac{\left(f\left(e^{4\pi i h/k - 4\pi i z/k}\right)\right)^2}{f\left(e^{2\pi i h/k - 2\pi z/k}\right)} e^{2\pi n z/k}\ dz, \tag{2.15}$$

where

$$\begin{aligned} s_{h,k} &= \frac{k}{k^2 + k_p^2} + \frac{k_p}{k^2 + k_p^2}i, \\ t_{h,k} &= \frac{k}{k^2 + k_s^2} - \frac{k_s}{k^2 + k_s^2}i \end{aligned} \tag{2.16}$$

are initial and terminal points, respectively.

2.4. Modular Transformation

Next, we note that

$$f(q) = f\left(e^{2\pi i\tau}\right) = \frac{e^{\pi i\tau/12}}{\eta(\tau)}, \tag{2.17}$$

where $\eta(\tau)$ is the Dedekind eta function. Rewriting modular functional equation [5, page 96] for $\eta(\tau)$ in terms of $f(q) = f(e^{2\pi i\tau}) = f(e^{2\pi ih/k-2\pi z/k})$ we get

$$f\left(e^{2\pi ih/k-2\pi z/k}\right) = \omega(h,k)\exp\left(\frac{\pi(z^{-1}-z)}{12k}\right)\sqrt{z}f\left(\exp\left(2\pi i\frac{iz^{-1}+H}{k}\right)\right), \tag{2.18}$$

with $hH \equiv -1(\text{mod}\, k)$, $(h,k) = 1$.

To evaluate (2.15) we would like to express

$$G(q) = G\left(e^{2\pi i\tau}\right) = G\left(e^{2\pi ih/k-2\pi z/k}\right) = \frac{\left(f\left(e^{4\pi ih/k-4\pi iz/k}\right)\right)^2}{f\left(e^{2\pi ih/k-2\pi z/k}\right)}, \tag{2.19}$$

in the same way we did for $f(q)$ above. Two cases have to be considered: $(k,2) = 1$ and $(k,2) = 2$. When $(k,2) = 1$ we will replace h by $2h$ and z by $2z$, and when $(k,2) = 2$, k will be replaced by $k/2$ in order to obtain $f(q^2)$ from $f(q)$. Hence, we have

$$G\left(e^{2\pi ih/k-2\pi z/k}\right) = \begin{cases} \dfrac{\omega^2(2h,k)e^{\pi((2z)^{-1}-2z)/6k}2zf^2\left(e^{2\pi i(i(2z)^{-1}+H_2)/k}\right)}{\omega(h,k)e^{\pi(z^{-1}-z)/12k}\sqrt{z}f\left(e^{2\pi i(iz^{-1}+H_2)/k}\right)}, & \text{if } (k,2) = 1, \\ \dfrac{\omega^2(h,k/2)e^{\pi(z^{-1}-z)/3k}zf^2\left(e^{4\pi i(iz^{-1}+H_1)/k}\right)}{\omega(h,k)e^{\pi(z^{-1}-z)/12k}\sqrt{z}f\left(e^{2\pi i(iz^{-1}+H_1)/k}\right)}, & \text{if } (k,2) = 2, \end{cases} \tag{2.20}$$

which simplifies to

$$G\left(e^{2\pi ih/k-2\pi z/k}\right) = \begin{cases} 2\dfrac{\omega^2(2h,k)}{\omega(h,k)}e^{-\pi z/4k}\sqrt{z}\dfrac{f^2\left(e^{2\pi i(i(2z)^{-1}+H_2)/k}\right)}{f\left(e^{2\pi i(iz^{-1}+H_2)/k}\right)}, & \text{if } (k,2) = 1, \\ \dfrac{\omega^2(h,k/2)}{\omega(h,k)}e^{\pi(z^{-1}-z)/4k}\sqrt{z}G\left(e^{2\pi i(iz^{-1}+H_1)/k}\right), & \text{if } (k,2) = 2, \end{cases} \tag{2.21}$$

where $hH_j \equiv -1(\text{mod}\, k)$ and $j \mid H_j$ for $j = 1,2$.

We return to evaluation of (2.15). To proceed we note that

$$G\left(e^{2\pi i(iz^{-1}+H_1)/k}\right) = 1 + \left\{G\left(e^{2\pi i(iz^{-1}+H_1)/k}\right) - 1\right\}. \tag{2.22}$$

Rewriting (2.15) in terms of (2.21) and (2.22) we obtain

$$\begin{aligned}
\widetilde{p}(n) &= i\sum_{k=1(k,2)=1}^{N} k^{-1} \sum_{0\le h<k(h,k)=1} 2\frac{\omega^2(2h,k)}{\omega(h,k)} e^{-2\pi inh/k} \int_{s_{h,k}}^{t_{h,k}} \sqrt{z}\frac{f^2\left(e^{2\pi i(i(2z)^{-1}+H_2)/k}\right)}{f\left(e^{2\pi i(iz^{-1}+H_2)/k}\right)} e^{\pi z/k(2n-1/4)}dz \\
&\quad + i\sum_{k=1(k,2)=2}^{N} k^{-1} \sum_{0\le h<k(h,k)=1} \frac{\omega^2(h,k/2)}{\omega(h,k)} e^{-2\pi inh/k} \int_{s_{h,k}}^{t_{h,k}} \sqrt{z}\left\{1+\left(G\left(e^{2\pi i(iz^{-1}+H_1)/k}\right)-1\right)\right\} \\
&\quad \times e^{((\pi z/k)(2n-1/4)+\pi/4zk)}\ dz \\
&= 2i\sum_{k=1(k,2)=1}^{N} k^{-1} \sum_{0\le h<k(h,k)=1} \frac{\omega^2(2h,k)}{\omega(h,k)} e^{-2\pi inh/k} \int_{s_{h,k}}^{t_{h,k}} \sqrt{z}\frac{f^2\left(e^{2\pi i(i(2z)^{-1}+H_2)/k}\right)}{f\left(e^{2\pi i(iz^{-1}+H_2)/k}\right)} e^{\pi z/k(2n-1/4)}dz \\
&\quad + i\sum_{k=1(k,2)=2}^{N} k^{-1} \sum_{0\le h<k(h,k)=1} \frac{\omega^2(h,k/2)}{\omega(h,k)} e^{-2\pi inh/k}(J_1(h,k)+J_2(h,k)),
\end{aligned} \tag{2.23}$$

where

$$\begin{aligned}
J_1(h,k) &= \int_{s_{h,k}}^{t_{h,k}} \sqrt{z} e^{(\pi z/k(2n-1/4)+\pi/4zk)}\ dz\ , \\
J_2(h,k) &= \int_{s_{h,k}}^{t_{h,k}} \sqrt{z}\left\{G\left(e^{2\pi i(iz^{-1}+H_1)/k}\right)-1\right\} e^{(\pi z/k(2n-1/4)+\pi/4zk)}\ dz.
\end{aligned} \tag{2.24}$$

2.5. Estimation of the First Term

We will estimate the first term in (2.23) and will show that it is small for large N. To do this we change variable again by letting $\xi = zk$. Then the first term in (2.23) becomes

$$2i\sum_{k=1(k,2)=1}^{N} k^{-5/2} \sum_{0\le h<k(h,k)=1} \frac{\omega^2(2h,k)}{\omega(h,k)} e^{-2\pi inh/k} \int_{s^*_{h,k}}^{t^*_{h,k}} \sqrt{\xi}\frac{f^2\left(e^{2\pi i(i(2\xi/k)^{-1}+H_2)/k}\right)}{f\left(e^{2\pi i(i(\xi/k)^{-1}+H_2)/k}\right)} e^{\pi\xi/k^2\ (2n-1/4)}\ d\xi, \tag{2.25}$$

where

$$s^*_{h,k} = \frac{k^2}{k^2+k_p^2} + \frac{kk_p}{k^2+k_p^2}i, \tag{2.26}$$

$$t^*_{h,k} = \frac{k^2}{k^2+k_s^2} - \frac{kk_s}{k^2+k_s^2}i, \tag{2.27}$$

are initial and terminal points obtained from (2.16), respectively. Under this change of variable circle C_k in z-plane with center at $1/2k$ and radius $1/2k$ is mapped to a circle C_k^*

in ξ-plane centered at $1/2$ with radius $1/2$. Note also that the mapping $w = 1/\xi$ maps the circle C_k^* and its interior onto a half-plane $\Re(w) \geq 1$ (where $\Re(w)$ denotes the real part of complex variable w and $\Im(w)$ is the imaginary part). From elementary complex analysis we have that $\Re(w) = x/(x^2+y^2)$ and $\Im(w) = -y/(x^2+y^2)$, where $x+iy = \xi$. It is readily seen that the segment $0 < x \leq 1$ in the ξ-plane is mapped to an infinite strip $[1,\infty)$ in the w-plane. So, it follows that inside and on the circle C_k^* we have that $0 < \Re(\xi) \leq 1$ and $\Re(1/\xi) \geq 1$. We now show that $\Re(1/\xi) = 1$ on the circle C_k^*. To see this note that in the polar form $\xi = 1/2+(1/2)e^{i\theta}$ on C_k^*, $0 \leq \theta \leq 2\pi$. From this we get that

$$\begin{aligned}\frac{1}{\xi} &= \frac{2}{1+e^{i\theta}} = \frac{2}{(1+\cos\theta)+i\sin\theta} \\ &= \frac{2[(1+\cos\theta)-i\sin\theta]}{(1+\cos\theta)^2+\sin^2\theta} \\ &= \frac{2(1+\cos\theta)}{2+2\cos\theta} - i\frac{2\sin\theta}{2+2\cos\theta} \\ &= 1 - i\frac{\sin\theta}{1+\cos\theta}.\end{aligned} \tag{2.28}$$

So, $\Re(1/\xi) = 1$.

Furthermore, we may move path of integration from the arc joining $s_{h,k}^*$ and $t_{h,k}^*$ to a segment connecting these two points on the circle C_k^*. By [5, page 104], Theorem 5.9 the length of the path of integration is bounded by $2\sqrt{2}k/N$, and on the segment connecting $s_{h,k}^*$ and $t_{h,k}^*$, $|\xi| < \sqrt{2}k/N$.

Next, let us define $\widetilde{p}^*(m)$ by

$$\sum_{m=0}^{\infty} \widetilde{p}^*(m) q^m = \frac{f^2\left(e^{2\pi i (i(2\xi/k)^{-1}+H_2)/k}\right)}{f\left(e^{2\pi i (i(\xi/k)^{-1}+H_2)/k}\right)}, \tag{2.29}$$

which is a part of the integrand in (2.25). Then, estimating the integrand in (2.25) we get

$$\begin{aligned}&\left|\sqrt{\xi} e^{(\pi\xi/k^2)\,(2n-1/4)}\right| \times \left| -1 + \frac{f^2\left(e^{2\pi i (i(2\xi/k^{-1})+H_2)/k}\right)}{f\left(e^{2\pi i (i(\xi/k)^{-1}+H_2)/k}\right)} \right| \\ &= |\xi|^{1/2} \left| e^{\pi\xi/k^2\,(2n-1/4)} \right| \times \left| \sum_{m=1}^{\infty} \widetilde{p}^*(m) \exp\left(\frac{2\pi i m \left(i(\xi/k)^{-1} + H_2 \right)}{k} \right) \right| \\ &= |\xi|^{1/2} \left| e^{\pi\xi/k^2\,(2n-1/4)} \right| \times \left| \sum_{m=1}^{\infty} \widetilde{p}^*(m) \exp\left(-\frac{2\pi m}{\xi} \right) \exp\left(\frac{2\pi i m H_2}{k} \right) \right|\end{aligned}$$

$$\begin{aligned}
&\leq |\xi|^{1/2} e^{\pi/k^2\,(2n-1/4)\Re(\xi)} \left| \sum_{m=1}^{\infty} \widetilde{p}^*(m) \exp\left(-\frac{2\pi m}{\xi}\right) \exp\left(\frac{2\pi i m H_2}{k}\right) \right| \\
&\leq |\xi|^{1/2} e^{2\pi n} \sum_{m=1}^{\infty} \left|\widetilde{p}^*(m)\right| \exp\left(-2\pi m \Re\left(\frac{1}{\xi}\right)\right) \\
&\leq |\xi|^{1/2} e^{2\pi n} \sum_{m=1}^{\infty} \left|\widetilde{p}^*(m)\right| e^{-2\pi m} \\
&= |\xi|^{1/2} e^{2\pi n} \sum_{m=1}^{\infty} \left|\widetilde{p}^*(m)\right| y^m, \qquad \left(\text{where } y = e^{-2\pi}\right) \\
&= c|\xi|^{1/2},
\end{aligned} \tag{2.30}$$

where

$$c = e^{2\pi n} \sum_{m=1}^{\infty} \left|\widetilde{p}^*(m)\right| y^m. \tag{2.31}$$

Note that c does not depend on ξ or N. It depends on n, but n remains fixed in the above analysis. So,

$$\begin{aligned}
\left| \int_{s^*(h,k)}^{t^*(h,k)} \sqrt{\xi} \frac{f^2\left(e^{2\pi i (i(2\xi/k)^{-1}+H_2)/k}\right)}{f\left(e^{2\pi i (i(\xi/k)^{-1}+H_2)/k}\right)} e^{(\pi\xi/k^2)(2n-1/4)}\, d\xi \right| &\leq c|\xi|^{1/2} \leq c\left(\frac{\sqrt{2}k}{N}\right)^{1/2} \frac{2\sqrt{2}N}{N} \\
&< \alpha k^{3/2} N^{-3/2},
\end{aligned} \tag{2.32}$$

for some constant α, and we have that

$$\begin{aligned}
&\left| 2i \sum_{k=1 (k,2)=1}^{N} k^{-5/2} \sum_{0\leq h<k (h,k)=1} \frac{\omega^2(2h,k)}{\omega(h,k)} e^{-2\pi i n h/k} \int_{s^*(h,k)}^{t^*(h,k)} \right. \\
&\qquad \left. \sqrt{\xi} \frac{f^2\left(e^{2\pi i (i(2\xi/k)^{-1}+H_2)/k}\right)}{f\left(e^{2\pi i (i(\xi/k)^{-1}+H_2)/k}\right)} e^{\pi\xi/k^2 (2n-1/4)} (d\xi) \right| \\
&\qquad \leq 2 \sum_{k=1 (k,2)=1}^{N} \sum_{0\leq h<k (h,k)=1} \alpha k^{-1} N^{-3/2} \\
&\qquad \leq 2\alpha N^{-3/2} \sum_{k=1 (k,2)=1}^{N} 1 = 2\alpha N^{-1/2}.
\end{aligned} \tag{2.33}$$

This completes the estimation of the first term in (2.23). We proceed to the second term.

2.6. Estimation of the Second Term

First, we will show that

$$J_2(h,k) = \int_{s_{h,k}}^{t_{h,k}} \sqrt{z}\left\{G\left(e^{2\pi i(iz^{-1}+H_1)/k}\right) - 1\right\}e^{(\pi z/k(2n-1/4)+\pi/4zk)}\ dz \tag{2.34}$$

is small for large N. Making change of variable $\xi = zk$ as before, we get that

$$J_2(h,k) = k^{-3/2}\int_{s^*_{h,k}}^{t^*_{h,k}} \sqrt{\xi}\left\{G\left(e^{2\pi i(i(\xi/k)^{-1}+H_1)/k}\right) - 1\right\}e^{(\pi\xi/k^2(2n-1/4)+\pi/4\xi)}\ d\xi, \tag{2.35}$$

where $s^*_{h,k}$ and $t^*_{h,k}$ are as in (2.26), respectively. As before, we define $\widetilde{p}^{**}(m)$ by

$$\sum_{m=0}^{\infty}\widetilde{p}^{**}(m)q^m = G\left(e^{2\pi i(i(\xi/k^{-1})+H_1)/k}\right) - 1. \tag{2.36}$$

Then, estimating the integrand, we see that

$$\begin{aligned}
&\left|\sqrt{\xi}e^{((\pi\xi/k^2)\ (2n-1/4)+\pi/4\xi)}\right| \times \left|G\left(e^{2\pi i(i(\xi/k)^{-1}+H_1)/k}\right) - 1\right| \\
&\quad= |\xi|^{1/2}\left|e^{\pi\xi/k^2\ (2n-1/4)}e^{\pi/4\xi}\right| \times \left|\sum_{m=0}^{\infty}\widetilde{p}^{**}(m)\exp\left(\frac{2\pi im(ik/\xi + H_1)}{k}\right) - 1\right| \\
&\quad\leq |\xi|^{1/2}e^{(\pi/k^2)\ (2n-1/4)\Re(\xi)}e^{\pi/4\Re(1/\xi)}\left|\sum_{m=1}^{\infty}\widetilde{p}^{**}(m)\exp\left(\frac{-2\pi m}{\xi}\right)\exp\left(\frac{2\pi imH_1}{k}\right)\right| \\
&\quad\leq |\xi|^{1/2}e^{2\pi n}e^{\pi/4\Re(1/\xi)}\sum_{m=1}^{\infty}\left|\widetilde{p}^{**}(m)\right|\exp\left(-2\pi m\Re\left(\frac{1}{\xi}\right)\right) \\
&\quad= |\xi|^{1/2}e^{2\pi n}\sum_{m=1}^{\infty}\left|\widetilde{p}^{**}(m)\right|\exp\left(\left(-2\pi m + \frac{\pi}{4}\right)\Re\left(\frac{1}{\xi}\right)\right) \\
&\quad\leq |\xi|^{1/2}e^{2\pi n}\sum_{m=1}^{\infty}\left|\widetilde{p}^{**}(m)\right|\exp\left(-2\pi m + \frac{\pi}{4}\right) \\
&\quad= |\xi|^{1/2}e^{2\pi n}\sum_{m=1}^{\infty}\left|\widetilde{p}^{**}(m)\right|e^{-\pi/4(8m-1)} \\
&\quad\leq |\xi|^{1/2}e^{2\pi n}\sum_{m=1}^{\infty}\left|\widetilde{p}^{**}(8m-1)\right|e^{-\pi/4(8m-1)}
\end{aligned}$$

$$\begin{aligned}&= |\xi|^{1/2} e^{2\pi n} \sum_{m=1}^{\infty} \left|\tilde{p}^{**}(8m-1)\right| x^{8m-1}, \quad \text{where } x = e^{-\pi/4} \\ &= b|\xi|^{1/2},\end{aligned} \tag{2.37}$$

where

$$b = e^{2\pi n} \sum_{m=1}^{\infty} \left|\tilde{p}^{**}(8m-1)\right| x^{8m-1}. \tag{2.38}$$

Note that b does not depend on ξ or N. It depends on n, but n is fixed. It follows, therefore, that

$$|J_2(h,k)| \leq b\left(\frac{\sqrt{2}k}{N}\right)^{1/2} \frac{2\sqrt{2}N}{N} < \beta k^{3/2} N^{-3/2}, \tag{2.39}$$

for some constant β. Then we have that

$$\begin{aligned}\left| i \sum_{k=1(k,2)=2}^{N} k^{-5/2} \sum_{0\leq h<k(h,k)=1} \frac{\omega^2(h,k/2)}{\omega(h,k)} e^{-\frac{2\pi inh}{k}} J_2(h,k) \right| &< \sum_{k=1(k,2)=20\leq h<k(h,k)=1}^{N} \sum \beta k^{-1} N^{-3/2} \\ &\leq \beta N^{-3/2} \sum_{k=1(k,2)=2}^{N} 1 = \beta N^{-1/2}.\end{aligned} \tag{2.40}$$

Combining the results from (2.33) and (2.40) we have that

$$\begin{aligned}\tilde{p}(n) &= i \sum_{k=1(k,2)=2}^{N} k^{-5/2} \sum_{0\leq h<k(h,k)=1} \frac{\omega^2(h,k/2)}{\omega(h,k)} e^{-2\pi inh/k} J_1(h,k) + O\left(\beta N^{-1/2} + 2\alpha N^{-1/2}\right) \\ &= i \sum_{k=1(k,2)=2}^{N} k^{-5/2} \sum_{0\leq h<k(h,k)=1} \frac{\omega^2(h,k/2)}{\omega(h,k)} e^{-2\pi inh/k} J_1(h,k) \ + O\left(N^{-1/2}\right).\end{aligned} \tag{2.41}$$

Finally, we turn our attention to

$$J_1(h,k) = k^{-3/2} \int_{s_{h,k}^*}^{t_{h,k}^*} \sqrt{\xi} e^{((\pi\xi/k^2)(2n-1/4)+\pi/4\xi)} \, d\xi. \tag{2.42}$$

We note that

$$J_1(h,k) = \int_{C_k^*} - \int_{s_{h,k}^*}^{0} - \int_{0}^{t_{h,k}^*} = \int_{C_k^*} -S_1 - S_2, \tag{2.43}$$

where C_k^* is a circle in the ξ-plane centered at $1/2$ with radius $1/2$, as before. It is easily seen that the length of the arc connecting 0 and $s_{h,k}^*$ is less then

$$2\pi\frac{\left|s_{h,k}^*\right|}{2}\leq\pi\left|s_{h,k}^*\right|\leq\pi\sqrt{2}\frac{k}{N}. \tag{2.44}$$

From the discussion above we know that $\Re(1/\xi)=1$ and $0<\Re(\xi)\leq 1$ on C_k^*. So, the integrand in S_1 could be estimated as

$$\begin{aligned}\left|\sqrt{\xi}e^{(\pi\xi/k^2\ (2n-1/4)+\pi/4\xi)}\right| &= |\xi|^{1/2}\left|e^{\pi\xi/k^2\ (2n-1/4)}\right|\left|e^{\pi/4\xi}\right| \\ &= |\xi|^{1/2}e^{\pi/k^2\ (2n-1/4)\Re(\xi)}e^{\pi/4\Re(1/\xi)} \\ &\leq 2^{1/4}\frac{k^{1/2}}{N^{1/2}}e^{2\pi n}e^{\pi/4}.\end{aligned} \tag{2.45}$$

2.7. Combining the Results

We combine the results in (2.44) and (2.45) to get

$$|S_1|<\gamma k^{3/2}N^{-3/2}, \tag{2.46}$$

where γ is a constant. We can obtain similar estimate for S_2 and, as before, we get an error term $O(N^{-1/2})$ in the formula for $\widetilde{p}(n)$. Therefore, we can write

$$\begin{aligned}\widetilde{p}(n) &= i\sum_{k=1(k,2)=2}^{N}k^{-5/2}\sum_{0\leq h<k(h,k)=1}\frac{\omega^2(h,k/2)}{\omega(h,k)}e^{-2\pi inh/k} \\ &\quad\times\int_{C_k^*}\sqrt{\xi}e^{((\pi\xi/k^2)(2n-1/4)+\pi/4\xi)}\ d\xi+O\left(N^{-1/2}\right).\end{aligned} \tag{2.47}$$

Letting $N\to\infty$ we have that

$$\begin{aligned}\widetilde{p}(n) &= i\sum_{k=1(k,2)=2}^{\infty}k^{-5/2}\sum_{0\leq h<k(h,k)=1}\frac{\omega^2(h,k/2)}{\omega(h,k)}e^{-2\pi inh/k} \\ &\quad\times\int_{C_k^*}\sqrt{\xi}e^{((\pi\xi/k^2)(2n-1/4)+\pi/4\xi)}\ d\xi.\end{aligned} \tag{2.48}$$

We introduce another change of variable

$$\xi=\frac{1}{w},\qquad d\xi=-\frac{1}{w^2}. \tag{2.49}$$

Then (2.48) becomes

$$\widetilde{p}(n) = \frac{1}{i}\sum_{k=1 (k,2)=2}^{\infty} k^{-5/2} \sum_{0\le h<k (h,k)=1} \frac{\omega^2(h,k/2)}{\omega(h,k)} e^{-2\pi inh/k} \times \int_{1-\infty i}^{1+\infty i} w^{-5/2} e^{((\pi/k^2)(2n-1/4)1/w+\pi w/4)}\, dw. \tag{2.50}$$

Let $t = \pi w/4$ in (2.50), then the above becomes

$$\widetilde{p}(n) = 2\pi\left(\frac{\pi^{3/2}}{8}\right)\sum_{k=1 (k,2)=2}^{\infty} k^{-5/2} \sum_{0\le h<k (h,k)=1} \frac{\omega^2(h,k/2)}{\omega(h,k)} e^{-2\pi inh/k} \frac{1}{2\pi i} \times \int_{\pi/4-\infty i}^{\pi/4+\infty i} t^{-5/2} e^{(t+(\pi^2/4k^2)(2n-1/4)(1/t))}\, dt. \tag{2.51}$$

2.8. Bessel Function

In Watson's Treatise on Bessel functions [6, page 181], we find a formula equivalent to the following:

$$I_\nu(z) = \frac{(1/2z)^\nu}{2\pi i}\int_{c-\infty i}^{c+\infty i} t^{-\nu-1} e^{t+(z^2/4t)}\, dt, \quad (\text{if } c>0, \quad \Re(\nu)>0). \tag{2.52}$$

Let

$$\frac{z}{2} = \left\{\frac{\pi^2}{4k^2}\left(2n-\frac{1}{4}\right)\right\}^{1/2} \tag{2.53}$$

and $\nu = 3/2$. Then we have

$$\begin{aligned}\widetilde{p}(n) &= 2\pi\left(\frac{\pi^{3/2}}{8}\right)\sum_{k=1 (k,2)=2}^{\infty} k^{-5/2} \sum_{0\le h<k (h,k)=1} \frac{\omega^2(h,k/2)}{\omega(h,k)} e^{-2\pi inh/k} \\ &\quad\times \frac{\pi^{-3/2}(2n-1/4)^{-3/4}}{4^{-3/4}k^{-3/2}} I_{3/2}\left(\frac{\pi}{k}\sqrt{2n-\frac{1}{4}}\right) \\ &= \frac{2\pi(2n-1/4)^{-3/4}}{\sqrt{8}}\sum_{k=1 (k,2)=2}^{\infty} k^{-1} \sum_{0\le h<k (h,k)=1} \frac{\omega^2(h,k/2)}{\omega(h,k)} e^{-2\pi inh/k} I_{3/2}\left(\frac{\pi}{k}\sqrt{2n-\frac{1}{4}}\right).\end{aligned} \tag{2.54}$$

Note that Bessel functions of this order can be expressed as

$$I_{3/2}(z) = \sqrt{\frac{2z}{\pi}} \frac{d}{dz}\left(\frac{\sinh z}{z}\right). \tag{2.55}$$

Expanding (2.55) we have that

$$I_{3/2}(z) = \sqrt{\frac{2z}{\pi}}\left(\frac{\cosh z}{z} - \frac{\sinh z}{z^2}\right). \tag{2.56}$$

Substituting (2.53) into (2.56), we get

$$\begin{aligned}
I_{3/2}(z) &= I_{3/2}\left(\frac{\pi}{k}\left(2n - \frac{1}{4}\right)^{1/2}\right) = I_{3/2}\left(\frac{\pi\sqrt{8n-1}}{2k}\right) \\
&= \sqrt{\frac{2\left(\pi\sqrt{8n-1}/2k\right)}{\pi}}\left(\frac{\cosh\left(\pi\sqrt{8n-1}/2k\right)}{\left(\pi\sqrt{8n-1}/2k\right)} - \frac{\sinh\left(\pi\sqrt{8n-1}/2k\right)}{\left(\pi\sqrt{8n-1}/2k\right)^2}\right) \\
&= \frac{(8n-1)^{1/4}}{\sqrt{k}}\left(\frac{2\cosh\left(\pi\sqrt{8n-1}/2k\right)}{\frac{\pi}{k}\sqrt{8n-1}} - \frac{4k/\pi \sinh\left(\pi\sqrt{8n-1}/2k\right)}{\frac{\pi}{k}(8n-1)}\right) \\
&= \frac{1}{\pi\sqrt{\sqrt{8n-1}/k}}\left(2\cosh\left(\frac{\pi\sqrt{8n-1}}{2k}\right) - \frac{4k\sinh\left(\pi\sqrt{8n-1}/2k\right)}{\pi\sqrt{8n-1}}\right).
\end{aligned} \tag{2.57}$$

Multiplying (2.57) by

$$\frac{2\pi(2n-1/4)^{-3/4}}{\sqrt{8}} = \frac{2\pi}{(8n-1)^{3/4}}, \tag{2.58}$$

we get

$$\frac{2\left(2\cosh\left(\pi\sqrt{8n-1}/2k\right) - 4k\sinh\left(\pi\sqrt{8n-1}/2k\right)/\pi\sqrt{8n-1}\right)}{(8n-1)^{3/4}\sqrt{\sqrt{8n-1}/k}}. \tag{2.59}$$

2.9. Final Form

Finally, we rewrite (2.54) in terms of (2.59) to get

$$\widetilde{p}(n) = \sum_{k=1(k,2)=2}^{\infty} k^{-1} \sum_{0\le h<k(h,k)=1} \frac{\omega^2(h,k/2)}{\omega(h,k)} e^{-2\pi i n h/k} \times \frac{2\left(2\cosh\left(\pi\sqrt{8n-1}/2k\right) - 4k\sinh\left(\pi\sqrt{8n-1}/2k\right)/\pi\sqrt{8n-1}\right)}{(8n-1)^{3/4}\sqrt{\sqrt{8n-1}/k}}. \tag{2.60}$$

Thus,

$$pod(n) = (-1)^n \sum_{k=1(k,2)=2}^{\infty} k^{-1} \sum_{0\le h<k(h,k)=1} \frac{\omega^2(h,k/2)}{\omega(h,k)} e^{-2\pi i n h/k} \times \frac{2\left(2\cosh\left(\pi\sqrt{8n-1}/2k\right) - 4k\sinh\left(\pi\sqrt{8n-1}/2k\right)/\pi\sqrt{8n-1}\right)}{(8n-1)^{3/4}\sqrt{\sqrt{8n-1}/k}}, \tag{2.61}$$

or equivalently

$$pod(n) = (-1)^n \frac{2}{\pi} \sum_{k=1(k,2)=2}^{\infty} \sqrt{k} \sum_{0\le h<k(h,k)=1} \frac{\omega^2(h,k/2)}{\omega(h,k)} e^{-2\pi i n h/k} \frac{d}{dn}\left(\frac{\sinh\left(\pi\sqrt{8n-1}/2k\right)}{\sqrt{8n-1}}\right). \tag{2.62}$$

Acknowledgment

The authors wishes to express gratitude and appreciation to Dr. A. Sills for his suggestions and criticism.

References

[1] H. Rademacher, "On the expansion of the partition function in a series," *Annals of Mathematics*, vol. 44, no. 2, pp. 416–422, 1943.
[2] H. Rademacher, "On the partition function $p(n)$," *Proceedings of the London Mathematical Society*, vol. 43, no. 2, pp. 241–254, 1937.
[3] J. Bruinier and K. Ono, "An algebraic formula for the partition function," http://www.aimath.org/news/partition/brunier-ono.
[4] A. V. Sills, "A Rademacher type formula for partitions and overpartitions," *International Journal of Mathematics and Mathematical Sciences*, vol. 2010, Article ID 630458, 21 pages, 2010.
[5] T. M. Apostol, *Modular Functions and Dirichlet Series in Number Theory*, vol. 41 of *Graduate Texts in Mathematics*, Springer, New York, NY, USA, 2nd edition, 1990.
[6] G. N. Watson, *A Treatise on the Theory of Bessel Functions*, Cambridge University Press, Cambridge, UK, 2nd edition, 1944.