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*Research Article*

# Bounds of Hankel Determinant for a Class of Univalent Functions

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The authors study the coefficient condition for the class $\mathfrak{H}_\alpha$ defined as the family of analytic functions $f$, $f(0) = 0$ and $f'(0) = 1$, which satisfy $\Re[(1-\alpha)f'(z)+\alpha(1+(zf''(z))/(f'(z)))] > \alpha,\ z \in E$, where $\alpha$ is a real number.

## 1. Introduction

Let $\mathcal{A}$ be the class of functions of the following form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the unit disc $E = \{z : |z| < 1\}$, and let $S$ be the subclass of $\mathcal{A}$ consisting of functions which are univalent in $E$. A function $f \in \mathcal{A}$ is said to be close to convex in the open unit disc $E$ if there exists a convex function $g$ (not necessarily normalized) such that

$$\Re\left(\frac{f'(z)}{g'(z)}\right) > 0, \quad z \in E. \tag{1.2}$$

For fixed real numbers $\alpha$, let $\mathfrak{H}_\alpha$ denote the family of functions $f$ in $\mathcal{A}$ which satisfy

$$\Re\left[(1-\alpha)f'(z) + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)\right] > \alpha, \quad z \in E. \tag{1.3}$$

In 2005, V. Singh et al. [1] established that, for $0 < \alpha < 1$, functions in $\mathfrak{H}_\alpha$ satisfy $\Re f'(z) > 0$ in $E$ and so are close to convex in $E$.

In [2], Noonan and Thomas defined the Hankel determinant $H_q(n)$ of the function $f$ for $q \geq 1$ and $n \geq 1$ by

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2(q-1)} \end{vmatrix}. \tag{1.4}$$

The determinant has been investigated by several authors with the subject of inquiry ranging from rate of growth of $H_q(n)$ as $n \to \infty$, to the determination of precise bounds on $H_q(n)$ for specific $q$ and $n$ for some special classes of functions. In a classical theorem, Fekete and Szegö [3] considered the Hankel determinant of $f \in S$ for $q = 2$ and $n = 1$

$$H_2(1) = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix}. \tag{1.5}$$

The well-known result due to them states that if $f \in S$, then

$$\left| a_3 - \mu a_2{}^2 \right| \leq \begin{cases} 3 - 4\mu & \text{if } \mu \leq 0, \\ 1 + 2\exp\left(\dfrac{-2\mu}{1-\mu}\right) & \text{if } 0 \leq \mu < 1, \\ 4\mu - 3 & \text{if } \mu \geq 1, \end{cases} \tag{1.6}$$

where $a_1 = 1$ and $\mu$ is a real number. In the present paper, we obtain a sharp bound for $H_2(2) = |a_2 a_4 - a_3^2|$ when $f \in \mathfrak{H}_\alpha$.

## 2. Preliminary Results

We denote by $P$ the family of all functions $p(z)$ given by

$$p(z) = 1 + c_1 z + c_2 z^2 + \cdots \tag{2.1}$$

analytic in $E$ for which $\Re\{p(z)\} > 0$ for $z \in E$. It is well known that for $p \in P$, $|c_k| \leq 2$ for each $k$.

**Lemma 2.1** (See [4]). *The power series for p(z) given in* (2.1) *converges in E to a function in P if and only if the Toeplitz determinants*

$$D_n = \begin{vmatrix} 2 & c_1 & c_2 & \cdots & c_n \\ c_{-1} & 2 & c_1 & \cdots & c_{n-1} \\ \vdots & & & & \vdots \\ c_{-n} & c_{-n+1} & c_{-n+2} & \cdots & 2 \end{vmatrix} \tag{2.2}$$

*$n = 1, 2, 3 \ldots$ and $c_{-k} = \overline{c}_k$ are all nonnegative. They are strictly positive except for*

$$p(z) = \sum_{k=1}^{m} \rho_k p_0\left(e^{it_k} z\right), \quad \rho_k > 0,\ t_k \text{ real}, \tag{2.3}$$

*and $t_k \neq t_j$ for $k \neq j$; in this case, $D_n > 0$ for $n < m - 1$ and $D_n = 0$ for $n \geq m$.*

**Lemma 2.2** (See [5, 6]). *Let $p \in P$. Then*

$$\begin{gathered} 2c_2 = c_1{}^2 + x\left(4 - c_1{}^2\right), \\ 4c_3 = c_1{}^3 + 2xc_1\left(4 - c_1{}^2\right) - x^2 c_1\left(4 - c_1{}^2\right) + 2z\left(1 - |x|^2\right)\left(4 - c_1{}^2\right) \end{gathered} \tag{2.4}$$

*for some $x, z$ such that $|x| \leq 1$ and $|z| \leq 1$.*

## 3. Main Result

**Theorem 3.1.** *Let $\alpha$, $0 \leq \alpha \leq 1$, be a real number. If $f \in \mathfrak{H}_\alpha$, then*

$$\left|a_2 a_4 - a_3{}^2\right| \leq \begin{cases} \dfrac{4}{9}\dfrac{(1-\alpha)^2}{(1+\alpha)^2} & 0 \leq \alpha \leq \alpha_0, \\ K(\alpha) & \alpha_0 \leq \alpha \leq 1, \end{cases} \tag{3.1}$$

*where $\alpha_0 = 0.4276891324\ldots$ is the root of the equation $10\alpha^3 - 5\alpha^2 + 12\alpha - 5 = 0$ and*

$$K(\alpha) = \frac{(1-\alpha)^2}{72(1+\alpha)^2(1+2\alpha)}\left[32(1+2\alpha) - \frac{1}{2}\frac{\left[10\alpha^3 - 5\alpha^2 + 12\alpha - 5\right]^2}{\left[4\alpha^5 - 6\alpha^4 - 18\alpha^3 + 29\alpha^2 - 20\alpha - 1\right]}\right]. \tag{3.2}$$

*Proof.* Since $f \in \mathfrak{H}_\alpha$, it follows from (1.3) that there exists a function $p \in P$ such that

$$(1-\alpha)f'(z) + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) = \alpha + (1-\alpha)p(z). \tag{3.3}$$

Equating coefficients in (3.3) yields

$$\begin{gathered} 2a_2 = (1-\alpha)c_1, \\ 3a_3(1+\alpha) = (1-\alpha)c_2 + \alpha(1-\alpha)^2 c_1{}^2, \\ 4a_4(1+2\alpha) = (1-\alpha)c_3 + \frac{3\alpha(1-\alpha)^2}{(1+\alpha)}c_1 c_2 + \frac{\alpha(1-\alpha)^3(2\alpha-1)}{(1+\alpha)}c_1{}^3. \end{gathered} \tag{3.4}$$

Thus, we can easily establish that

$$\left|a_2a_4 - a_3^2\right| = \frac{(1-\alpha)^2}{72(1+\alpha)^2(1+2\alpha)} \times \left|\left\{9(1+\alpha)^2 c_1c_3 - 8(1+2\alpha)c_2^2 - \alpha(1-\alpha)(11-5\alpha)c_1^2c_2 \right.\right.$$
$$\left.\left. +\alpha(1-\alpha)^2\left(2\alpha^2+\alpha-9\right)c_1^4\right\}\right|. \tag{3.5}$$

Using (2.4), in view of Lemma 2.2, we obtain that

$$\begin{aligned}
&\left|a_2a_4 - a_3^2\right| \\
&\quad = \frac{(1-\alpha)^2}{72(1+\alpha)^2(1+2\alpha)}\left|\left\{\frac{c_1^4}{4}\left[\left(9\alpha^2+2\alpha+1\right)+4\alpha\left(2\alpha^2+\alpha-9\right)(1-\alpha)^2+2\alpha(1-\alpha)(11-5\alpha)\right]\right.\right. \\
&\qquad + \frac{1}{2}\left(4-c_1^2\right)c_1^2x\left[\left(9\alpha^2+2\alpha+1\right)+\alpha(1-\alpha)(11-5\alpha)\right] \\
&\qquad + \frac{9}{2}(1+\alpha)^2\left(4-c_1^2\right)c_1\left(1-|x|^2\right)z - \frac{1}{4}x^2\left(4-c_1^2\right) \\
&\qquad \left.\left. \times\left[32(1+2\alpha)+c_1^2\left(9\alpha^2+2\alpha+1\right)\right]\right\}\right|.
\end{aligned} \tag{3.6}$$

Since $p \in P$, so $|c_1| \le 2$. Letting $c_1 = c$, we may assume without restriction that $c \in [0,2]$. Thus, applying the triangle inequality on (3.6), with $\rho = |x| \le 1$, we obtain

$$\begin{aligned}
&\left|a_2a_4 - a_3^2\right| \\
&\quad \le \frac{(1-\alpha)^2}{72(1+\alpha)^2(1+2\alpha)}\left\{\frac{c^4}{4}\left[\left(9\alpha^2+2\alpha+1\right)+4\alpha\left(2\alpha^2+\alpha-9\right)(1-\alpha)^2+2\alpha(1-\alpha)(11-5\alpha)\right]\right. \\
&\qquad + \frac{1}{2}\left(4-c^2\right)c^2\rho\left[\left(9\alpha^2+2\alpha+1\right)+\alpha(1-\alpha)(11-5\alpha)\right] \\
&\qquad + \frac{9}{2}(1+\alpha)^2\left(4-c^2\right)c + \frac{1}{4}\left(4-c^2\right)(c-2)\rho^2 \\
&\qquad \left. \times\left[c\left(9\alpha^2+2\alpha+1\right)-16(2\alpha+1)\right]\right\} \equiv F(\rho).
\end{aligned} \tag{3.7}$$

Differentiating $F(\rho)$, we get the following:

$$F'(\rho) = \frac{(1-\alpha)^2}{72(1+\alpha)^2(1+2\alpha)}\left\{\frac{1}{2}\left(4-c^2\right)c^2\left[\left(9\alpha^2+2\alpha+1\right)+\alpha(1-\alpha)(11-5\alpha)\right]\right.$$
$$\left.+\frac{1}{2}\left(4-c^2\right)(c-2)\rho\left[c\left(9\alpha^2+2\alpha+1\right)-16(2\alpha+1)\right]\right\}. \tag{3.8}$$

Using elementary calculus, one can show that $F'(\rho) > 0$ for $\rho > 0$. It implies that $F$ is an increasing function, and, thus, the upper bound for $F(\rho)$ corresponds to $\rho = 1$, in which case

$$F(\rho) \leq \frac{(1-\alpha)^2}{72(1+\alpha)^2(1+2\alpha)}$$
$$\times\left\{\frac{c^4}{4}\left[4\alpha\left(2\alpha^2+\alpha-9\right)(1-\alpha)^2-2\left(9\alpha^2+2\alpha+1\right)\right]\right.$$
$$\left.+c^2\left[3\left(9\alpha^2+2\alpha+1\right)+2\alpha(1-\alpha)(11-5\alpha)-8(2\alpha+1)\right]+32(2\alpha+1)\right\} \equiv G(c). \tag{3.9}$$

Then,

$$G'(c) = \frac{(1-\alpha)^2}{72(1+\alpha)^2(1+2\alpha)}c\left\{2c^2\left[2\alpha\left(2\alpha^2+\alpha-9\right)(1-\alpha)^2-\left(9\alpha^2+2\alpha+1\right)\right]\right.$$
$$\left.+2\left[3\left(9\alpha^2+2\alpha+1\right)+2\alpha(1-\alpha)(11-5\alpha)-8(2\alpha+1)\right]\right\}. \tag{3.10}$$

Setting $G'(c) = 0$, since $0 \leq c \leq 2$, we have

$$c_0 = \sqrt{\frac{-(10\alpha^3-5\alpha^2+12\alpha-5)}{4\alpha^5-6\alpha^4-18\alpha^3+29\alpha^2-20\alpha-1}} \tag{3.11}$$

provided $\alpha \geq \alpha_0$, where $\alpha_0 = 0.4276891324\ldots$ is the root of the equation $10\alpha^3 - 5\alpha^2 + 12\alpha - 5 = 0$. □

*Case 1.* When $0 \leq \alpha \leq \alpha_0$, then the maximum value of $G(c)$ corresponds to $c = 0$. Therefore, we have

$$\max_{0\leq c\leq 2} G(c) = G(0) = \frac{4}{9}\frac{(1-\alpha)^2}{(1+\alpha)^2}. \tag{3.12}$$

*Case 2.* When $\alpha_0 \leq \alpha \leq 1$, the maximum value of $G(c)$ corresponds to $c = c_0$. Therefore, we have

$$\max_{0\leq c\leq 2} G(c) = G(c_0) = K(\alpha), \tag{3.13}$$

where $K(\alpha)$ is given by (3.2). This completes the proof of the Theorem.

Setting $\alpha = 0$ in above theorem, we get the following result of Janteng et al. [7].

**Corollary 3.2.** *If an analytic function $f$ is such that $\Re\{f'(z)\} > 0$, $z \in E$, then*

$$\left|a_2 a_4 - {a_3}^2\right| \leq \frac{4}{9}. \tag{3.14}$$

*The result is sharp.*

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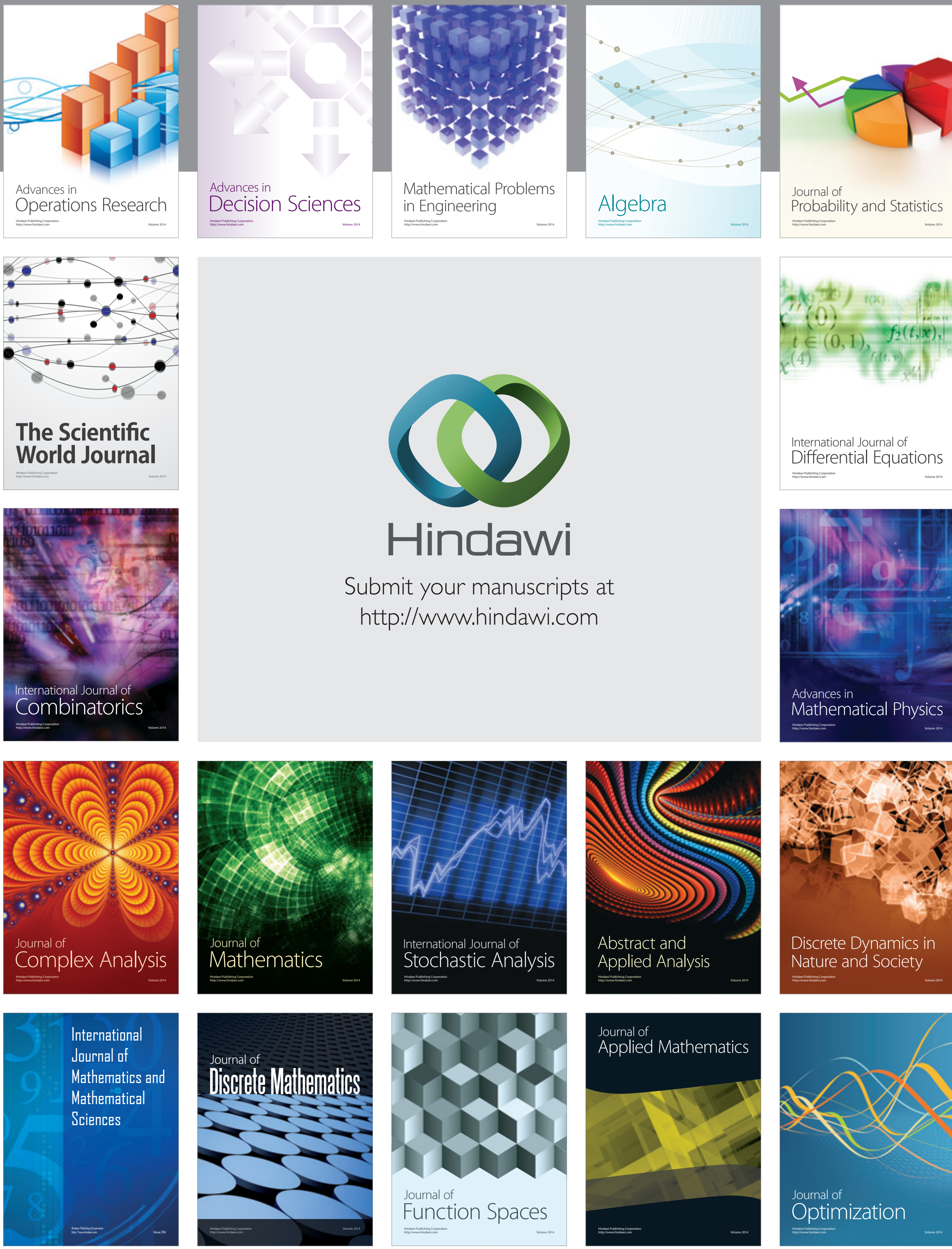
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