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Research Article

Product Summability Transform of Conjugate Series of Fourier Series

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A known theorem, Nigam (2010) dealing with the degree of approximation of conjugate of a signal belonging to Lip$\xi(t)$-class by $(E,1)(C,1)$ product summability means of conjugate series of Fourier series has been generalized for the weighted $W(L_r,\xi(t))$, $(r \geq 1)$, $(t > 0)$-class, where $\xi(t)$ is nonnegative and increasing function of t, by $\widetilde{E_n^1 C_n^1}$ which is in more general form of Theorem 2 of Nigam and Sharma (2011).

1. Introduction

Khan [1, 2] has studied the degree of approximation of a function belonging to $\text{Lip}(\alpha, r)$ and $W(L_r,\xi(t))$ classes by Nörlund and generalized Nörlund means. Working in the same direction Rhoades [3], Mittal et al. [4], Mittal and Mishra [5], and Mishra [6, 7] have studied the degree of approximation of a function belonging to $W(L_r,\xi(t))$ class by linear operators. Thereafter, Nigam [8] and Nigam and Sharma [9] discussed the degree of approximation of conjugate of a function belonging to $\text{Lip}(\xi(t), r)$ class and $W(L_r,\xi(t))$ by $(E,1)(C,1)$ product summability means, respectively. Recently, Rhoades et al. [10] have determined very interesting result on the degree of approximation of a function belonging to $\text{Lip}\alpha$ class by Hausdorff means. Summability techniques were also applied on some engineering problems like Chen and Jeng [11] who implemented the Cesàro sum of order $(C,1)$ and $(C,2)$, in order to accelerate the convergence rate to deal with the Gibbs phenomenon, for the dynamic response of a finite elastic body subjected to boundary traction. Chen et al. [12] applied regularization with Cesàro sum technique for the derivative of the double layer potential. Similarly, Chen and Hong [13] used Cesàro sum regularization technique for hypersingularity of dual integral equation.

The generalized weighted $W(L_r, \xi(t)), (r \geq 1)$-class is generalization of Lipα, Lip(α, r) and Lip$(\xi(t), r)$ classes. Therefore, in the present paper, a theorem on degree of approximation of conjugate of signals belonging to the generalized weighted $W(L_r, \xi(t)), r \geq 1$ class by $(E,1)(C,1)$ product summability means of conjugate series of Fourier series has been established which is in more general form than that of Nigam and Sharma [9]. We also note some errors appearing in the paper of Nigam [8], Nigam and Sharma [9] and rectify the errors pointed out in Remarks 2.2, 2.3 and 2.4.

Let $f(x)$ be a 2π-periodic function and integrable in the sense of Lebesgue. The Fourier series of $f(x)$ at any point x is given by

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \equiv \sum_{n=0}^{\infty} A_n(x), \tag{1.1}$$

with nth partial sum$s_n(f; x)$.

The conjugate series of Fourier series (1.1) is given by

$$\sum_{n=1}^{\infty}(b_n \cos nx - a_n \sin nx) \equiv \sum_{n=1}^{\infty} B_n(x). \tag{1.2}$$

Let $\sum_{n=0}^{\infty} u_n$ be a given infinite series with sequence of its nth partial sums $\{s_n\}$. The $(E,1)$ transform is defined as the nth partial sum of $(E,1)$ summability, and we denote it by E_n^1.

If

$$E_n^1 = \frac{1}{(2)^n}\sum_{k=0}^{n}\binom{n}{k} s_k \longrightarrow s, \quad \text{as } n \longrightarrow \infty, \tag{1.3}$$

then the infinite series $\sum_{n=0}^{\infty} u_n$ is summable $(E,1)$ to a definite number s, Hardy [14].

If

$$\tau_n = \frac{s_0 + s_1 + s_2 + \cdots + s_n}{n+1} = \frac{1}{n+1}\sum_{k=0}^{n} s_k \longrightarrow s, \quad \text{as } n \longrightarrow \infty, \tag{1.4}$$

then the infinite series $\sum_{n=0}^{\infty} u_n$ is summable to the definite number s by $(C,1)$ method. The $(E,1)$ transform of the $(C,1)$ transform defines $(E,1)(C,1)$ product transform and denotes it by $E_n^1 C_n^1$. Thus, if

$$E_n^1 C_n^1 = \frac{1}{(2)^n}\sum_{k=0}^{n}\binom{n}{k} C_k^1 \longrightarrow s, \quad \text{as } n \longrightarrow \infty, \tag{1.5}$$

then the infinite series $\sum_{n=0}^{\infty} u_n$ is said to be summable by $(E,1)(C,1)$ method or summable $(E,1)(C,1)$ to a definite number s. The $(E,1)$ is regular method of summability

$$\begin{aligned} s_n \longrightarrow s \Longrightarrow C_n^1(s_n) = \tau_n = \frac{1}{n+1}\sum_{k=0}^{n} s_k \longrightarrow s, &\quad \text{as } n \longrightarrow \infty, C_n^1 \text{ method is regular,} \\ \Longrightarrow E_n^1\left(C_n^1(s_n)\right) = E_n^1 C_n^1 \longrightarrow s, &\quad \text{as } n \longrightarrow \infty, E_n^1 \text{ method is regular,} \\ \Longrightarrow E_n^1 C_n^1 \text{ method is regular.} & \end{aligned} \tag{1.6}$$

A function $f(x) \in \mathrm{Lip}\alpha$, if

$$f(x+t) - f(x) = O(|t^{\alpha}|) \quad \text{for } 0 < \alpha \leq 1,\ t > 0, \tag{1.7}$$

and $f(x) \in \mathrm{Lip}(\alpha, r)$, for $0 \leq x \leq 2\pi$, if

$$\omega_r(t; f) = \left(\int_0^{2\pi} \left|f(x+t) - f(x)\right|^r dx\right)^{1/r} = O\left(|t|^{\alpha}\right), \quad 0 < \alpha \leq 1,\ r \geq 1,\ t > 0. \tag{1.8}$$

Given a positive increasing function $\xi(t)$, $f(x) \in \mathrm{Lip}(\xi(t), r)$, if

$$\omega_r(t; f) = \left(\int_0^{2\pi} \left|f(x+t) - f(x)\right|^r dx\right)^{1/r} = O(\xi(t)), \quad r \geq 1,\ t > 0. \tag{1.9}$$

Given positive increasing function $\xi(t)$, an integer $r \geq 1$, $f \in W(L_r, \xi(t))$, ([2]), if

$$\omega_r(t; f) = \left\{\int_0^{2\pi} \left|\{f(x+t) - f(x)\}\sin^{\beta} x\right|^r dx\right\}^{1/r} = O(\xi(t)), \quad (\beta \geq 0),\ t > 0. \tag{1.10}$$

For our convenience to evaluate I_2 without error, we redefine the weighted class as follows:

$$\omega_r(t; f) = \left(\int_0^{2\pi} \left|f(x+t) - f(x)\right|^r \sin^{\beta r}\left(\frac{x}{2}\right) dx\right)^{1/r} = O(\xi(t)), \quad \beta \geq 0,\ t > 0 \ ([16]). \tag{1.11}$$

If $\beta = 0$, then the weighted class $W(L_r, \xi(t))$ coincides with the class $\mathrm{Lip}(\xi(t), r)$, we observe that

$$W(L_r, \xi(t)) \xrightarrow{\beta=0} \mathrm{Lip}(\xi(t), r) \xrightarrow{\xi(t)=t^{\alpha}} \mathrm{Lip}(\alpha, r) \xrightarrow{r \to \infty} \mathrm{Lip}\alpha \quad \text{for } 0 < \alpha \leq 1,\ r \geq 1,\ t > 0. \tag{1.12}$$

L_r-norm of a function is defined by

$$\|f\|_r = \left(\int_0^{2\pi} |f(x)|^r dx\right)^{1/r}, \quad r \geq 1. \tag{1.13}$$

A signal f is approximated by trigonometric polynomials τ_n of order n, and the degree of approximation $E_n(f)$ is given by Rhoades [3]

$$E_n(f) = \min_n \|f(x) - \tau_n(f; x)\|_r, \tag{1.14}$$

in terms of n, where $\tau_n(f; x)$ is a trigonometric polynomial of degree n. This method of approximation is called trigonometric Fourier approximation (TFA) [4].

We use the following notations throughout this paper:

$$\psi(t) = f(x+t) - f(x-t),$$
$$\widetilde{G_n(t)} = \frac{1}{2^{n+1}\pi}\left[\sum_{k=0}^{n}\binom{n}{k}\frac{1}{1+k}\sum_{v=0}^{k}\frac{\cos(v+1/2)t}{\sin(t/2)}\right]. \tag{1.15}$$

2. Previous Result

Nigam [8] has proved a theorem on the degree of approximation of a function $\widetilde{f(x)}$, conjugate to a periodic function $f(x)$ with period 2π and belonging to the class $\mathrm{Lip}(\xi(t), r)$ $(r \geq 1)$, by $(E,1)(C,1)$ product summability means of conjugate series of Fourier series. He has proved the following theorem.

Theorem 2.1 (see [8]). *If $\widetilde{f(x)}$, conjugate to a 2π-periodic function $f(x)$, belongs to $\mathrm{Lip}(\xi(t), r)$ class, then its degree of approximation by $(E,1)(C,1)$ product summability means of conjugate series of Fourier series is given by*

$$\left\|\widetilde{E_n^1C_n^1} - \widetilde{f}\right\|_r = O\left\{(n+1)^{1/r}\xi\left(\frac{1}{n+1}\right)\right\}, \tag{2.1}$$

provided $\xi(t)$ satisfies the following conditions:

$$\left(\int_0^{1/(n+1)}\left(\frac{t|\psi(t)|}{\xi(t)}\right)^r dt\right)^{1/r} = O\left(\frac{1}{n+1}\right), \tag{2.2}$$

$$\left(\int_{1/(n+1)}^{\pi}\left(\frac{t^{-\delta}|\psi(t)|}{\xi(t)}\right)^r dt\right)^{1/r} = O\left\{(n+1)^{\delta}\right\}, \tag{2.3}$$

where δ is an arbitrary number such that $s(1-\delta)-1>0$, $r^{-1}+s^{-1}=1, 1 \leq r \leq \infty$, conditions (2.2) *and* (2.3) *hold uniformly in x and $\widetilde{E_n^1C_n^1}$ is $(E,1)(C,1)$ means of the series* (1.2).

Remark 2.2. The proof proceeds by estimating $|\widetilde{E_n^1C_n^1} - \widetilde{f}|$, which is represented in terms of an integral. The domain of integration is divided into two parts—from $[0, 1/(n+1)]$ and $[1/(n+1), \pi]$. Referring to second integral as I_2, and using Hölder inequality, the author [8] obtains

$$\begin{aligned}
|I_2| &\leq \left\{\int_{1/(n+1)}^{\pi}\left(\frac{t^{-\delta}|\psi(t)|}{\xi(t)}\right)^r dt\right\}^{1/r}\left\{\int_{1/(n+1)}^{\pi}\left(\frac{\xi(t)\left|\widetilde{G_n(t)}\right|}{t^{-\delta}}\right)^s dt\right\}^{1/s} \\
&= O\left\{(n+1)^{\delta}\right\}\left\{\int_{1/(n+1)}^{\pi}\left(\frac{\xi(t)\left|\widetilde{G_n(t)}\right|}{t^{-\delta}}\right)^s dt\right\}^{1/s}.
\end{aligned} \tag{2.4}$$

The author then makes the substitution $y = 1/t$ to obtain

$$= O\left\{(n+1)^{\delta}\right\}\left[\int_{1/\pi}^{n+1}\left(\frac{\xi(1/y)}{y^{\delta-1}}\right)^{s}\frac{dy}{y^2}\right]^{1/s}. \tag{2.5}$$

In the next step $\xi(1/y)$ is removed from the integrand by replacing it with $O(\xi(1/(n+1)))$, while $\xi(t)$ is an increasing function, $\xi(1/y)$ is now a decreasing function. Therefore, this step is invalid.

Remark 2.3. The proof follows by obtaining $|\widetilde{(EC)}_n^1 - \tilde{f}|$, in Theorem 2 of Nigam and Sharma [9], which is expressed in terms of an integral. The domain of integration is divided into two parts—from $[0, 1/(n+1)]$ and $[1/(n+1), \pi]$. Referring to second integral as $I_{2.2}$, and using Hölder inequality, the authors [9] obtain the following:

$$|I_{2.2}| \le \left\{\int_{1/(n+1)}^{\pi}\left(\frac{t^{-\delta}|\psi(t)|\sin^{\beta}t}{\xi(t)}\right)^{r}dt\right\}^{1/r}\left\{\int_{1/(n+1)}^{\pi}\left(\frac{\xi(t)\left|\widetilde{G_n(t)}\right|}{t^{-\delta}\sin^{\beta}t}\right)^{s}dt\right\}^{1/s} \tag{2.6}$$

$$= O\left\{(n+1)^{\delta}\right\}\left\{\int_{1/(n+1)}^{\pi}\left(\frac{\xi(t)}{t^{1-\delta+\beta}}\right)^{s}dt\right\}^{1/s}. \tag{2.7}$$

The authors then make the substitution $y = 1/t$ to get

$$= O\left\{(n+1)^{\delta}\right\}\left[\int_{1/\pi}^{n+1}\left(\frac{\xi(1/y)}{y^{\delta-1-\beta}}\right)^{s}\frac{dy}{y^2}\right]^{1/s}. \tag{2.8}$$

In the next step, $\xi(1/y)$ is removed from the integrand by replacing it with $O(\xi(1/(n+1)))$, while $\xi(t)$ is an increasing function, $\xi(1/y)$ is now a decreasing function. Therefore, in view of second mean value theorem of integral, this step is invalid.

Remark 2.4. The condition $1/\sin^{\beta}(t) = O(1/t^{\beta}), 1/(n+1) \le t \le \pi$ used by Nigam and Sharma [9] is not valid, since $\sin t \to 0$ as $t \to \pi$.

3. Main Result

It is well known that the theory of approximation, that is, TFA, which originated from a theorem of Weierstrass, has become an exciting interdisciplinary field of study for the last 130 years. These approximations have assumed important new dimensions due to their wide applications in signal analysis [15], in general, and in digital signal processing [16] in particular, in view of the classical Shannon sampling theorem. Broadly speaking, signals are treated as function of one variable and images are represented by functions of two variables.

This has motivated Mittal and Rhoades [17–20] and Mittal et al. [4, 21] to obtain many results on TFA using summability methods without rows of the matrix. In this paper, we prove the following theorem.

Theorem 3.1. *If $\widetilde{f(x)}$, conjugate to a 2π-periodic function f, belongs to the generalized weighted $W(L_r, \xi(t))(r \geq 1)$-class, then its degree of approximation by $(E,1)(C,1)$ product summability means of conjugate series of Fourier series is given by*

$$\left\| \widetilde{E_n^1 C_n^1} - \widetilde{f} \right\|_r = O\left\{ (n+1)^{\beta+1/r} \xi\left(\frac{1}{n+1}\right) \right\}, \tag{3.1}$$

provided $\xi(t)$ satisfies the following conditions:

$$\left(\int_0^{\pi/(n+1)} \left(\frac{t|\psi(t)|}{\xi(t)} \right)^r sin^{\beta r}\left(\frac{t}{2}\right) dt \right)^{1/r} = O\left(\frac{1}{n+1}\right), \tag{3.2}$$

$$\left(\int_{\pi/(n+1)}^{\pi} \left(\frac{t^{-\delta}|\psi(t)|}{\xi(t)} \right)^r dt \right)^{1/r} = O(n+1)^{\delta}, \tag{3.3}$$

$$\left\{ \frac{\xi(t)}{t} \right\} \text{ is nonincreasing sequence in "}t\text{"}, \tag{3.4}$$

where δ is an arbitrary number such that $s(1-\delta)-1>0$, $r^{-1}+s^{-1}=1, 1 \leq r \leq \infty$, conditions (3.2) *and* (3.3) *hold uniformly in x and $\widetilde{E_n^1 C_n^1}$ is $\widetilde{(E,1)(C,1)}$ means of the series* (1.2) *and the conjugate function $\widetilde{f}(x)$ is defined for almost every x by*

$$\widetilde{f}(x) = -\frac{1}{2\pi} \int_0^{\pi} \psi(t) \cot\left(\frac{t}{2}\right) dt = \lim_{h \to 0}\left(-\frac{1}{2\pi} \int_h^{\pi} \psi(t) \cot\left(\frac{t}{2}\right) dt \right). \tag{3.5}$$

Note 1. $\xi(\pi/(n+1)) \leq \pi\xi(1/(n+1))$, for $(\pi/(n+1)) \geq (1/(n+1))$.

Note 2. Also for $\beta = 0$, Theorem 3.1 reduces to Theorem 2.1, and thus generalizes the theorem of Nigam [8]. Also our Theorem 3.1 in the modified form of Theorem 2 of Nigam and Sharma [9].

Note 3. The product transform $(E,1)(C,1)$ plays an important role in signal theory as a double digital filter [6] and the theory of machines in mechanical engineering.

4. Lemmas

For the proof of our theorem, the following lemmas are required.

Lemma 4.1. *Consider $|\widetilde{G_n(t)}| = O[1/t]$ for $0 < t \leq \pi/(n+1)$.*

Proof. For $0 < t \leq \pi/(n+1), \sin(t/2) \geq (t/\pi)$ and $|\cos nt| \leq 1$.

$$\begin{aligned}
\left|\widetilde{G_n(t)}\right| &= \frac{1}{2\pi(2)^n}\left|\sum_{k=0}^{n}\left[\binom{n}{k}\left(\frac{1}{1+k}\right)\sum_{v=0}^{k}\frac{\cos(v+1/2)t}{\sin(t/2)}\right]\right| \\
&\leq \frac{1}{2\pi(2)^n}\sum_{k=0}^{n}\left[\binom{n}{k}\left(\frac{1}{1+k}\right)\sum_{v=0}^{k}\frac{|\cos(v+1/2)t|}{|\sin(t/2)|}\right] \\
&= \frac{1}{2t(2)^n}\sum_{k=0}^{n}\left[\binom{n}{k}\frac{1}{(k+1)}\sum_{v=0}^{k}(1)\right] \\
&= \frac{1}{2t(2)^n}\sum_{k=0}^{n}\left[\binom{n}{k}\right] \\
&= \frac{1}{2t(2)^n}(2)^n = O\left[\frac{1}{t}\right], \quad \text{since } \sum_{k=0}^{n}\binom{n}{k} = (2)^n.
\end{aligned} \tag{4.1}$$

This completes the proof of Lemma 4.1. □

Lemma 4.2. *Consider* $|\widetilde{G_n(t)}| = O[1/t]$, *for* $0 < t \leq \pi$ *and any* n.

Proof. For $0 < \pi/(n+1) \leq t \leq \pi, \sin(t/2) \geq (t/\pi)$.

$$\begin{aligned}
\left|\widetilde{G_n(t)}\right| &= \frac{1}{2\pi(2)^n}\left|\sum_{k=0}^{n}\left[\binom{n}{k}\left(\frac{1}{1+k}\right)\sum_{v=0}^{k}\frac{\cos(v+1/2)t}{\sin(t/2)}\right]\right| \\
&\leq \frac{1}{2t(2)^n}\left|\sum_{k=0}^{n}\left[\binom{n}{k}\left(\frac{1}{1+k}\right)\operatorname{Re}\left\{\sum_{v=0}^{k}e^{i(v+1/2)t}\right\}\right]\right| \\
&\leq \frac{1}{2t(2)^n}\left|\sum_{k=0}^{n}\left[\binom{n}{k}\left(\frac{1}{1+k}\right)\operatorname{Re}\left\{\sum_{v=0}^{k}e^{ivt}\right\}\right]\right|\left|e^{i(t/2)}\right| \\
&\leq \frac{1}{2t(2)^n}\left|\sum_{k=0}^{n}\left[\binom{n}{k}\left(\frac{1}{1+k}\right)\operatorname{Re}\left\{\sum_{v=0}^{k}e^{ivt}\right\}\right]\right| \\
&\leq \frac{1}{2t(2)^n}\left|\sum_{k=0}^{\tau-1}\left[\binom{n}{k}\left(\frac{1}{1+k}\right)\operatorname{Re}\left\{\sum_{v=0}^{k}e^{ivt}\right\}\right]\right| \\
&\quad + \frac{1}{2t(2)^n}\left|\sum_{k=\tau}^{n}\left[\binom{n}{k}\left(\frac{1}{1+k}\right)\operatorname{Re}\left\{\sum_{v=0}^{k}e^{ivt}\right\}\right]\right|.
\end{aligned} \tag{4.2}$$

Now considering first term of (4.2)

$$\begin{aligned}
\frac{1}{2t(2)^n}\left|\sum_{k=0}^{\tau-1}\left[\binom{n}{k}\left(\frac{1}{1+k}\right)\operatorname{Re}\left\{\sum_{v=0}^{k}e^{ivt}\right\}\right]\right| &\leq \frac{1}{2t(2)^n}\left|\sum_{k=0}^{\tau-1}\left[\binom{n}{k}\left(\frac{1}{1+k}\right)\left\{\sum_{v=0}^{k}1\right\}\right]\right|\left|e^{ivt}\right| \\
&\leq \frac{1}{2t(2)^n}\left|\sum_{k=0}^{\tau-1}\left[\binom{n}{k}\right]\right|.
\end{aligned} \tag{4.3}$$

Now considering second term of (4.2) and using Abel's lemma

$$\begin{aligned}\frac{1}{2t(2)^n}\left|\sum_{k=\tau}^{n}\left[\binom{n}{k}\left(\frac{1}{1+k}\right)\operatorname{Re}\left\{\sum_{v=0}^{k}e^{ivt}\right\}\right]\right| &\le \frac{1}{2t(2)^n}\sum_{k=\tau}^{n}\binom{n}{k}\left(\frac{1}{1+k}\right)\max_{0\le m\le k}\left|\sum_{v=0}^{m}e^{ivt}\right| \\ &\le \frac{1}{2t(2)^n}\sum_{k=\tau}^{n}\left[\binom{n}{k}\left(\frac{1}{1+k}\right)(1+k)\right] \\ &= \frac{1}{2t(2)^n}\sum_{k=\tau}^{n}\left[\binom{n}{k}\right].\end{aligned} \tag{4.4}$$

On combining (4.2), (4.3), and (4.4)

$$\begin{gathered}\left|\widetilde{G_n(t)}\right| \le \frac{1}{2t(2)^n}\sum_{k=0}^{\tau-1}\left[\binom{n}{k}\right] + \frac{1}{2t(2)^n}\sum_{k=\tau}^{n}\binom{n}{k}, \\ \left|\widetilde{G_n(t)}\right| = O\left[\frac{1}{t}\right].\end{gathered} \tag{4.5}$$

This completes the proof of Lemma 4.2. □

5. Proof of Theorem

Let $\widetilde{s_n(x)}$ denotes the nth partial sum of series (1.2). Then following Nigam [8], we have

$$\widetilde{s_n(x)} - \widetilde{f(x)} = \frac{1}{2\pi}\int_0^{\pi}\psi(t)\frac{\cos(n+1/2)t}{\sin(t/2)}dt. \tag{5.1}$$

Therefore, using (1.2), the $(C,1)$ transform C_n^1 of $\widetilde{s_n}$ is given by

$$\widetilde{C_n^1} - \widetilde{f(x)} = \frac{1}{2\pi(n+1)}\int_0^{\pi}\psi(t)\sum_{k=0}^{n}\frac{\cos(n+1/2)t}{\sin(t/2)}dt. \tag{5.2}$$

Now denoting $\widetilde{(E,1)(C,1)}$ transform of $\widetilde{s_n}$ as $\widetilde{(E_n^1, C_n^1)}$, we write

$$\begin{aligned}\widetilde{(E_n^1, C_n^1)} - \widetilde{f(x)} &= \frac{1}{2\pi(2)^n}\sum_{k=0}^{n}\left[\binom{n}{k}\int_0^{\pi}\frac{\psi(t)}{\sin(t/2)}\left(\frac{1}{1+k}\right)\left\{\sum_{v=0}^{k}\cos(v+1/2)t\right\}dt\right] \\ &= \int_0^{\pi}\psi(t)\widetilde{G_n(t)}dt \\ &= \left[\int_0^{\pi/(n+1)} + \int_{\pi/(n+1)}^{\pi}\right]\psi(t)\widetilde{G_n(t)}dt \\ &= I_1 + I_2 \text{ (say)}.\end{aligned} \tag{5.3}$$

We consider, $|I_1| \le \int_0^{\pi/(n+1)} |\psi(t)||\widetilde{G_n(t)}|dt$.

Using Hölder's inequality

$$\begin{aligned}
|I_1| &\leq \left[\int_0^{\pi/(n+1)} \left(\frac{t|\psi(t)|}{\xi(t)}\right)^r \sin^{\beta r}\left(\frac{t}{2}\right) dt\right]^{1/r} \left[\int_0^{\pi/(n+1)} \left(\frac{\xi(t)\left|\widetilde{G_n(t)}\right|}{t\sin^{\beta}(t/2)}\right)^s dt\right]^{1/s} \\
&= O\left(\frac{1}{n+1}\right)\left[\int_0^{\pi/(n+1)} \left(\frac{\xi(t)\left|\widetilde{G_n(t)}\right|}{t\sin^{\beta}(t/2)}\right)^s dt\right]^{1/s} \quad \text{by (3.2)} \\
&= O\left(\frac{1}{n+1}\right)\left[\int_0^{\pi/(n+1)} \left(\frac{\xi(t)}{t^2\sin^{\beta}(t/2)}\right)^s dt\right]^{1/s} \quad \text{by Lemma 4.1} \\
&= O\left(\frac{1}{n+1}\right)\left[\int_0^{\pi/(n+1)} \left(\frac{\xi(t)(t/2)^{\beta}}{t^2\sin^{\beta}(t/2)\cdot(t/2)^{\beta}}\right)^s dt\right]^{1/s} \\
&= O\left(\frac{1}{n+1}\right)\left[\left(\frac{2(\pi/2(n+1))}{\sin(\pi/2(n+1))}\right)^{\beta s} \int_h^{\pi/(n+1)} \left(\frac{\xi(t)}{t^{2+\beta}}\right)^s dt\right]^{1/s}, \quad \text{as } h \longrightarrow 0 \\
&= O\left(\frac{1}{n+1}\right)\left[\int_h^{\pi/(n+1)} \left(\frac{\xi(t)}{t^{2+\beta}}\right)^s dt\right]^{1/s}.
\end{aligned} \tag{5.4}$$

Since $\xi(t)$ is a positive increasing function and by using second mean value theorem for integrals, we have

$$\begin{aligned}
I_1 &= O\left\{\left(\frac{1}{n+1}\right)\xi\left(\frac{\pi}{n+1}\right)\right\}\left[\int_{\in}^{\pi/(n+1)} \left(\frac{1}{t^{2+\beta}}\right)^s dt\right]^{1/s}, \quad \text{for some } 0 < \in < \frac{\pi}{n+1} \\
&= O\left\{\left(\frac{1}{n+1}\right)\pi\xi\left(\frac{1}{n+1}\right)\right\}\left[\int_{\in}^{\pi/(n+1)} t^{-\beta s-2s} dt\right]^{1/s}.
\end{aligned} \tag{5.5}$$

Note that $\xi(\pi/(n+1)) \leq \pi\xi(1/(n+1))$,

$$\begin{aligned}
I_1 &= O\left\{\left(\frac{1}{n+1}\right)\xi\left(\frac{1}{n+1}\right)\right\}\left[\left\{\frac{t^{-\beta s-2s+1}}{-\beta s-2s+1}\right\}_{\in}^{\pi/(n+1)}\right]^{1/s} \\
&= O\left[\left(\frac{1}{n+1}\right)\xi\left(\frac{1}{n+1}\right)(n+1)^{\beta+2-1/s}\right] \\
&= O\left[\xi\left(\frac{1}{n+1}\right)(n+1)^{\beta+1-1/s}\right] \\
&= O\left[\xi\left(\frac{1}{n+1}\right)(n+1)^{\beta+1/r}\right] \quad \because r^{-1}+s^{-1}=1,\ 1\leq r\leq\infty.
\end{aligned} \tag{5.6}$$

Now, we consider

$$|I_2| \leq \int_{\pi/(n+1)}^{\pi} |\psi(t)|\left|\widetilde{G_n(t)}\right| dt. \tag{5.7}$$

Using Hölder's inequality

$$
\begin{aligned}
|I_2| &\le \left[\int_{\pi/(n+1)}^{\pi}\left(\frac{t^{-\delta}\sin^{\beta}(t/2)|\psi(t)|}{\xi(t)}\right)^{r}dt\right]^{1/r}\left[\int_{\pi/(n+1)}^{\pi}\left(\frac{\xi(t)\left|\widetilde{G_n(t)}\right|}{t^{-\delta}\sin^{\beta}(t/2)}\right)^{s}dt\right]^{1/s}\\
&= O\left\{(n+1)^{\delta}\right\}\left[\int_{\pi/(n+1)}^{\pi}\left(\frac{\xi(t)\left|\widetilde{G_n(t)}\right|}{t^{-\delta}\sin^{\beta}(t/2)}\right)^{s}dt\right]^{1/s} \quad \text{by (3.3)}\\
&= O\left\{(n+1)^{\delta}\right\}\left[\int_{\pi/(n+1)}^{\pi}\left(\frac{\xi(t)}{t^{-\delta}t\sin^{\beta}(t/2)}\right)^{s}dt\right]^{1/s} \quad \text{by Lemma 4.2}\\
&= O\left\{(n+1)^{\delta}\right\}\left[\int_{\pi/(n+1)}^{\pi}\left(\frac{\xi(t)}{t^{-\delta+1}\sin^{\beta}(t/2)}\right)^{s}dt\right]^{1/s}\\
&= O\left\{(n+1)^{\delta}\right\}\left[\int_{\pi/(n+1)}^{\pi}\left(\frac{\xi(t)}{t^{-\delta+\beta+1}}\right)^{s}dt\right]^{1/s}.
\end{aligned}
\tag{5.8}
$$

Now putting $t = 1/y$, we have

$$
I_2 = O\left\{(n+1)^{\delta}\right\}\left[\int_{1/\pi}^{(n+1)/\pi}\left(\frac{\xi(1/y)}{y^{\delta-\beta-1}}\right)^{s}\frac{dy}{y^2}\right]^{1/s}. \tag{5.9}
$$

Since $\xi(t)$ is a positive increasing function, so $\xi(1/y)/(1/y)$ is also a positive increasing function and using second mean value theorem for integrals, we have

$$
\begin{aligned}
&= O\left\{(n+1)^{\delta}\frac{\xi(\pi/(n+1))}{\pi/(n+1)}\right\}\left[\int_{\eta}^{(n+1)/\pi}\left(\frac{dy}{y^{-\beta s+\delta s+2}}\right)\right]^{1/s}, \quad \text{for some } \frac{1}{\pi}\le\eta\le\frac{n+1}{\pi}\\
&= O\left\{(n+1)^{\delta+1}\xi\left(\frac{1}{n+1}\right)\right\}\left\{\left[\frac{y^{-\delta s-2+\beta s+1}}{-\delta s-2+\beta s+1}\right]_{1}^{(n+1)/\pi}\right\}^{1/s}, \quad \text{for some } \frac{1}{\pi}\le 1\le\frac{n+1}{\pi}\\
&= O\left\{(n+1)^{\delta+1}\xi\left(\frac{1}{n+1}\right)\right\}\left\{\left[y^{-\delta s-1+\beta s}\right]_{1}^{n+1/\pi}\right\}^{1/s}\\
&= O\left\{(n+1)^{\delta+1}\xi\left(\frac{1}{n+1}\right)\right\}(n+1)^{-\delta-1/s+\beta}\\
&= O\left\{\xi\left(\frac{1}{n+1}\right)(n+1)^{\delta+1-\delta-1/s+\beta}\right\}\\
&= O\left\{\xi\left(\frac{1}{n+1}\right)(n+1)^{\beta+1/r}\right\} \quad \because r^{-1}+s^{-1}=1,\ 1\le r\le\infty.
\end{aligned}
\tag{5.10}
$$

Combining I_1 and I_2 yields

$$
\left|\widetilde{E_n^1C_n^1}-\tilde{f}\right| = O\left\{(n+1)^{1/r+\beta}\xi\left(\frac{1}{n+1}\right)\right\}. \tag{5.11}
$$

Now, using the L_r-norm of a function, we get

$$\begin{aligned}\left\|\widetilde{E_n^1C_n^1}-\tilde{f}\right\|_r &= \left\{\int_0^{2\pi}\left|\widetilde{E_n^1C_n^1}-\tilde{f}\right|^r dx\right\}^{1/r}\\ &= O\left\{\int_0^{2\pi}\left((n+1)^{\beta+1/r}\xi\left(\frac{1}{n+1}\right)\right)^r dx\right\}^{1/r}\\ &= O\left\{(n+1)^{\beta+1/r}\xi\left(\frac{1}{n+1}\right)\left(\int_0^{2\pi}dx\right)^{1/r}\right\}\\ &= O\left((n+1)^{\beta+1/r}\xi\left(\frac{1}{n+1}\right)\right).\end{aligned} \tag{5.12}$$

This completes the proof of Theorem 3.1.

6. Applications

The theory of approximation is a very extensive field, which has various applications, and the study of the theory of trigonometric Fourier approximation is of great mathematical interest and of great practical importance. From the point of view of the applications, Sharper estimates of infinite matrices [22] are useful to get bounds for the lattice norms (which occur in solid state physics) of matrix valued functions and enables to investigate perturbations of matrix valued functions and compare them.

The following corollaries may be derived from Theorem 3.1.

Corollary 6.1. *If $\xi(t) = t^\alpha, 0 < \alpha \le 1$, then the weighted class $W(L_r, \xi(t)), r \ge 1$ reduces to the class $\mathrm{Lip}(\alpha, r)$ and the degree of approximation of a function $\tilde{f}(x)$ conjugate to a 2π-periodic function f belonging to the class $\mathrm{Lip}(\alpha, r)$, is given by*

$$\left|\widetilde{E_n^1C_n^1}-\tilde{f}\right| = O\left(\frac{1}{(n+1)^{\alpha-1/r}}\right). \tag{6.1}$$

Proof. The result follows by setting $\beta = 0$ in (3.1), we have

$$\begin{aligned}\left\|\widetilde{E_n^1C_n^1}-\tilde{f}\right\|_r &= \left\{\int_0^{2\pi}\left|\widetilde{E_n^1C_n^1}-\tilde{f}\right|^r dx\right\}^{1/r} = O\left((n+1)^{1/r}\xi\left(\frac{1}{n+1}\right)\right)\\ &= O\left(\frac{1}{(n+1)^{\alpha-1/r}}\right), \quad r \ge 1.\end{aligned} \tag{6.2}$$

Thus, we get

$$\left|\widetilde{E_n^1C_n^1}-\tilde{f}\right| \le \left\{\int_0^{2\pi}\left|\widetilde{E_n^1C_n^1}-\tilde{f}\right|^r dx\right\}^{1/r} = O\left(\frac{1}{(n+1)^{\alpha-1/r}}\right), \quad r \ge 1. \tag{6.3}$$

This completes the proof of Corollary 6.1. □

Corollary 6.2. *If $\xi(t) = t^{\alpha}$ for $0 < \alpha < 1$ and $r \to \infty$ in Corollary 6.1, then $f \in \mathrm{Lip}\alpha$ and*

$$\left|\widetilde{E_n^1 C_n^1} - \tilde{f}\right| = O\left(\frac{1}{(n+1)^{\alpha}}\right). \tag{6.4}$$

Proof. For $r = \infty$ in Corollary 6.1, we get

$$\begin{aligned}\left\|\widetilde{E_n^1 C_n^1} - \tilde{f}\right\|_{\infty} &= \sup_{0 \le x \le 2\pi}\left|\widetilde{E_n^1 C_n^1}(x) - \tilde{f}(x)\right| \\ &= O\left(\frac{1}{(n+1)^{\alpha}}\right).\end{aligned} \tag{6.5}$$

Thus, we get

$$\begin{aligned}\left|\widetilde{E_n^1 C_n^1} - \tilde{f}\right| &\le \left\|\widetilde{E_n^1 C_n^1} - \tilde{f}\right\|_{\infty} \\ &= \sup_{0 \le x \le 2\pi}\left|\widetilde{E_n^1 C_n^1}(x) - \tilde{f}(x)\right| \\ &= O\left(\frac{1}{(n+1)^{\alpha}}\right).\end{aligned} \tag{6.6}$$

This completes the proof of Corollary 6.2. □

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