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Research Article

The Real and Complex Hermitian Solutions to a System of Quaternion Matrix Equations with Applications

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We establish necessary and sufficient conditions for the existence of and the expressions for the general real and complex Hermitian solutions to the classical system of quaternion matrix equations $A_1X = C_1, XB_1 = C_2$, and $A_3XA_3^* = C_3$. Moreover, formulas of the maximal and minimal ranks of four real matrices X_1, X_2, X_3, and X_4 in solution $X = X_1 + X_2i + X_3j + X_4k$ to the system mentioned above are derived. As applications, we give necessary and sufficient conditions for the quaternion matrix equations $A_1X = C_1, XB_1 = C_2, A_3XA_3^* = C_3$, and $A_4XA_4^* = C_4$ to have real and complex Hermitian solutions.

1. Introduction

Throughout this paper, we denote the real number field by $\mathbb{R}$; the complex field by $\mathbb{C}$; the set of all $m \times n$ matrices over the quaternion algebra

$$\mathbb{H} = \left\{ a_0 + a_1 i + a_2 j + a_3 k \mid i^2 = j^2 = k^2 = ijk = -1, a_0, a_1, a_2, a_3 \in \mathbb{R} \right\} \tag{1.1}$$

by $\mathbb{H}^{m\times n}$; the identity matrix with the appropriate size by I; the transpose, the conjugate transpose, the column right space, the row left space of a matrix A over $\mathbb{H}$ by $A^T, A^*, \mathcal{R}(A)$, $\mathcal{N}(A)$, respectively; the dimension of $\mathcal{R}(A)$ by $\dim \mathcal{R}(A)$. By [1], for a quaternion matrix $A, \dim \mathcal{R}(A) = \dim \mathcal{N}(A)$. $\dim \mathcal{R}(A)$ is called the rank of a quaternion matrix A and denoted by $r(A)$. The Moore-Penrose inverse of matrix A over $\mathbb{H}$ by $A^\dagger$ which satisfies four Penrose equations $AA^\dagger A = A, A^\dagger AA^\dagger = A^\dagger, (AA^\dagger)^* = AA^\dagger$, and $(A^\dagger A)^* = A^\dagger A$. In this case, $A^\dagger$ is unique and $(A^\dagger)^* = (A^*)^\dagger$. Moreover, R_A and L_A stand for the two projectors $L_A = I - A^\dagger A$, and $R_A = I - AA^{\dagger}$ induced by A. Clearly, R_A and L_A are idempotent and satisfies $(R_A)^* = R_A$, $(L_A)^* = L_A$, $R_A = L_{A^*}$, and $R_{A^*} = L_A$.

Hermitian solutions to some matrix equations were investigated by many authors. In 1976, Khatri and Mitra [2] gave necessary and sufficient conditions for the existence of the Hermitian solutions to the matrix equations $AX = B$, $AXB = C$ and

$$A_1X = C_1, \qquad XB_2 = C_2, \tag{1.2}$$

over the complex field $\mathbb{C}$, and presented explicit expressions for the general Hermitian solutions to them by generalized inverses when the solvability conditions were satisfied. Matrix equation that has symmetric patterns with Hermitian solutions appears in some application areas, such as vibration theory, statistics, and optimal control theory ([3–7]). Groß in [8], and Liu et al. in [9] gave the solvability conditions for Hermitian solution and its expressions of

$$AXA^* = B \tag{1.3}$$

over $\mathbb{C}$ in terms of generalized inverses, respectively. In [10], Tian and Liu established the solvability conditions for

$$A_3XA_3^* = C_3, \qquad A_4XA_4^* = C_4 \tag{1.4}$$

to have a common Hermitian solution over $\mathbb{C}$ by the ranks of coefficient matrices. In [11], Tian derived the general common Hermitian solution of (1.4). Wang and Wu in [12] gave some necessary and sufficient conditions for the existence of the common Hermitian solution to equations

$$A_1X = C_1, \qquad XB_2 = C_2, \qquad A_3XA_3^* = C_3, \tag{1.5}$$

$$A_1X = C_1, \qquad XB_2 = C_2, \qquad A_3XA_3^* = C_3, \qquad A_4XA_4^* = C_4, \tag{1.6}$$

for operators between Hilbert C^*-modules by generalized inverses and range inclusion of matrices.

As is known to us, extremal ranks of some matrix expressions can be used to characterize nonsingularity, rank invariance, range inclusion of the corresponding matrix expressions, as well as solvability conditions of matrix equations ([4, 7, 9–24]). Real matrices and its extremal ranks in solutions to some complex matrix equation have been investigated by Tian and Liu ([9, 13–15]). Tian [13] gave the maximal and minimal ranks of two real matrices X_0 and X_1 in solution $X = X_0 + iX_1$ to $AXB = C$ over $\mathbb{C}$ with its applications. Liu et al. [9] derived the maximal and minimal ranks of the two real matrices X_0 and X_1 in a Hermitian solution $X = X_0 + iX_1$ of (1.3), where $B^* = B$. In order to investigate the real and complex solutions to quaternion matrix equations, Wang and his partners have been studying the real matrices in solutions to some quaternion matrix equations such as $AXB = C$,

$$\begin{aligned} A_1XB_1 = C_1, \qquad A_2XB_2 = C_2, \\ AXA^* + BXB^* = C, \end{aligned} \tag{1.7}$$

recently ([24–27]). To our knowledge, the necessary and sufficient conditions for (1.5) over $\mathbb{H}$ to have the real and complex Hermitian solutions have not been given so far. Motivated by the work mentioned above, we in this paper investigate the real and complex Hermitian solutions to system (1.5) over $\mathbb{H}$ and its applications.

This paper is organized as follows. In Section 2, we first derive formulas of extremal ranks of four real matrices X_1, X_2, X_3, and X_4 in quaternion solution $X = X_1 + X_2 i + X_3 j + X_4 k$ to (1.5) over $\mathbb{H}$, then give necessary and sufficient conditions for (1.5) over $\mathbb{H}$ to have real and complex solutions as well as the expressions of the real and complex solutions. As applications, we in Section 3 establish necessary and sufficient conditions for (1.6) over $\mathbb{H}$ to have real and complex solutions.

2. The Real and Complex Hermitian Solutions to System (1.5) Over $\mathbb{H}$

In this section, we first give a solvability condition and an expression of the general Hermitian solution to (1.5) over $\mathbb{H}$, then consider the maximal and minimal ranks of four real matrices X_1, X_2, X_3, and X_4 in solution $X = X_1 + X_2 i + X_3 j + X_4 k$ to (1.5) over $\mathbb{H}$, last, investigate the real and complex Hermitian solutions to (1.5) over $\mathbb{H}$.

For an arbitrary matrix $M_t = M_{t1} + M_{t2} i + M_{t3} j + M_{t4} k \in \mathbb{H}^{m\times n}$ where M_{t1}, M_{t2}, M_{t3}, and M_{t4} are real matrices, we define a map $\phi(\cdot)$ from $\mathbb{H}^{m\times n}$ to $\mathbb{R}^{4m\times 4n}$ by

$$\phi(M_t) = \begin{bmatrix} M_{t1} & M_{t2} & M_{t3} & M_{t4} \\ -M_{t2} & M_{t1} & M_{t4} & -M_{t3} \\ -M_{t3} & -M_{t4} & M_{t1} & M_{t2} \\ -M_{t4} & M_{t3} & -M_{t2} & M_{t1} \end{bmatrix}. \tag{2.1}$$

By (2.1), it is easy to verify that $\phi(\cdot)$ satisfies the following properties.

(a) $M = N \Leftrightarrow \phi(M) = \phi(N)$.

(b) $\phi(kM + lN) = k\phi(M) + l\phi(N)$, $\phi(MN) = \phi(M)\phi(N)$, $k, l \in \mathbb{R}$.

(c) $\phi(M^*) = \phi^T(M)$, $\phi(M^\dagger) = \phi^\dagger(M)$.

(d) $\phi(M) = T_m^{-1}\phi(M)T_n = R_m^{-1}\phi(M)R_n = S_m^{-1}\phi(M)S_n$, where $t = m, n$,

$$T_t = \begin{bmatrix} 0 & -I_t & 0 & 0 \\ I_t & 0 & 0 & 0 \\ 0 & 0 & 0 & I_t \\ 0 & 0 & -I_t & 0 \end{bmatrix}, \quad R_t = \begin{bmatrix} 0 & 0 & -I_t & 0 \\ 0 & 0 & 0 & -I_t \\ I_t & 0 & 0 & 0 \\ 0 & I_t & 0 & 0 \end{bmatrix}, \quad S_t = \begin{bmatrix} 0 & 0 & 0 & -I_t \\ 0 & 0 & I_t & 0 \\ 0 & -I_t & 0 & 0 \\ I_t & 0 & 0 & 0 \end{bmatrix}. \tag{2.2}$$

(e) $r[\phi(M)] = 4r(M)$.

(f) $M^* = M \Leftrightarrow \phi^T(M) = \phi(M)$, $M^* = -M \Leftrightarrow \phi^T(M) = -\phi(M)$.

The following lemmas provide us with some useful results over $\mathbb{C}$, which can be generalized to $\mathbb{H}$.

Lemma 2.1 (see [2, Lemma 2.1]). *Let $A \in \mathbb{H}^{m\times n}, B = B^* \in \mathbb{H}^{m\times m}$ be known, $X \in \mathbb{H}^{n\times n}$ unknown; then the system* (1.3) *has a Hermitian solution if and only if*

$$AA^{\dagger}B = B. \tag{2.3}$$

In that case, the general Hermitian solution of (1.3) *can be expressed as*

$$X = A^{\dagger}B\left(A^{\dagger}\right)^* + L_AV + V^*L_A, \tag{2.4}$$

where V is arbitrary matrix over $\mathbb{H}$ with compatible size.

Lemma 2.2 (see [12, Corollary 3.4]). *Let $A_1, C_1 \in \mathbb{H}^{m\times n}; B_1, C_2 \in \mathbb{H}^{n\times s}; A_3 \in \mathbb{H}^{r\times n};\ C_3 \in \mathbb{H}^{r\times r}$ be known, $X \in \mathbb{H}^{n\times n}$ unknown, and $F = B_1^*L_{A_1},\ M = SL_F,\ S = A_3L_{A_1},\ D = C_2^* - B_1^*A_1^{\dagger}C_1,\ J = A_1^{\dagger}C_1 + F^{\dagger}D, G = C_3 - A_3(J + L_{A_1}L_F^*J^*)A_3^*,\ C_3 = C_3^*$; then the system* (1.5) *have a Hermitian solution if and only if*

$$\begin{gathered} A_1C_2 = C_1B_1, \qquad A_1C_1^* = C_1A_1^*, \qquad B_1^*C_2 = C_2^*B_1, \\ R_{A_1}C_1 = 0, \qquad R_FD = 0, \qquad R_MG = 0. \end{gathered} \tag{2.5}$$

In that case, the general Hermitian solution of (1.5) *can be expressed as*

$$X = J + L_{A_1}L_FJ^* + L_{A_1}L_FM^{\dagger}G\left(M^{\dagger}\right)^*L_FL_{A_1} + L_{A_1}L_FL_MVL_FL_{A_1} + L_{A_1}L_FV^*L_ML_FL_{A_1}, \tag{2.6}$$

where V is arbitrary matrix over $\mathbb{H}$ with compatible size.

Lemma 2.3 (see [21, Lemma 2.4]). *Let $A \in \mathbb{H}^{m\times n},\ B \in \mathbb{H}^{m\times k},\ C \in \mathbb{H}^{l\times n},\ D \in \mathbb{H}^{j\times k}$, and $E \in \mathbb{H}^{l\times i}$. Then they satisfy the following rank equalities.*

(a) $r(CL_A) = r\begin{bmatrix} A \\ C \end{bmatrix} - r(A)$.

(b) $r\begin{bmatrix} B & AL_C \end{bmatrix} = r\begin{bmatrix} B & A \\ 0 & C \end{bmatrix} - r(C)$.

(c) $r\begin{bmatrix} C \\ R_BA \end{bmatrix} = r\begin{bmatrix} C & 0 \\ A & B \end{bmatrix} - r(B)$.

(d) $r\begin{bmatrix} A & BL_D \\ R_EC & 0 \end{bmatrix} = r\begin{bmatrix} A & B & 0 \\ C & 0 & E \\ 0 & D & 0 \end{bmatrix} - r(D) - r(E)$.

Lemma 2.3 plays an important role in simplifying ranks of various block matrices.

Lemma 2.4 (see [11, Theorem 4.1, Corollary 4.2]). *Let $A = \pm A^* \in \mathbb{H}^{m\times m},\ B \in \mathbb{H}^{m\times n}$, and $C \in \mathbb{H}^{p\times m}$ be given; then*

$$\begin{gathered} \max_{X\in\mathbb{H}^{n\times p}} r\left[A - BXC \mp (BXC)^*\right] = \min\left\{ r\begin{bmatrix} A & B & C^* \end{bmatrix},\ r\begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix},\ r\begin{bmatrix} A & C^* \\ C & 0 \end{bmatrix} \right\}, \\ \min_{X\in\mathbb{H}^{n\times p}} r\left[A - BXC \mp (BXC)^*\right] = 2r\begin{bmatrix} A & B & C^* \end{bmatrix} + \max\{s_1, s_2\}, \end{gathered} \tag{2.7}$$

where

$$s_1 = r\begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix} - 2r\begin{bmatrix} A & B & C^* \\ B^* & 0 & 0 \end{bmatrix},$$
$$s_2 = r\begin{bmatrix} A & C^* \\ C & 0 \end{bmatrix} - 2r\begin{bmatrix} A & B & C^* \\ C & 0 & 0 \end{bmatrix}. \tag{2.8}$$

If $\mathcal{R}(B) \subseteq \mathcal{R}(C^*)$,

$$\max_{X\in\mathbb{H}^{n\times p}} r\left[A - BXC - (BXC)^*\right] = \min\left\{ r\begin{bmatrix} A & C^* \end{bmatrix},\ r\begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix} \right\}, \tag{2.9}$$

$$\min_{X\in\mathbb{H}^{n\times p}} r\left[A - BXC - (BXC)^*\right] = 2r\begin{bmatrix} A & C^* \end{bmatrix} + r\begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix} - 2r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}. \tag{2.10}$$

Lemma 2.5 (see [28, Theorem 3.1]). *Let* $A \in \mathbb{H}^{m\times n}$, $B_1 \in \mathbb{H}^{m\times p_1}$, $B_3 \in \mathbb{H}^{m\times p_3}$, $B_4 \in \mathbb{H}^{m\times p_4}$, $C_2 \in \mathbb{H}^{q_2\times n}$, $C_3 \in \mathbb{H}^{q_3\times n}$, *and* $C_4 \in \mathbb{H}^{q_4\times n}$ *be given. Then the matrix equation*

$$B_1X_1 + X_2C_2 + B_3X_3C_3 + B_4X_4C_4 = A \tag{2.11}$$

is consistent if and only if

$$r\begin{bmatrix} A & B_1 \\ C_2 & 0 \\ C_3 & 0 \\ C_4 & 0 \end{bmatrix} = r\begin{bmatrix} 0 & B_1 \\ C_2 & 0 \\ C_3 & 0 \\ C_4 & 0 \end{bmatrix}, \qquad r\begin{bmatrix} A & B_1 & B_3 & B_4 \\ C_2 & 0 & 0 & 0 \end{bmatrix} = r\begin{bmatrix} 0 & B_1 & B_3 & B_4 \\ C_2 & 0 & 0 & 0 \end{bmatrix},$$
$$r\begin{bmatrix} A & B_1 & B_3 \\ C_2 & 0 & 0 \\ C_4 & 0 & 0 \end{bmatrix} = r\begin{bmatrix} 0 & B_1 & B_3 \\ C_2 & 0 & 0 \\ C_4 & 0 & 0 \end{bmatrix}, \qquad r\begin{bmatrix} A & B_1 & B_4 \\ C_2 & 0 & 0 \\ C_3 & 0 & 0 \end{bmatrix} = r\begin{bmatrix} 0 & B_1 & B_4 \\ C_2 & 0 & 0 \\ C_3 & 0 & 0 \end{bmatrix}. \tag{2.12}$$

Theorem 2.6. *System* (1.5) *has a Hermitian solution over* $\mathbb{H}$ *if and only if the system of matrix equations*

$$\phi(A_1)\left(Y_{ij}\right)_{4\times 4} = \phi(C_1), \qquad \left(Y_{ij}\right)_{4\times 4}\phi(B_1) = \phi(C_2), \qquad \phi(A_3)\left(Y_{ij}\right)_{4\times 4}\phi^T(A_3) = \phi(C_3),$$
$$i, j = 1, 2, 3, 4, \tag{2.13}$$

has a symmetric solution over $\mathbb{R}$. *In that case, the general Hermitian solution of* (1.5) *over* $\mathbb{H}$ *can be written as*

$$\begin{aligned} X &= X_1 + X_2 i + X_3 j + X_4 k \\ &= \frac{1}{4}(Y_{11} + Y_{22} + Y_{33} + Y_{44}) + \frac{1}{4}\left(Y_{12} - Y_{12}^T + Y_{34} - Y_{34}^T\right)i \\ &\quad + \frac{1}{4}\left(Y_{13} - Y_{13}^T + Y_{24}^T - Y_{24}\right)j + \frac{1}{4}\left(Y_{14} - Y_{14}^T + Y_{23} - Y_{23}^T\right)k, \end{aligned} \tag{2.14}$$

where $Y_{tt} = Y_{tt}^T;\ t = 1,2,3,4;\ Y_{1j}^T = Y_{j1};\ j = 2,3,4;\ Y_{2j}^T = Y_{j2};\ j = 3,4;\ Y_{34}^T = Y_{43}$ *are the general solutions of* (2.13) *over* $\mathbb{R}$. *Written in an explicit form,* X_1, X_2, X_3, *and* X_4 *in* (2.14) *are*

$$\begin{aligned} X_1 &= \frac{1}{4}P_1\phi(X_0)P_1^T + \frac{1}{4}P_2\phi(X_0)P_2^T + \frac{1}{4}P_3\phi(X_0)P_3^T + \frac{1}{4}P_4\phi(X_0)P_4^T \\ &\quad + [P_1, P_2, P_3, P_4]L_{\phi(A_1)}L_{\phi(F)}L_{\phi(M)}V\begin{bmatrix} L_{\phi(F)}L_{\phi(A_1)}P_1^T \\ L_{\phi(F)}L_{\phi(A_1)}P_2^T \\ L_{\phi(F)}L_{\phi(A_1)}P_3^T \\ L_{\phi(F)}L_{\phi(A_1)}P_4^T \end{bmatrix} \\ &\quad + [P_1, P_2, P_3, P_4]L_{\phi(A_1)}L_{\phi(F)}V^T\begin{bmatrix} L_{\phi(M)}L_{\phi(F)}L_{\phi(A_1)}P_1^T \\ L_{\phi(M)}L_{\phi(F)}L_{\phi(A_1)}P_2^T \\ L_{\phi(M)}L_{\phi(F)}L_{\phi(A_1)}P_3^T \\ L_{\phi(M)}L_{\phi(F)}L_{\phi(A_1)}P_4^T \end{bmatrix}, \end{aligned} \tag{2.15}$$

$$\begin{aligned} X_2 &= \frac{1}{4}P_1\phi(X_0)P_2^T - \frac{1}{4}P_2\phi(X_0)P_1^T + \frac{1}{4}P_3\phi(X_0)P_4^T - \frac{1}{4}P_4\phi(X_0)P_3^T \\ &\quad + [P_1, -P_2, P_3, -P_4]L_{\phi(A_1)}L_{\phi(F)}L_{\phi(M)}V\begin{bmatrix} L_{\phi(F)}L_{\phi(A_1)}P_2^T \\ L_{\phi(F)}L_{\phi(A_1)}P_1^T \\ L_{\phi(F)}L_{\phi(A_1)}P_4^T \\ L_{\phi(F)}L_{\phi(A_1)}P_3^T \end{bmatrix} \\ &\quad + [P_2, P_1, P_4, P_3]L_{\phi(A_1)}L_{\phi(F)}V^T\begin{bmatrix} L_{\phi(M)}L_{\phi(F)}L_{\phi(A_1)}P_1^T \\ -L_{\phi(M)}L_{\phi(F)}L_{\phi(A_1)}P_2^T \\ L_{\phi(M)}L_{\phi(F)}L_{\phi(A_1)}P_3^T \\ -L_{\phi(M)}L_{\phi(F)}L_{\phi(A_1)}P_4^T \end{bmatrix}, \end{aligned} \tag{2.16}$$

$$X_3 = \frac{1}{4}P_1\phi(X_0)P_3^T - \frac{1}{4}P_3\phi(X_0)P_1^T + \frac{1}{4}P_4\phi(X_0)P_2^T - \frac{1}{4}P_2\phi(X_0)P_4^T + [P_1, -P_3, P_4, -P_2]L_{\phi(A_1)}L_{\phi(F)}L_{\phi(M)}V\begin{bmatrix} L_{\phi(F)}L_{\phi(A_1)}P_3^T \\ L_{\phi(F)}L_{\phi(A_1)}P_1^T \\ L_{\phi(F)}L_{\phi(A_1)}P_2^T \\ L_{\phi(F)}L_{\phi(A_1)}P_4^T \end{bmatrix} + [P_3, P_1, P_2, P_4]L_{\phi(A_1)}L_{\phi(F)}V^T\begin{bmatrix} L_{\phi(M)}L_{\phi(F)}L_{\phi(A_1)}P_1^T \\ -L_{\phi(M)}L_{\phi(F)}L_{\phi(A_1)}P_3^T \\ L_{\phi(M)}L_{\phi(F)}L_{\phi(A_1)}P_4^T \\ -L_{\phi(M)}L_{\phi(F)}L_{\phi(A_1)}P_2^T \end{bmatrix}, \tag{2.17}$$

$$X_4 = \frac{1}{4}P_1\phi(X_0)P_4^T - \frac{1}{4}P_4\phi(X_0)P_1^T + \frac{1}{4}P_2\phi(X_0)P_3^T - \frac{1}{4}P_3\phi(X_0)P_2^T + [P_1, -P_4, P_2, -P_3]L_{\phi(A_1)}L_{\phi(F)}L_{\phi(M)}V\begin{bmatrix} L_{\phi(F)}L_{\phi(A_1)}P_4^T \\ L_{\phi(F)}L_{\phi(A_1)}P_1^T \\ L_{\phi(F)}L_{\phi(A_1)}P_3^T \\ L_{\phi(F)}L_{\phi(A_1)}P_2^T \end{bmatrix} + [P_4, P_1, P_3, P_2]L_{\phi(A_1)}L_{\phi(F)}V^T\begin{bmatrix} L_{\phi(M)}L_{\phi(F)}L_{\phi(A_1)}P_1^T \\ -L_{\phi(M)}L_{\phi(F)}L_{\phi(A_1)}P_4^T \\ L_{\phi(M)}L_{\phi(F)}L_{\phi(A_1)}P_2^T \\ -L_{\phi(M)}L_{\phi(F)}L_{\phi(A_1)}P_3^T \end{bmatrix}, \tag{2.18}$$

where

$$P_1 = [I_n, 0, 0, 0], \qquad P_2 = [0, I_n, 0, 0], \qquad P_3 = [0, 0, I_n, 0], \qquad P_4 = [0, 0, 0, I_n], \tag{2.19}$$

$\phi(X_0)$ *is a particular symmetric solution to* (2.13)*, and* V *is arbitrary real matrices with compatible sizes.*

Proof. Suppose that (1.5) has a Hermitian solution X over $\mathbb{H}$. Applying properties (a) and (b) of $\phi(\cdot)$ to (1.5) yields

$$\phi(A_1)\phi(X) = \phi(C_1), \qquad \phi(X)\phi(B_2) = \phi(C_2), \qquad \phi(A_3)\phi(X)\phi^T(A_3) = \phi(C_3), \tag{2.20}$$

implying that $\phi(X)$ is a real symmetric solution to (2.13).

Conversely, suppose that (2.13) has a real symmetric solution

$$\widehat{Y} = \widehat{Y}^T = \left(Y_{ij}\right)_{4\times 4}, \quad i, j = 1, 2, 3, 4. \tag{2.21}$$

That is,

$$\phi(A_1)\widehat{Y} = \phi(C_1), \qquad \widehat{Y}\phi(B_2) = \phi(C_2), \qquad \phi(A_3)\widehat{Y}\phi^T(A_3) = \phi(C_3), \tag{2.22}$$

then by property (d) of $\phi(\cdot)$,

$$\begin{gathered}
T_m^{-1}\phi(A_1)T_n\widehat{Y} = T_m^{-1}\phi(C_1)T_n, \qquad \widehat{Y}T_n^{-1}\phi(B_2)T_s = T_n^{-1}\phi(C_2)T_s,\\
T_r^{-1}\phi(A_3)T_n\widehat{Y}T_n^{-1}\phi^T(A_3)T_r = T_r^{-1}\phi(C_3)T_r,\\
R_m^{-1}\phi(A_1)R_n\widehat{Y} = R_m^{-1}\phi(C_1)R_n, \qquad \widehat{Y}R_n^{-1}\phi(B_2)R_s = R_n^{-1}\phi(C_2)R_s,\\
R_r^{-1}\phi(A_3)R_n\widehat{Y}R_n\phi^T(A_3)R_r = R_r^{-1}\phi(C_3)R_r,\\
S_m^{-1}\phi(A_1)S_n\widehat{Y} = S_m^{-1}\phi(C_1)S_n, \qquad \widehat{Y}S_n^{-1}\phi(B_2)S_s = S_n^{-1}\phi(C_2)S_s,\\
S_r^{-1}\phi(A_3)S_n\widehat{Y}S_n^{-1}\phi^T(A_3)S_r = S_r^{-1}\phi(C_3)S_r.
\end{gathered} \tag{2.23}$$

Hence,

$$\begin{gathered}
\phi(A_1)T_n\widehat{Y}T_n^{-1} = \phi(C_1), \qquad T_n\widehat{Y}T_n^{-1}\phi(B_2) = \phi(C_2), \qquad \phi(A_3)T_n\widehat{Y}T_n^{-1}\phi^T(A_3) = \phi(C_3),\\
\phi(A_1)R_n\widehat{Y}R_n^{-1} = \phi(C_1), \qquad R_n\widehat{Y}R_n^{-1}\phi(B_2) = \phi(C_2), \qquad \phi(A_3)R_n\widehat{Y}R_n^{-1}\phi^T(A_3) = \phi(C_3),\\
\phi(A_1)S_n\widehat{Y}S_n^{-1} = \phi(C_1), \qquad S_n\widehat{Y}S_n^{-1}\phi(B_2) = \phi(C_2), \qquad \phi(A_3)S_n\widehat{Y}S_n^{-1}\phi^T(A_3) = \phi(C_3),
\end{gathered} \tag{2.24}$$

implying that $T_n\widehat{Y}T_n^{-1}$, $R_n\widehat{Y}R_n^{-1}$, and $S_n\widehat{Y}S_n^{-1}$ are also symmetric solutions of (2.13). Thus,

$$\frac{1}{4}\left(\widehat{Y} + T_n\widehat{Y}T_n^{-1} + R_n\widehat{Y}R_n^{-1} + S_n\widehat{Y}S_n^{-1}\right) \tag{2.25}$$

is a symmetric solution of (2.13), where

$$\begin{gathered}
\widehat{Y} + T_n\widehat{Y}T_n^{-1} + R_n\widehat{Y}R_n^{-1} + S_n\widehat{Y}S_n^{-1} = \left(\widetilde{Y_{ij}}\right)_{4\times 4}, \quad i = 1,2,3,4,\\
\widetilde{Y_{11}} = Y_{11} + Y_{22} + Y_{33} + Y_{44}, \qquad \widetilde{Y_{12}} = Y_{12} - Y_{12}^T + Y_{34} - Y_{34}^T,\\
\widetilde{Y_{13}} = Y_{13} - Y_{13}^T + Y_{24}^T - Y_{24}, \qquad \widetilde{Y_{14}} = Y_{14} - Y_{14}^T + Y_{23} - Y_{23}^T,\\
\widetilde{Y_{21}} = Y_{12}^T - Y_{12} + Y_{34}^T - Y_{34}, \qquad \widetilde{Y_{22}} = Y_{11} + Y_{22} + Y_{33} + Y_{44},
\end{gathered}$$

$$\begin{aligned}
\widetilde{Y_{23}} &= Y_{14} - Y_{14}^T + Y_{23} - Y_{23}^T, & \widetilde{Y_{24}} &= Y_{13} - Y_{13}^T + Y_{24}^T - Y_{24}, \\
\widetilde{Y_{31}} &= Y_{13}^T - Y_{13} + Y_{24} - Y_{24}^T, & \widetilde{Y_{32}} &= Y_{14} - Y_{14}^T + Y_{23} - Y_{23}^T, \\
\widetilde{Y_{33}} &= Y_{11} + Y_{22} + Y_{33} + Y_{44}, & \widetilde{Y_{34}} &= Y_{12} - Y_{12}^T + Y_{34} - Y_{34}^T, \\
\widetilde{Y_{41}} &= Y_{14}^T - Y_{14} + Y_{23}^T - Y_{23}, & \widetilde{Y_{42}} &= Y_{13} - Y_{13}^T + Y_{24}^T - Y_{24}, \\
\widetilde{Y_{43}} &= Y_{12} - Y_{12}^T + Y_{34} - Y_{34}^T, & \widetilde{Y_{44}} &= Y_{11} + Y_{22} + Y_{33} + Y_{44}.
\end{aligned} \tag{2.26}$$

Let

$$\begin{aligned}
\widehat{X} = {} & \frac{1}{4}(Y_{11} + Y_{22} + Y_{33} + Y_{44}) + \frac{1}{4}\left(Y_{12} - Y_{12}^T + Y_{34} - Y_{34}^T\right)i \\
& + \frac{1}{4}\left(Y_{13} - Y_{13}^T + Y_{24}^T - Y_{24}\right)j + \frac{1}{4}\left(Y_{14} - Y_{14}^T + Y_{23} - Y_{23}^T\right)k.
\end{aligned} \tag{2.27}$$

Then by (2.1),

$$\phi\left(\widehat{X}\right) = \frac{1}{4}\left(\widehat{Y} + T_n\widehat{Y}T_n^{-1} + R_n\widehat{Y}R_n^{-1} + S_n\widehat{Y}S_n^{-1}\right). \tag{2.28}$$

Hence, by the property (a), we know that $\widehat{X}$ is a Hermitian solution of (1.5). Observe that Y_{ij}, $i, j = 1, 2, 3, 4$, in (2.13) can be written as

$$Y_{ij} = P_i\widehat{Y}P_j^T. \tag{2.29}$$

From Lemma 2.2, the general Hermitian solution to (2.13) can be written as

$$\widehat{Y} = \phi(X_0) + 4L_{\phi(A_1)}L_{\phi(F)}L_{\phi(M)}VL_{\phi(A_1)}L_{\phi(F)} + 4L_{\phi(F)}L_{\phi(A_1)}V^TL_{\phi(M)}L_{\phi(F)}L_{\phi(A_1)}, \tag{2.30}$$

where $V \in \mathbb{R}$ is arbitrary. Hence,

$$\begin{aligned}
Y_{ij} = {} & P_i\phi(X_0)P_j^T + 4P_iL_{\phi(A_1)}L_{\phi(F)}L_{\phi(M)}VL_{\phi(A_1)}L_{\phi(F)}P_j^T \\
& + 4P_iL_{\phi(F)}L_{\phi(A_1)}V^TL_{\phi(M)}L_{\phi(F)}L_{\phi(A_1)}P_j^T,
\end{aligned} \tag{2.31}$$

where $i, j = 1, 2, 3, 4$, substituting them into (2.14), yields the four real matrices X_1, X_2, X_3, and X_4 in (2.15)–(2.18). □

Now we consider the maximal and minimal ranks of four real matrices X_1, X_2, X_3, and X_4 in solution $X = X_1 + X_2i + X_3j + X_4k$ to (1.5) over $\mathbb{H}$.

Theorem 2.7. *Suppose that system* (1.5) *over* $\mathbb{H}$ *has a Hermitian solution, and* $A_1 = A_{11} + A_{12}i + A_{13}j + A_{14}k, C_1 = C_{11} + C_{12}i + C_{13}j + C_{14}k \in \mathbb{H}^{m\times n} B_1 = B_{11} + B_{12}i + B_{13}j + B_{14}k,\ C_2 = C_{21} + C_{22}i + C_{23}j + C_{24}k \in \mathbb{H}^{n\times s},\ A_3 = A_{31} + A_{32}i + A_{33}j + A_{34}k \in \mathbb{H}^{r\times n},\ C_3 = C_{31} + C_{32}i + C_{33}j + C_{34}k \in \mathbb{H}^{r\times r}$

$$S_1 = \left\{ X_1 \in \mathbb{R}^{n\times n} \mid \begin{matrix} A_1X = C_1,\ XB_1 = C_2,\ A_3XA_3^* = C_3 \\ X = X_1 + X_2i + X_3j + X_4k \end{matrix} \right\},$$

$$S_2 = \left\{ X_2 \in \mathbb{R}^{p\times q} \mid \begin{matrix} A_1X = C_1,\ XB_1 = C_2,\ A_3XA_3^* = C_3 \\ X = X_1 + X_2i + X_3j + X_4k \end{matrix} \right\},$$

$$S_3 = \left\{ X_3 \in \mathbb{R}^{p\times q} \mid \begin{matrix} A_1X = C_1,\ XB_1 = C_2,\ A_3XA_3^* = C_3 \\ X = X_1 + X_2i + X_3j + X_4k \end{matrix} \right\},$$

$$S_4 = \left\{ X_4 \in \mathbb{R}^{p\times q} \mid \begin{matrix} A_1X = C_1,\ XB_1 = C_2,\ A_3XA_3^* = C_3 \\ X = X_1 + X_2i + X_3j + X_4k \end{matrix} \right\},$$

$$L_{21} = \begin{bmatrix} C_{21} \\ C_{22} \\ C_{23} \\ C_{24} \end{bmatrix}, \quad L_{11} = \begin{bmatrix} C_{11} \\ -C_{12} \\ -C_{13} \\ -C_{14} \end{bmatrix}, \quad M_{31} = \begin{bmatrix} A_{32} & A_{33} & A_{34} \\ A_{31} & A_{34} & -A_{33} \\ -A_{34} & A_{31} & A_{32} \\ A_{33} & -A_{32} & A_{31} \end{bmatrix},$$

$$M_{11} = \begin{bmatrix} A_{12} & A_{13} & A_{14} \\ A_{11} & A_{14} & -A_{13} \\ -A_{14} & A_{11} & A_{12} \\ A_{13} & -A_{12} & A_{11} \end{bmatrix}, \quad M_{12} = \begin{bmatrix} A_{11} & A_{13} & A_{14} \\ -A_{12} & A_{14} & -A_{13} \\ -A_{13} & A_{11} & A_{12} \\ -A_{14} & -A_{12} & A_{11} \end{bmatrix}, \tag{2.32}$$

$$M_{13} = \begin{bmatrix} A_{11} & A_{12} & A_{14} \\ -A_{12} & A_{11} & -A_{13} \\ -A_{13} & -A_{14} & A_{12} \\ -A_{14} & A_{13} & A_{11} \end{bmatrix}, \quad M_{14} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ -A_{12} & A_{11} & A_{14} \\ -A_{13} & -A_{14} & A_{11} \\ -A_{14} & A_{13} & -A_{12} \end{bmatrix},$$

$$N_{11} = \begin{bmatrix} -B_{12} & B_{11} & B_{14} & -B_{13} \\ -B_{13} & -B_{14} & B_{11} & B_{12} \\ -B_{14} & B_{13} & -B_{12} & B_{11} \end{bmatrix}, \quad N_{12} = \begin{bmatrix} B_{11} & B_{12} & B_{13} & B_{14} \\ -B_{13} & -B_{14} & B_{11} & B_{12} \\ -B_{14} & B_{13} & -B_{12} & B_{11} \end{bmatrix},$$

$$N_{13} = \begin{bmatrix} B_{11} & B_{12} & B_{13} & B_{14} \\ -B_{12} & B_{11} & B_{14} & -B_{13} \\ -B_{14} & B_{13} & -B_{12} & B_{11} \end{bmatrix}, \quad N_{14} = \begin{bmatrix} B_{11} & B_{12} & B_{13} & B_{14} \\ -B_{12} & B_{11} & B_{14} & -B_{13} \\ -B_{13} & -B_{14} & B_{11} & B_{12} \end{bmatrix}.$$

Then the maximal and minimal ranks of X_i, $i = 1, 2, 3, 4$, *in Hermitian solution* $X = X_1 + X_2 i + X_3 j + X_4 k$ *to* (1.5) *are given by*

$$\max_{X_i \in S_i} r(X_i) = \min\{t_{1i}, t\}, \tag{2.33}$$

$$\min_{X_i \in S_i} r(X_i) = 2t_{1i} + t - 2t_{2i}, \tag{2.34}$$

where

$$t_{1i} = r\begin{bmatrix} L_{21} & N_{1i}^T \\ L_{11} & M_{1i} \end{bmatrix} - 4r\begin{bmatrix} B_1^* \\ A_1 \end{bmatrix} + n,$$

$$t = r\begin{bmatrix} 0 & M_{31}^T & N_{11} & M_{11}^T \\ M_{31} & \phi(C_3) & \phi(A_3)\phi(C_2) & \phi(A_3)\phi^T(C_1) \\ N_{11}^T & \phi^T(C_2)\phi^T(A_3) & \phi^T(C_2)\phi(B_1) & \phi^T(C_2)\phi^T(A_1) \\ M_{11} & \phi(C_1)\phi^T(A_3) & \phi(C_1)\phi(B_1) & \phi(C_1)\phi^T(A_1) \end{bmatrix} - 8r\begin{bmatrix} A_3 \\ B_1^* \\ A_1 \end{bmatrix} + 2n,$$

$$t_{2i} = r\begin{bmatrix} 0 & N_{1i} & M_{1i}^T \\ M_{31} & \phi(A_3)\phi(C_2) & \phi(A_3)\phi^T(C_1) \\ N_{11}^T & \phi^T(C_2)\phi(B_1) & \phi^T(C_2)\phi^T(A_1) \\ M_{11} & \phi(C_1)\phi(B_1) & \phi(C_1)\phi^T(A_1) \end{bmatrix} - 4r\begin{bmatrix} A_3 \\ B_1^* \\ A_1 \end{bmatrix} - 4r\begin{bmatrix} B_1^* \\ A_1 \end{bmatrix} + 2n. \tag{2.35}$$

Proof. We only prove the case that $i = 1$. Similarly, we can get the results that $i = 2, 3, 4$. Let

$$\frac{1}{4}P_1\phi(X_0)P_1^T + \frac{1}{4}P_2\phi(X_0)P_2^T + \frac{1}{4}P_3\phi(X_0)P_3^T + \frac{1}{4}P_4\phi(X_0)P_4^T = A,$$
$$[P_1, P_2, P_3, P_4]L_{\phi(A_1)}L_{\phi(F)}L_{\phi(M)} = B,$$
$$\begin{bmatrix} L_{\phi(F)}L_{\phi(A_1)}P_1^T \\ L_{\phi(F)}L_{\phi(A_1)}P_2^T \\ L_{\phi(F)}L_{\phi(A_1)}P_3^T \\ L_{\phi(F)}L_{\phi(A_1)}P_4^T \end{bmatrix} = C; \tag{2.36}$$

note that L_M is Hermtian; then $L_{\phi(M)}$ is symmetric; hence (2.15) can be written as

$$X_1 = A + BVC + (BVC)^*. \tag{2.37}$$

Note that $A = A^*$ and $\mathcal{R}(B) \subseteq \mathcal{R}(C^*)$; applying (2.9) and (2.10) to (2.37) yields

$$\max_{X_1 \in S_1} r(X_1) = \min\left\{ r[A, C^*],\ r\begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix} \right\}, \tag{2.38}$$

$$\min_{X_1 \in S_1} r(X_1) = 2r[A, C^*] + r\begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix} - 2r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}. \tag{2.39}$$

Let

$$[P_1, P_2, P_3, P_4] = P,$$

$$a_i = \begin{bmatrix} \phi(A_i) & 0 & 0 & 0 \\ 0 & \phi(A_i) & 0 & 0 \\ 0 & 0 & \phi(A_i) & 0 \\ 0 & 0 & 0 & \phi(A_i) \end{bmatrix}, \quad i = 1, 3, \tag{2.40}$$

$$b_1 = \begin{bmatrix} \phi(B_1) & 0 & 0 & 0 \\ 0 & \phi(B_1) & 0 & 0 \\ 0 & 0 & \phi(B_1) & 0 \\ 0 & 0 & 0 & \phi(B_1) \end{bmatrix}.$$

Note that $\phi(X_0)$ is a particular solution to (2.13), it is not difficult to find by Lemma 2.3, block Gaussian elimination, and property (e) of $\phi(\cdot)$ that

$$\begin{aligned} r[A, C^*] &= r\begin{bmatrix} A & P \\ 0 & b_1^T \\ 0 & a_1 \end{bmatrix} - 4r[\phi(A_1)] - 4r[\phi(F)] \\ &= r\begin{bmatrix} 0 & P \\ -\frac{1}{4}\phi^T(C_2)P_1^T & b_1^T \\ -\frac{1}{4}\phi(C_1)P_1^T & a_1 \end{bmatrix} - 4r\begin{bmatrix} \phi(B_1^*) \\ \phi(A_1) \end{bmatrix} \\ &= r\begin{bmatrix} 0 & [P_1, 0, 0, 0] \\ \phi^T(C_2)P_1^T & b_1^T \\ \phi(C_1)P_1^T & a_1 \end{bmatrix} - 4r\begin{bmatrix} \phi(B_1^*) \\ \phi(A_1) \end{bmatrix} \\ &= r\begin{bmatrix} L_{21} & N_{1i}^T \\ L_{11} & M_{1i} \end{bmatrix} - 4r\begin{bmatrix} \phi(B_1^*) \\ \phi(A_1) \end{bmatrix} + 3r\begin{bmatrix} \phi(B_1^*) \\ \phi(A_1) \end{bmatrix} + n \\ &= r\begin{bmatrix} L_{21} & N_{1i}^T \\ L_{11} & M_{1i} \end{bmatrix} - 4r\begin{bmatrix} B_1^* \\ A_1 \end{bmatrix} + n. \end{aligned} \tag{2.41}$$

Note that $L_A = R_{A^*}$, then $L_{\phi(A)} = R_{\phi^*(A)}$; hence

$$r\begin{bmatrix} A & B \\ B^* & 0 \end{bmatrix} = r\begin{bmatrix} A & P & 0 & 0 & 0 \\ P^T & 0 & a_3^T & b_1 & a_1^T \\ 0 & a_3 & 0 & 0 & 0 \\ 0 & b_1^T & 0 & 0 & 0 \\ 0 & a_1 & 0 & 0 & 0 \end{bmatrix} - 8r[\phi(M)] - 8r[\phi(F)] - 8r[\phi(A_1)]$$

$$= r \begin{bmatrix} 0 & M_{31}^T & N_{11} & M_{11}^T \\ M_{31} & \phi(C_3) & \phi(A_3)\phi(C_2) & \phi(A_3)\phi^T(C_1) \\ N_{11}^T & \phi^T(C_2)\phi^T(A_3) & \phi^T(C_2)\phi(B_1) & \phi^T(C_2)\phi^T(A_1) \\ M_{11} & \phi(C_1)\phi^T(A_3) & \phi(C_1)\phi(B_1) & \phi(C_1)\phi^T(A_1) \end{bmatrix}$$

$$- 8r \begin{bmatrix} \phi(A_3) \\ \phi(B_1^*) \\ \phi(A_1) \end{bmatrix} + 6r \begin{bmatrix} \phi(A_3) \\ \phi(B_1^*) \\ \phi(A_1) \end{bmatrix} + 2n$$

$$= r \begin{bmatrix} 0 & M_{31}^T & N_{11} & M_{11}^T \\ M_{31} & \phi(C_3) & \phi(A_3)\phi(C_2) & \phi(A_3)\phi^T(C_1) \\ N_{11}^T & \phi^T(C_2)\phi^T(A_3) & \phi^T(C_2)\phi(B_1) & \phi^T(C_2)\phi^T(A_1) \\ M_{11} & \phi(C_1)\phi^T(A_3) & \phi(C_1)\phi(B_1) & \phi(C_1)\phi^T(A_1) \end{bmatrix} - 8r \begin{bmatrix} A_3 \\ B_1^* \\ A_1 \end{bmatrix} + 2n. \tag{2.42}$$

Similarly, we can obtain

$$r \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = r \begin{bmatrix} 0 & N_{11} & M_{11}^T \\ M_{31} & \phi(A_3)\phi(C_2) & \phi(A_3)\phi^T(C_1) \\ N_{11}^T & \phi^T(C_2)\phi(B_1) & \phi^T(C_2)\phi^T(A_1) \\ M_{11} & \phi(C_1)\phi(B_1) & \phi(C_1)\phi^T(A_1) \end{bmatrix} - 4r \begin{bmatrix} A_3 \\ B_1^* \\ A_1 \end{bmatrix} - 4r \begin{bmatrix} B_1^* \\ A_1 \end{bmatrix} + 2n, \tag{2.43}$$

Substituting (2.41) and (2.43) into (2.38) and (2.39) yields (2.33) and (2.34), that is $i = 1$. □

Corollary 2.8. *Suppose system* (1.5) *over* $\mathbb{H}$ *have a Hermitian solution. Then we have the following.*

(a) *System* (1.5) *has a real hermtian solution if and only if*

$$2r \begin{bmatrix} L_{21} & N_{1i}^T \\ L_{11} & M_{1i} \end{bmatrix} + r \begin{bmatrix} 0 & M_{31}^T & N_{11} & M_{11}^T \\ M_{31} & \phi(C_3) & \phi(A_3)\phi(C_2) & \phi(A_3)\phi^T(C_1) \\ N_{11}^T & \phi^T(C_2)\phi^T(A_3) & \phi^T(C_2)\phi(B_1) & \phi^T(C_2)\phi^T(A_1) \\ M_{11} & \phi(C_1)\phi^T(A_3) & \phi(C_1)\phi(B_1) & \phi(C_1)\phi^T(A_1) \end{bmatrix}$$
$$= 2r \begin{bmatrix} 0 & N_{1i} & M_{1i}^T \\ M_{31} & \phi(A_3)\phi(C_2) & \phi(A_3)\phi^T(C_1) \\ N_{11}^T & \phi^T(C_2)\phi(B_1) & \phi^T(C_2)\phi^T(A_1) \\ M_{11} & \phi(C_1)\phi(B_1) & \phi(C_1)\phi^T(A_1) \end{bmatrix} \tag{2.44}$$

hold when $i = 2, 3, 4$. *In that case, the real solution of* (1.5) *can be expressed as* $X = X_1$ *in* (2.15).

(b) *System* (1.5) *has a complex solution if and only if* (2.44) *hold when* $i = 3, 4$ *or* $i = 2, 4$ *or* $i = 2, 3$. *In that case, the complex solutions of* (1.5) *can be expressed as* $X = X_1 + X_2 i$ *or* $X = X_1 + X_3 j$

or $X = X_1 + X_4k$, *where* X_1, X_2, X_3, *and* X_4 *are expressed as* (2.15), (2.16), (2.17), *and* (2.18), *respectively.*

Proof. From (2.34) we can get the necessary and sufficient conditions for $X_i = 0,\ i = 1,2,3,4$. Thus we can get the results of this Corollary. □

3. Solvability Conditions for Real and Complex Hermitian Solutions to (1.6) Over $\mathbb{H}$

In this section, using the results of Theorem 2.6, Theorem 2.7, and Corollary 2.8, we give necessary and sufficient conditions for (1.6) over $\mathbb{H}$ to have real and complex Hermitian solutions.

Theorem 3.1. *Let* A_1, A_3, B_1, C_1, C_2, *and* C_3 *be defined in Lemma 2.2,* $A_4 \in \mathbb{H}^{l\times n}, C_4 \in \mathbb{H}^{l\times l}$, *and suppose that system* (1.5) *and the matrix equation* $A_4 Y A_4^* = C_4$ *over* $\mathbb{H}$ *have Hermitian solutions* X *and* $Y \in \mathbb{H}^{n\times n}$, *respectively. Then system* (1.6) *over* $\mathbb{H}$ *has a real Hermitian solution if and only if* (2.44) *hold when* $i = 2,3,4$, *and*

$$r\begin{bmatrix} 0 & M_{31}^T \\ M_{31} & \phi(C_3) \end{bmatrix} = 2r(M_{31}), \tag{3.1}$$

$$r\begin{bmatrix} 0 & M_{41}^T & 0 & 0 \\ 0 & M_{411}^T & \phi(B_1) & \phi^T(A_1) \\ M_{41} & \phi(C_4) & \phi(A_4)\phi(C_2) & \phi(A_4)\phi^T(C_1) \end{bmatrix} = r(M_{41}) + r\begin{bmatrix} M_{41}^T & 0 & 0 \\ M_{411}^T & \phi(B_1) & \phi^T(A_1) \end{bmatrix}, \tag{3.2}$$

$$r\begin{bmatrix} 0 & 0 & M_{41}^T \\ M_{41} & M_{411} & \phi(C_4) \\ 0 & \phi^T(B_1) & \phi^T(C_2)\phi^T(A_4) \\ 0 & \phi(A_1) & \phi(C_1)\phi^T(A_4) \end{bmatrix} = r(M_{41}) + r\begin{bmatrix} M_{41} & M_{411} \\ 0 & \phi^T(B_1) \\ 0 & \phi(A_1) \end{bmatrix},$$

$$r\begin{bmatrix} 0 & 0 & M_{41}^T & 0 & 0 & 0 \\ 0 & 0 & M_{411}^T & \phi^T(A_3) & \phi(B_1) & \phi^T(A_1) \\ M_{41} & M_{411} & \phi(C_4) & 0 & 0 & 0 \\ 0 & \phi(A_3) & 0 & \phi(C_3) & \phi(A_3)\phi(C_2) & \phi(A_3)\phi^T(C_1) \\ 0 & \phi^T(B_1) & 0 & \phi^T(C_2)\phi^T(A_3) & \phi^T(C_2)\phi(B_1) & \phi^T(C_2)\phi^T(A_1) \\ 0 & \phi(A_1) & 0 & \phi(C_1)\phi^T(A_3) & \phi(C_1)\phi(B_1) & \phi(C_1)\phi^T(A_1) \end{bmatrix} = 2r\begin{bmatrix} M_{41} & M_{411} \\ 0 & \phi(A_3) \\ 0 & \phi^T(B_1) \\ 0 & \phi(A_1) \end{bmatrix},$$

$$r\begin{bmatrix} 0 & 0 & M_{41}^T & 0 & 0 \\ 0 & 0 & M_{411}^T & \phi(B_1) & \phi^T(A_1) \\ M_{41} & M_{411} & \phi(C_4) & 0 & 0 \\ 0 & \phi^T(B_1) & 0 & \phi^T(B_1)\phi(C_2) & \phi^T(B_1)\phi^T(C_1) \\ 0 & \phi(A_1) & 0 & \phi(C_1)\phi(B_1) & \phi(C_1)\phi^T(A_1) \end{bmatrix} = 2r\begin{bmatrix} M_{41} & M_{411} \\ 0 & \phi^T(B_1) \\ 0 & \phi(A_1) \end{bmatrix}, \tag{3.3}$$

where

$$M_{41}=\begin{bmatrix}A_{42}&A_{43}&A_{44}\\A_{41}&A_{44}&-A_{43}\\-A_{44}&A_{41}&A_{42}\\A_{43}&-A_{42}&A_{41}\end{bmatrix},\qquad M_{411}=\begin{bmatrix}A_{21}&0&0&0\\-A_{22}&0&0&0\\-A_{23}&0&0&0\\-A_{24}&0&0&0\end{bmatrix}. \tag{3.4}$$

Proof. From Corollary 2.8, system (1.5) over $\mathbb{H}$ has a real Hermitian solution if and only if (2.44) hold when $i=2,3,4$. By (2.15), the real Hermitian solutions of (1.5) over $\mathbb{H}$ can be expressed as

$$\begin{aligned}X_1={}&\frac{1}{4}P_1\phi(X_0)P_1^T+\frac{1}{4}P_2\phi(X_0)P_2^T+\frac{1}{4}P_3\phi(X_0)P_3^T+\frac{1}{4}P_4\phi(X_0)P_4^T\\&+[P_1,P_2,P_3,P_4]L_{\phi(A_1)}L_{\phi(F)}L_{\phi(M)}V\begin{bmatrix}L_{\phi(F)}L_{\phi(A_1)}P_1^T\\L_{\phi(F)}L_{\phi(A_1)}P_2^T\\L_{\phi(F)}L_{\phi(A_1)}P_3^T\\L_{\phi(F)}L_{\phi(A_1)}P_4^T\end{bmatrix}\\&+[P_1,P_2,P_3,P_4]L_{\phi(A_1)}L_{\phi(F)}V^T\begin{bmatrix}L_{\phi(M)}L_{\phi(F)}L_{\phi(A_1)}P_1^T\\L_{\phi(M)}L_{\phi(F)}L_{\phi(A_1)}P_2^T\\L_{\phi(M)}L_{\phi(F)}L_{\phi(A_1)}P_3^T\\L_{\phi(M)}L_{\phi(F)}L_{\phi(A_1)}P_4^T\end{bmatrix},\end{aligned} \tag{3.5}$$

where V is arbitrary matrices with compatible sizes.

Let $A_1, C_1=0; B_1, C_2=0;\ A_3=A_4; C_3=C_4$ in Corollary 2.8 and (2.15). It is easy to verify that the matrix equation $A_4YA_4^*=C_4$ over $\mathbb{H}$ has a real Hermitian solution if and only if (3.1) hold and the real Hermitian solution can be expressed as

$$\begin{aligned}Y_1={}&\frac{1}{4}P_1\phi(Y_0)P_1^T+\frac{1}{4}P_2\phi(Y_0)P_2^T+\frac{1}{4}P_3\phi(Y_0)P_3^T+\frac{1}{4}P_4\phi(Y_0)P_4^T\\&+[P_1,P_2,P_3,P_4]L_{\phi(A_4)}U+U^T\begin{bmatrix}L_{\phi(A_1)}P_1^T\\L_{\phi(A_1)}P_2^T\\L_{\phi(A_1)}P_3^T\\L_{\phi(A_1)}P_4^T\end{bmatrix},\end{aligned} \tag{3.6}$$

where $\phi(Y_0)$ is a particular solution to $\phi(A_4)(Y_{ij})_{4\times 4}\phi^T(A_4) = \phi(C_4)$ and U is arbitrary matrices with compatible sizes. The expression of Y_1 can also be obtained from Lemma 2.1. Let

$$\begin{aligned} &[P_1, P_2, P_3, P_4] = P, \\ G &= \frac{1}{4}P_1\phi(X_0)P_1^T + \frac{1}{4}P_2\phi(X_0)P_2^T + \frac{1}{4}P_3\phi(X_0)P_3^T + \frac{1}{4}P_4\phi(X_0)P_4^T \\ &\quad -\frac{1}{4}P_1\phi(Y_0)P_1^T - \frac{1}{4}P_2\phi(Y_0)P_2^T - \frac{1}{4}P_3\phi(Y_0)P_3^T - \frac{1}{4}P_4\phi(Y_0)P_4^T. \end{aligned} \tag{3.7}$$

Equating X_1 and Y_1, we obtain the following equation:

$$\begin{aligned} X_1 - Y_1 &= G + PL_{\phi(A_1)}L_{\phi(F)}L_{\phi(M)}V\begin{bmatrix} L_{\phi(F)}L_{\phi(A_1)}P_1^T \\ L_{\phi(F)}L_{\phi(A_1)}P_2^T \\ L_{\phi(F)}L_{\phi(A_1)}P_3^T \\ L_{\phi(F)}L_{\phi(A_1)}P_4^T \end{bmatrix} \\ &\quad + PL_{\phi(A_1)}L_{\phi(F)}V^T\begin{bmatrix} L_{\phi(M)}L_{\phi(F)}L_{\phi(A_1)}P_1^T \\ L_{\phi(M)}L_{\phi(F)}L_{\phi(A_1)}P_2^T \\ L_{\phi(M)}L_{\phi(F)}L_{\phi(A_1)}P_3^T \\ L_{\phi(M)}L_{\phi(F)}L_{\phi(A_1)}P_4^T \end{bmatrix} - PL_{\phi(A_4)}U - U^T\begin{bmatrix} L_{\phi(A_4)}P_1^T \\ L_{\phi(A_4)}P_2^T \\ L_{\phi(A_4)}P_3^T \\ L_{\phi(A_4)}P_4^T \end{bmatrix}. \end{aligned} \tag{3.8}$$

It is obvious that system (1.5) and the matrix equation $A_4YA_4^* = C_4$ over $\mathbb{H}$ have common real Hermitian solution if and only if $\min r(X_1 - Y_1) = 0$, that is, $X_1 - Y_1 = 0$. Hence, we have the matrix equation

$$\begin{aligned} G &= PL_{\phi(A_4)}U + U^T\begin{bmatrix} L_{\phi(A_4)}P_1^T \\ L_{\phi(A_4)}P_2^T \\ L_{\phi(A_4)}P_3^T \\ L_{\phi(A_4)}P_4^T \end{bmatrix} - PL_{\phi(A_1)}L_{\phi(F)}L_{\phi(M)}V\begin{bmatrix} L_{\phi(F)}L_{\phi(A_1)}P_1^T \\ L_{\phi(F)}L_{\phi(A_1)}P_2^T \\ L_{\phi(F)}L_{\phi(A_1)}P_3^T \\ L_{\phi(F)}L_{\phi(A_1)}P_4^T \end{bmatrix} \\ &\quad - PL_{\phi(A_1)}L_{\phi(F)}V^T\begin{bmatrix} L_{\phi(M)}L_{\phi(F)}L_{\phi(A_1)}P_1^T \\ L_{\phi(M)}L_{\phi(F)}L_{\phi(A_1)}P_2^T \\ L_{\phi(M)}L_{\phi(F)}L_{\phi(A_1)}P_3^T \\ L_{\phi(M)}L_{\phi(F)}L_{\phi(A_1)}P_4^T \end{bmatrix}. \end{aligned} \tag{3.9}$$

We know by Lemma 2.5 that (3.9) is solvable if and only if the following four rank equalities hold

$$r\begin{bmatrix} G & PL_{\phi(A_4)} \\ R_{\phi(A_4)}P^T & 0 \\ R_{\phi(F)}R_{\phi(A_1)}P^T & 0 \end{bmatrix} = r\begin{bmatrix} 0 & PL_{\phi(A_4)} \\ R_{\phi(A_4)}P^T & 0 \\ R_{\phi(F)}R_{\phi(A_1)}P^T & 0 \end{bmatrix},$$
$$r\begin{bmatrix} G & PL_{\phi(A_4)} & PL_{\phi(A_1)}L_{\phi(F)} \\ R_{\phi(A_4)}P^T & 0 & 0 \end{bmatrix} = r\begin{bmatrix} 0 & PL_{\phi(A_4)} & PL_{\phi(A_1)}L_{\phi(F)} \\ R_{\phi(A_4)}P^T & 0 & 0 \end{bmatrix},$$
$$r\begin{bmatrix} G & PL_{\phi(A_4)} & PL_{\phi(A_1)}L_{\phi(F)}L_{\phi(M)} \\ R_{\phi(A_4)}P^T & 0 & 0 \\ R_{\phi(M)}R_{\phi(F)}R_{\phi(A_1)}P^T & 0 & 0 \end{bmatrix}$$
$$= r\begin{bmatrix} 0 & PL_{\phi(A_4)} & PL_{\phi(A_1)}L_{\phi(F)}L_{\phi(M)} \\ R_{\phi(A_4)}P^T & 0 & 0 \\ R_{\phi(M)}R_{\phi(F)}R_{\phi(A_1)}P^T & 0 & 0 \end{bmatrix},$$
$$r\begin{bmatrix} G & PL_{\phi(A_4)} & PL_{\phi(A_1)}L_{\phi(F)} \\ R_{\phi(A_4)}P^T & 0 & 0 \\ R_{\phi(F)}R_{\phi(A_1)}P^T & 0 & 0 \end{bmatrix} = r\begin{bmatrix} 0 & PL_{\phi(A_4)} & PL_{\phi(A_1)}L_{\phi(F)} \\ R_{\phi(A_4)}P^T & 0 & 0 \\ R_{\phi(F)}R_{\phi(A_1)}P^T & 0 & 0 \end{bmatrix}. \tag{3.10}$$

Under the conditions that the system (1.5) and the matrix equation $A_4YA_4^* = C_4$ over $\mathbb{H}$ have Hermitian solutions, it is not difficult to show by Lemma 2.3 and block Gaussian elimination that (3.10) are equivalent to the four rank equalities (3.2) and (3.3), respectively. Note that the processes are too much tedious; we omit them here. Obviously, the system (1.5) and the matrix equation $A_4YA_4^* = C_4$ over $\mathbb{H}$ have a common real Hermitian solution if and only if (3.2) and (3.3) hold. Thus, the system (1.6) over $\mathbb{H}$ has a real Hermitian solution if and only if (2.44) hold when $i = 2, 3, 4$, and (3.1)–(3.3) hold.

Similarly, from Corollary 2.8, we know that the system (1.5) over $\mathbb{H}$ has a complex Hermitian solution if and only if (2.44) hold when $i = 3, 4$, $i = 2, 4$, or; $i = 2, 3$; its complex Hermitian solutions can be expressed as $X = X_1 + X_2i$, $X = X_1 + X_3j$, or $X = X_1 + X_4k$. It is also easy to derive the necessary and sufficient condition for the matrix equation $A_4YA_4^* = C_4$ over $\mathbb{H}$ to have a complex Hermitian solution; its complex Hermitian solution can be expressed as $Y = Y_1 + Y_2i$, $Y = Y_1 + Y_3j$, or $Y = Y_1 + Y_4k$. By equating X_1 and Y_1, X_2 and Y_2, X_3, and Y_3, X_4 and Y_4, respectively, we can derive the necessary and sufficient conditions for the system (1.6) over $\mathbb{H}$ to have a complex Hermitian solution. □

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References

[1] T. W. Hungerford, *Algebra*, vol. 73 of *Graduate Texts in Mathematics*, Springer, New York, NY, USA, 1980.

[2] C. G. Khatri and S. K. Mitra, "Hermitian and nonnegative definite solutions of linear matrix equations," *SIAM Journal on Applied Mathematics*, vol. 31, no. 4, pp. 579–585, 1976.

[3] D. Hua and P. Lancaster, "Linear matrix equations from an inverse problem of vibration theory," *Linear Algebra and Its Applications*, vol. 246, pp. 31–47, 1996.

[4] Y. H. Liu and Y. G. Tian, "More on extremal ranks of the matrix expressions $A - BX \pm X^*B^*$ with statistical applications," *Numerical Linear Algebra with Applications*, vol. 15, no. 4, pp. 307–325, 2008.

[5] M. Wei and Q. Wang, "On rank-constrained Hermitian nonnegative-definite least squares solutions to the matrix equation $AXA^* = B$," *International Journal of Computer Mathematics*, vol. 84, no. 6, pp. 945–952, 2007.

[6] X. Zhang and M.-Y. Cheng, "The rank-constrained Hermitian nonnegative-definite and positive-definite solutions to the matrix equation $AXA^* = B$," *Linear Algebra and Its Applications*, vol. 370, pp. 163–174, 2003.

[7] Y. H. Liu and Y. G. Tian, "Max-min problems on the ranks and inertias of the matrix expressions $A - BXC \pm (BXC)^*$ with applications," *Journal of Optimization Theory and Applications*, vol. 148, no. 3, pp. 593–622, 2011.

[8] J. Groß, "A note on the general Hermitian solution to $AXA^* = B$," *Bulletin of the Malaysian Mathematical Society. Second Series*, vol. 21, no. 2, pp. 57–62, 1998.

[9] Y. H. Liu, Y. G. Tian, and Y. Takane, "Ranks of Hermitian and skew-Hermitian solutions to the matrix equation $AXA^* = B$," *Linear Algebra and Its Applications*, vol. 431, no. 12, pp. 2359–2372, 2009.

[10] Y. G. Tian and Y. H. Liu, "Extremal ranks of some symmetric matrix expressions with applications," *SIAM Journal on Matrix Analysis and Applications*, vol. 28, no. 3, pp. 890–905, 2006.

[11] Y. G. Tian, "Maximization and minimization of the rank and inertia of the Hermitian matrix expression $A - BX - (BX)^*$ with applications," *Linear Algebra and Its Applications*, vol. 434, no. 10, pp. 2109–2139, 2011.

[12] Q. W. Wang and Z.-C. Wu, "Common Hermitian solutions to some operator equations on Hilbert C^*-modules," *Linear Algebra and its Applications*, vol. 432, no. 12, pp. 3159–3171, 2010.

[13] Y. G. Tian, "Ranks of solutions of the matrix equation $AXB = C$," *Linear and Multilinear Algebra*, vol. 51, no. 2, pp. 111–125, 2003.

[14] Y. H. Liu, "Ranks of solutions of the linear matrix equation $AX + YB = C$," *Computers & Mathematics with Applications*, vol. 52, no. 6-7, pp. 861–872, 2006.

[15] Y. H. Liu, "Ranks of least squares solutions of the matrix equation $AXB = C$," *Computers & Mathematics with Applications*, vol. 55, no. 6, pp. 1270–1278, 2008.

[16] Y. G. Tian, "Upper and lower bounds for ranks of matrix expressions using generalized inverses," *Linear Algebra and Its Applications*, vol. 355, pp. 187–214, 2002.

[17] Y. G. Tian and S. Cheng, "The maximal and minimal ranks of $A - BXC$ with applications," *New York Journal of Mathematics*, vol. 9, pp. 345–362, 2003.

[18] Y. G. Tian, "The maximal and minimal ranks of some expressions of generalized inverses of matrices," *Southeast Asian Bulletin of Mathematics*, vol. 25, no. 4, pp. 745–755, 2002.

[19] Y. G. Tian, "The minimal rank of the matrix expression $A - BX - YC$," *Missouri Journal of Mathematical Sciences*, vol. 14, no. 1, pp. 40–48, 2002.

[20] Q. W. Wang, Z.-C. Wu, and C.-Y. Lin, "Extremal ranks of a quaternion matrix expression subject to consistent systems of quaternion matrix equations with applications," *Applied Mathematics and Computation*, vol. 182, no. 2, pp. 1755–1764, 2006.

[21] Q. W. Wang, G.-J. Song, and C.-Y. Lin, "Extreme ranks of the solution to a consistent system of linear quaternion matrix equations with an application," *Applied Mathematics and Computation*, vol. 189, no. 2, pp. 1517–1532, 2007.

[22] Q. W. Wang, S.-W. Yu, and C.-Y. Lin, "Extreme ranks of a linear quaternion matrix expression subject to triple quaternion matrix equations with applications," *Applied Mathematics and Computation*, vol. 195, no. 2, pp. 733–744, 2008.

[23] S. W. Yu and G. J. Song, "Extreme ranks of Hermitian solution to a pair of matrix equations," in *Proceedings of the 6th International Workshop on Matrix and Operators*, pp. 253–255, Chengdu, China, July 2011.

[24] Q. W. Wang and J. Jiang, "Extreme ranks of (skew-)Hermitian solutions to a quaternion matrix equation," *Electronic Journal of Linear Algebra*, vol. 20, pp. 552–573, 2010.

[25] Q. W. Wang, S. W. Yu, and W. Xie, "Extreme ranks of real matrices in solution of the quaternion matrix equation $AXB = C$ with applications," *Algebra Colloquium*, vol. 17, no. 2, pp. 345–360, 2010.

[26] Q. W. Wang, S.-W. Yu, and Q. Zhang, "The real solutions to a system of quaternion matrix equations with applications," *Communications in Algebra*, vol. 37, no. 6, pp. 2060–2079, 2009.

[27] Q. W. Wang, H.-S. Zhang, and S.-W. Yu, "On solutions to the quaternion matrix equation $AXB + CYD = E$," *Electronic Journal of Linear Algebra*, vol. 17, pp. 343–358, 2008.

[28] Y. G. Tian, "The solvability of two linear matrix equations," *Linear and Multilinear Algebra*, vol. 48, no. 2, pp. 123–147, 2000.