Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2012, Article ID 315697, 15 pages
doi:10.1155/2012/315697

Research Article

Refinements of Inequalities among Difference of Means

Huan-Nan Shi, Da-Mao Li, and Jian Zhang

Department of Electronic Information, Teacher's College, Beijing Union University, Beijing 100011, China

Correspondence should be addressed to Huan-Nan Shi, shihuannan@yahoo.com.cn

Received 21 June 2012; Accepted 10 September 2012

Academic Editor: Janusz Matkowski

In this paper, for the difference of famous means discussed by Taneja in 2005, we study the Schur-geometric convexity in $(0,\infty)\times(0,\infty)$ of the difference between them. Moreover some inequalities related to the difference of those means are obtained.

1. Introduction

In 2005, Taneja [1] proved the following chain of inequalities for the binary means for $(a,b)\in R_+^2=(0,\infty)\times(0,\infty)$:

$$H(a,b)\le G(a,b)\le N_1(a,b)\le N_3(a,b)\le N_2(a,b)\le A(a,b)\le S(a,b), \tag{1.1}$$

where

$$\begin{aligned} A(a,b)&=\frac{a+b}{2},\\ G(a,b)&=\sqrt{ab},\\ H(a,b)&=\frac{2ab}{a+b},\\ N_1(a,b)&=\left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)^2=\frac{A(a,b)+G(a,b)}{2}, \end{aligned} \tag{1.2}$$

$$N_3(a,b) = \frac{a+\sqrt{ab}+b}{3} = \frac{2A(a,b)+G(a,b)}{3},$$
$$N_2(a,b) = \left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)\left(\sqrt{\frac{a+b}{2}}\right), \tag{1.3}$$
$$S(a,b) = \sqrt{\frac{a^2+b^2}{2}}.$$

The means A, G, H, S, N_1 and N_3 are called, respectively, the arithmetic mean, the geometric mean, the harmonic mean, the root-square mean, the square-root mean, and Heron's mean. The N_2 one can be found in Taneja [2, 3].

Furthermore Taneja considered the following difference of means:

$$\begin{aligned}
M_{SA}(a,b) &= S(a,b) - A(a,b),\\
M_{SN_2}(a,b) &= S(a,b) - N_2(a,b),\\
M_{SN_3}(a,b) &= S(a,b) - N_3(a,b),\\
M_{SN_1}(a,b) &= S(a,b) - N_1(a,b),\\
M_{SG}(a,b) &= S(a,b) - G(a,b),\\
M_{SH}(a,b) &= S(a,b) - H(a,b),\\
M_{AN_2}(a,b) &= A(a,b) - N_2(a,b),\\
M_{AG}(a,b) &= A(a,b) - G(a,b),\\
M_{AH}(a,b) &= A(a,b) - H(a,b),\\
M_{N_2N_1}(a,b) &= N_2(a,b) - N_1(a,b),\\
M_{N_2G}(a,b) &= N_2(a,b) - G(a,b)
\end{aligned} \tag{1.4}$$

and established the following.

Theorem A. *The difference of means given by* (1.4) *is nonnegative and convex in* $R_+^2 = (0,\infty) \times (0,\infty)$.

Further, using Theorem A, Taneja proved several chains of inequalities; they are refinements of inequalities in (1.1).

Theorem B. *The following inequalities among the mean differences hold:*

$$M_{SA}(a,b) \le \frac{1}{3}M_{SH}(a,b) \le \frac{1}{2}M_{AH}(a,b) \le \frac{1}{2}M_{SG}(a,b) \le M_{AG}(a,b), \tag{1.5}$$

$$\frac{1}{8}M_{AH}(a,b) \le M_{N_2N_1}(a,b) \le \frac{1}{3}M_{N_2G}(a,b) \le \frac{1}{4}M_{AG}(a,b) \le M_{AN_2}(a,b), \tag{1.6}$$

$$M_{SA}(a,b) \le \frac{4}{5}M_{SN_2}(a,b) \le 4M_{AN_2}(a,b), \tag{1.7}$$

$$M_{SH}(a,b) \le 2M_{SN_1}(a,b) \le \frac{3}{2}M_{SG}(a,b), \tag{1.8}$$

$$M_{SA}(a,b) \le \frac{3}{4}M_{SN_3}(a,b) \le \frac{2}{3}M_{SN_1}(a,b). \tag{1.9}$$

For the difference of means given by (1.4), we study the Schur-geometric convexity of difference between these differences in order to further improve the inequalities in (1.1). The main result of this paper reads as follows.

Theorem I. *The following differences are Schur-geometrically convex in* $R_+^2 = (0,\infty)\times(0,\infty)$*:*

$$\begin{aligned}
D_{SH-SA}(a,b) &= \frac{1}{3}M_{SH}(a,b) - M_{SA}(a,b),\\
D_{AH-SH}(a,b) &= \frac{1}{2}M_{AH}(a,b) - \frac{1}{3}M_{SH}(a,b),\\
D_{SG-AH}(a,b) &= M_{SG}(a,b) - M_{AH}(a,b),\\
D_{AG-SG}(a,b) &= M_{AG}(a,b) - \frac{1}{2}M_{SG}(a,b),\\
D_{N_2N_1-AH}(a,b) &= M_{N_2N_1}(a,b) - \frac{1}{8}M_{AH}(a,b),\\
D_{N_2G-N_2N_1}(a,b) &= \frac{1}{3}M_{N_2G}(a,b) - M_{N_2N_1}(a,b),\\
D_{AG-N_2G}(a,b) &= \frac{1}{4}M_{AG}(a,b) - \frac{1}{3}M_{N_2G}(a,b),\\
D_{AN_2-AG}(a,b) &= M_{AN_2}(a,b) - \frac{1}{4}M_{AG}(a,b),\\
D_{SN_2-SA}(a,b) &= \frac{4}{5}M_{SN_2}(a,b) - M_{SA}(a,b),\\
D_{AN_2-SN_2}(a,b) &= 4M_{AN_2}(a,b) - \frac{4}{5}M_{SN_2}(a,b),\\
D_{SN_1-SH}(a,b) &= 2M_{SN_1}(a,b) - M_{SH}(a,b),\\
D_{SG-SN_1}(a,b) &= \frac{3}{2}M_{SG}(a,b) - 2M_{SN_1}(a,b),
\end{aligned} \tag{1.10}$$

$$D_{SN_3-SA}(a,b) = \frac{3}{4}M_{SN_3}(a,b) - M_{SA}(a,b),$$
$$D_{SN_1-SN_3}(a,b) = \frac{2}{3}M_{SN_1}(a,b) - \frac{3}{4}M_{SN_3}(a,b). \tag{1.11}$$

The proof of this theorem will be given in Section 3. Applying this result, in Section 4, we prove some inequalities related to the considered differences of means. Obtained inequalities are refinements of inequalities (1.5)–(1.9).

2. Definitions and Auxiliary Lemmas

The Schur-convex function was introduced by Schur in 1923, and it has many important applications in analytic inequalities, linear regression, graphs and matrices, combinatorial optimization, information-theoretic topics, Gamma functions, stochastic orderings, reliability, and other related fields (cf. [4–14]).

In 2003, Zhang first proposed concepts of "Schur-geometrically convex function" which is extension of "Schur-convex function" and established corresponding decision theorem [15]. Since then, Schur-geometric convexity has evoked the interest of many researchers and numerous applications and extensions have appeared in the literature (cf. [16–19]).

In order to prove the main result of this paper we need the following definitions and auxiliary lemmas.

Definition 2.1 (see [4, 20]). Let $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $\mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n$.

(i) $\mathbf{x}$ is said to be majorized by $\mathbf{y}$ (in symbols $\mathbf{x} \prec \mathbf{y}$) if $\sum_{i=1}^{k} x_{[i]} \le \sum_{i=1}^{k} y_{[i]}$ for $k = 1, 2, \ldots, n-1$ and $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$, where $x_{[1]} \ge \cdots \ge x_{[n]}$ and $y_{[1]} \ge \cdots \ge y_{[n]}$ are rearrangements of $\mathbf{x}$ and $\mathbf{y}$ in a descending order.

(ii) $\Omega \subseteq \mathbb{R}^n$ is called a convex set if $(\alpha x_1 + \beta y_1, \ldots, \alpha x_n + \beta y_n) \in \Omega$ for every $\mathbf{x}$ and $\mathbf{y} \in \Omega$, where α and $\beta \in [0,1]$ with $\alpha + \beta = 1$.

(iii) Let $\Omega \subseteq \mathbb{R}^n$. The function φ: $\Omega \to \mathbb{R}$ is said to be a Schur-convex function on Ω if $\mathbf{x} \prec \mathbf{y}$ on Ω implies $\varphi(\mathbf{x}) \le \varphi(\mathbf{y})$. φ is said to be a Schur-concave function on Ω if and only if $-\varphi$ is Schur-convex.

Definition 2.2 (see [15]). Let $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $\mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n_+$.

(i) $\Omega \subseteq \mathbb{R}^n_+$ is called a geometrically convex set if $(x_1^{\alpha} y_1^{\beta}, \ldots, x_n^{\alpha} y_n^{\beta}) \in \Omega$ for all $\mathbf{x}, \mathbf{y} \in \Omega$ and $\alpha, \beta \in [0,1]$ such that $\alpha + \beta = 1$.

(ii) Let $\Omega \subseteq \mathbb{R}^n_+$. The function φ: $\Omega \to \mathbb{R}_+$ is said to be Schur-geometrically convex function on Ω if $(\ln x_1, \ldots, \ln x_n) \prec (\ln y_1, \ldots, \ln y_n)$ on Ω implies $\varphi(\mathbf{x}) \le \varphi(\mathbf{y})$. The function φ is said to be a Schur-geometrically concave on Ω if and only if $-\varphi$ is Schur-geometrically convex.

Definition 2.3 (see [4, 20]). (i) The set $\Omega \subseteq \mathbb{R}^n$ is called symmetric set, if $x \in \Omega$ implies $Px \in \Omega$ for every $n \times n$ permutation matrix P.

(ii) The function $\varphi : \Omega \to \mathbb{R}$ is called symmetric if, for every permutation matrix P, $\varphi(Px) = \varphi(x)$ for all $x \in \Omega$.

Lemma 2.4 (see [15]). *Let $\Omega \subseteq \mathbb{R}_+^n$ be a symmetric and geometrically convex set with a nonempty interior Ω^0. Let $\varphi : \Omega \to \mathbb{R}_+$ be continuous on Ω and differentiable in Ω^0. If φ is symmetric on Ω and*

$$(\ln x_1 - \ln x_2)\left(x_1 \frac{\partial \varphi}{\partial x_1} - x_2 \frac{\partial \varphi}{\partial x_2}\right) \geq 0 \quad (\leq 0) \tag{2.1}$$

holds for any $\mathbf{x} = (x_1, \dots, x_n) \in \Omega^0$, then φ is a Schur-geometrically convex (Schur-geometrically concave) function.

Lemma 2.5. *For $(a,b) \in R_+^2 = (0,\infty) \times (0,\infty)$ one has*

$$1 \geq \frac{a+b}{\sqrt{2(a^2+b^2)}} \geq \frac{1}{2} + \frac{2ab}{(a+b)^2}, \tag{2.2}$$

$$\frac{a+b}{\sqrt{2(a^2+b^2)}} - \frac{ab}{(a+b)^2} \leq \frac{3}{4}, \tag{2.3}$$

$$\frac{3}{2} \geq \frac{\sqrt{a+b}}{\sqrt{2}\left(\sqrt{a}+\sqrt{b}\right)} + \frac{\sqrt{a}+\sqrt{b}}{\sqrt{2}\sqrt{a+b}} \geq \frac{5}{4} + \frac{ab}{(a+b)^2}. \tag{2.4}$$

Proof. It is easy to see that the left-hand inequality in (2.2) is equivalent to $(a-b)^2 \geq 0$, and the right-hand inequality in (2.2) is equivalent to

$$\frac{\sqrt{2(a^2+b^2)} - (a+b)}{\sqrt{2(a^2+b^2)}} \leq \frac{(a+b)^2 - 4ab}{2(a+b)^2}, \tag{2.5}$$

that is,

$$\frac{(a-b)^2}{2(a^2+b^2) + \sqrt{2(a^2+b^2)}(a+b)} \leq \frac{(a-b)^2}{2(a+b)^2}. \tag{2.6}$$

Indeed, from the left-hand inequality in (2.2) we have

$$2\left(a^2+b^2\right) + \sqrt{2(a^2+b^2)}(a+b) \geq 2\left(a^2+b^2\right) + (a+b)^2 \geq 2(a+b)^2, \tag{2.7}$$

so the right-hand inequality in (2.2) holds.

The inequality in (2.3) is equivalent to

$$\frac{\sqrt{2(a^2+b^2)} - (a+b)}{\sqrt{2(a^2+b^2)}} \geq \frac{(a-b)^2}{4(a+b)^2}. \tag{2.8}$$

Since

$$\frac{\sqrt{2(a^2+b^2)}-(a+b)}{\sqrt{2(a^2+b^2)}} = \frac{2(a^2+b^2)-(a+b)^2}{\sqrt{2(a^2+b^2)}\left(\sqrt{2(a^2+b^2)}+(a+b)\right)} = \frac{(a-b)^2}{2(a^2+b^2)+(a+b)\sqrt{2(a^2+b^2)}}, \tag{2.9}$$

so it is sufficient prove that

$$2\left(a^2+b^2\right)+(a+b)\sqrt{2(a^2+b^2)} \le 4(a+b)^2, \tag{2.10}$$

that is,

$$(a+b)\sqrt{2(a^2+b^2)} \le 2\left(a^2+b^2+4ab\right), \tag{2.11}$$

and, from the left-hand inequalities in (2.2), we have

$$(a+b)\sqrt{2(a^2+b^2)} \le 2\left(a^2+b^2\right) \le 2\left(a^2+b^2+4ab\right), \tag{2.12}$$

so the inequality in (2.3) holds.

Notice that the functions in the inequalities (2.4) are homogeneous. So, without loss of generality, we may assume $\sqrt{a}+\sqrt{b}=1$, and set $t=\sqrt{ab}$. Then $0<t\le 1/4$ and (2.4) reduces to

$$\frac{3}{2} \ge \frac{\sqrt{1-2t}}{\sqrt{2}} + \frac{1}{\sqrt{2}\sqrt{1-2t}} \ge \frac{5}{4} + \frac{t^2}{(1-2t)^2}. \tag{2.13}$$

Squaring every side in the above inequalities yields

$$\frac{9}{4} \ge \frac{1-2t}{2} + \frac{1}{2-4t} + 1 \ge \frac{25}{16} + \frac{t^4}{(1-2t)^4} + \frac{5t^2}{2(1-2t)^2}. \tag{2.14}$$

Reducing to common denominator and rearranging, the right-hand inequality in (2.14) reduces to

$$\frac{(1-2t)\left(16t^2(2t-1)^2+(1/8)(16t-7)^2+(7/8)\right)}{16(2t-1)^4} \ge 0, \tag{2.15}$$

and the left-hand inequality in (2.14) reduces to

$$\frac{2(1-2t)^2+2-5(1-2t)}{2(1-2t)}=-\frac{1+2t}{2}\leq 0, \tag{2.16}$$

so two inequalities in (2.4) hold. □

Lemma 2.6 (see [16]). *Let $a \leq b$, $u(t)=ta+(1-t)b$, $v(t)=tb+(1-t)a$. If $1/2 \leq t_2 \leq t_1 \leq 1$ or $0 \leq t_1 \leq t_2 \leq 1/2$, then*

$$\left(\frac{a+b}{2},\frac{a+b}{2}\right)\prec (u(t_2),v(t_2))\prec (u(t_1),v(t_1))\prec (a,b). \tag{2.17}$$

3. Proof of Main Result

Proof of Theorem I. Let $(a,b)\in R_+^2$.

(1) For

$$D_{SH-SA}(a,b)=\frac{1}{3}M_{SH}(a,b)-M_{SA}(a,b)=\frac{a+b}{2}-\frac{2ab}{3(a+b)}-\frac{2}{3}\sqrt{\frac{a^2+b^2}{2}}, \tag{3.1}$$

we have

$$\begin{aligned}\frac{\partial D_{SH-SA}(a,b)}{\partial a}&=\frac{1}{2}-\frac{2b^2}{3(a+b)^2}-\frac{2}{3}\frac{a}{\sqrt{2(a^2+b^2)}},\\ \frac{\partial D_{SH-SA}(a,b)}{\partial b}&=\frac{1}{2}-\frac{2a^2}{3(a+b)^2}-\frac{2}{3}\frac{b}{\sqrt{2(a^2+b^2)}},\end{aligned} \tag{3.2}$$

whence

$$\begin{aligned}\Lambda &:= (\ln a-\ln b)\left(a\frac{\partial D_{SH-SA}(a,b)}{\partial a}-b\frac{\partial D_{SH-SA}(a,b)}{\partial b}\right)\\ &=(a-b)(\ln a-\ln b)\left(\frac{1}{2}+\frac{2ab}{3(a+b)^2}-\frac{2}{3}\frac{a+b}{\sqrt{2(a^2+b^2)}}\right).\end{aligned} \tag{3.3}$$

From (2.3) we have

$$\frac{1}{2}+\frac{2ab}{3(a+b)^2}-\frac{2}{3}\frac{a+b}{\sqrt{2(a^2+b^2)}}\geq 0, \tag{3.4}$$

which implies $\Lambda \geq 0$ and, by Lemma 2.4, it follows that D_{SH-SA} is Schur-geometrically convex in R_+^2.

(2) For

$$D_{AH-SH}(a,b)=\frac{1}{2}M_{AH}(a,b)-\frac{1}{3}M_{SH}(a,b)=\frac{a+b}{4}-\frac{ab}{3(a+b)}-\frac{1}{3}\sqrt{\frac{a^2+b^2}{2}}. \tag{3.5}$$

To prove that the function D_{AH-SH} is Schur-geometrically convex in R_+^2 it is enough to notice that $D_{AH-SH}(a,b) = (1/2)D_{SH-SA}(a,b)$.

(3) For

$$D_{SG-AH}(a,b) = M_{SG}(a,b) - M_{AH}(a,b) = \sqrt{\frac{a^2+b^2}{2}} - \sqrt{ab} - \frac{a+b}{2} + \frac{2ab}{a+b}, \tag{3.6}$$

we have

$$\begin{aligned}\frac{\partial D_{SG-AH}(a,b)}{\partial a} &= \frac{a}{\sqrt{2(a^2+b^2)}} - \frac{b}{2\sqrt{ab}} - \frac{1}{2} + \frac{2b^2}{(a+b)^2},\\ \frac{\partial D_{SG-AH}(a,b)}{\partial b} &= \frac{b}{\sqrt{2(a^2+b^2)}} - \frac{a}{2\sqrt{ab}} - \frac{1}{2} + \frac{2a^2}{(a+b)^2},\end{aligned} \tag{3.7}$$

and then

$$\begin{aligned}\Lambda &:= (\ln a - \ln b)\left(a\frac{\partial D_{SH-SA}(a,b)}{\partial a} - b\frac{\partial D_{SH-SA}(a,b)}{\partial b}\right)\\ &= (a-b)(\ln a - \ln b)\left(\frac{a+b}{\sqrt{2(a^2+b^2)}} - \frac{1}{2} - \frac{2ab}{(a+b)^2}\right).\end{aligned} \tag{3.8}$$

From (2.2) we have $\Lambda \geq 0$, so by Lemma 2.4, it follows that D_{SH-SA} is Schur-geometrically convex in R_+^2.

(4) For

$$D_{AG-SG}(a,b) = M_{AG}(a,b) - \frac{1}{2}M_{SG}(a,b) = \frac{1}{2}\left(a + b - \sqrt{ab} - \sqrt{\frac{a^2+b^2}{2}}\right), \tag{3.9}$$

we have

$$\begin{aligned}\frac{\partial D_{AG-SG}(a,b)}{\partial a} &= \frac{1}{2}\left(1 - \frac{b}{2\sqrt{ab}} - \frac{a}{\sqrt{2(a^2+b^2)}}\right),\\ \frac{\partial D_{AG-SG}(a,b)}{\partial b} &= \frac{1}{2}\left(1 - \frac{a}{2\sqrt{ab}} - \frac{b}{\sqrt{2(a^2+b^2)}}\right),\end{aligned} \tag{3.10}$$

and then

$$\begin{aligned}\Lambda &:= (\ln a - \ln b)\left(a\frac{\partial D_{SH-SA}(a,b)}{\partial a} - b\frac{\partial D_{SH-SA}(a,b)}{\partial b}\right)\\ &= (a-b)(\ln a - \ln b)\left(1 - \frac{a+b}{\sqrt{2(a^2+b^2)}}\right).\end{aligned} \tag{3.11}$$

By (2.2) we infer that

$$1-\frac{a+b}{\sqrt{2(a^2+b^2)}}\geq 0, \tag{3.12}$$

so $\Lambda \geq 0$. By Lemma 2.4, we get that D_{AG-SG} is Schur-geometrically convex in R_+^2.

(5) For

$$\begin{aligned} D_{N_2N_1-AH}(a,b) &= M_{N_2N_1}(a,b)-\frac{1}{8}M_{AH}(a,b) \\ &= \left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)\left(\sqrt{\frac{a+b}{2}}\right)-\frac{1}{4}(a+b)-\frac{1}{2}\sqrt{ab}-\frac{1}{8}\left(\frac{a+b}{2}-\frac{2ab}{a+b}\right), \end{aligned} \tag{3.13}$$

we have

$$\begin{aligned} \frac{\partial D_{N_2N_1-AH}(a,b)}{\partial a} &= \frac{1}{4\sqrt{a}}\sqrt{\frac{a+b}{2}}+\frac{1}{4}\left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)\left(\frac{a+b}{2}\right)^{-1/2} \\ &\quad -\frac{1}{4}-\frac{b}{4\sqrt{ab}}-\frac{1}{8}\left(\frac{1}{2}-\frac{2b^2}{(a+b)^2}\right), \\ \frac{\partial D_{N_2N_1-AH}(a,b)}{\partial b} &= \frac{1}{4\sqrt{b}}\sqrt{\frac{a+b}{2}}+\frac{1}{4}\left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)\left(\frac{a+b}{2}\right)^{-1/2} \\ &\quad -\frac{1}{4}-\frac{a}{4\sqrt{ab}}-\frac{1}{8}\left(\frac{1}{2}-\frac{2a^2}{(a+b)^2}\right), \end{aligned} \tag{3.14}$$

and then

$$\begin{aligned} \Lambda &= (\ln a-\ln b)\left(a\frac{\partial D_{N_2N_1-AH}(a,b)}{\partial a}-b\frac{\partial D_{N_2N_1-AH}(a,b)}{\partial b}\right) \\ &= \frac{1}{4}(a-b)(\ln a-\ln b)\left(\frac{\sqrt{a+b}}{\sqrt{2}\left(\sqrt{a}+\sqrt{b}\right)}+\frac{\sqrt{a}+\sqrt{b}}{\sqrt{2}\sqrt{a+b}}-\frac{5}{4}-\frac{ab}{(a+b)^2}\right). \end{aligned} \tag{3.15}$$

From (2.4) we have

$$\frac{\sqrt{a+b}}{\sqrt{2}\left(\sqrt{a}+\sqrt{b}\right)}+\frac{\sqrt{a}+\sqrt{b}}{\sqrt{2}\sqrt{a+b}}-\frac{5}{4}-\frac{ab}{(a+b)^2}\geq 0, \tag{3.16}$$

so $\Lambda \geq 0$; it follows that $D_{N_2N_1-AH}$ is Schur-geometrically convex in R_+^2.

(6) For

$$\begin{aligned} D_{N_2G-N_2N_1}(a,b) &= \frac{1}{3}M_{N_2G}(a,b) - M_{N_2N_1}(a,b) \\ &= \frac{a+b}{4} + \frac{\sqrt{ab}}{6} - \frac{2}{3}\left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)\left(\sqrt{\frac{a+b}{2}}\right), \end{aligned} \tag{3.17}$$

we have

$$\begin{aligned} \frac{\partial D_{N_2G-N_2N_1}(a,b)}{\partial a} &= \frac{1}{4} + \frac{b}{12\sqrt{ab}} - \frac{1}{6\sqrt{a}}\sqrt{\frac{a+b}{2}} - \frac{1}{6}\left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)\left(\frac{a+b}{2}\right)^{-1/2}, \\ \frac{\partial D_{N_2G-N_2N_1}(a,b)}{\partial b} &= \frac{1}{4} + \frac{a}{12\sqrt{ab}} - \frac{1}{6\sqrt{b}}\sqrt{\frac{a+b}{2}} - \frac{1}{6}\left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)\left(\frac{a+b}{2}\right)^{-1/2}, \end{aligned} \tag{3.18}$$

and then

$$\begin{aligned} \Lambda &= (\ln a - \ln b)\left(a\frac{\partial D_{N_2G-N_2N_1}(a,b)}{\partial a} - b\frac{\partial D_{N_2G-N_2N_1}(a,b)}{\partial b}\right) \\ &= (\ln a - \ln b)\left(\frac{1}{4}(a-b) - \frac{\sqrt{a}-\sqrt{b}}{6}\sqrt{\frac{a+b}{2}} - \frac{(a-b)\left(\sqrt{a}+\sqrt{b}\right)}{12}\left(\frac{a+b}{2}\right)^{-1/2}\right) \\ &= \frac{1}{6}(a-b)(\ln a - \ln b)\left(\frac{3}{2} - \frac{\sqrt{a+b}}{\sqrt{2}\left(\sqrt{a}+\sqrt{b}\right)} - \frac{\sqrt{a}+\sqrt{b}}{\sqrt{2}\sqrt{a+b}}\right). \end{aligned} \tag{3.19}$$

By (2.4) we infer that $\Lambda \geq 0$, which proves that $D_{N_2G-N_2N_1}$ is Schur-geometrically convex in R_+^2.

(7) For

$$\begin{aligned} D_{AG-N_2G}(a,b) &= \frac{1}{4}M_{AG}(a,b) - \frac{1}{3}M_{N_2G}(a,b) \\ &= \frac{a+b}{8} + \frac{1}{12}\sqrt{ab} - \frac{1}{3}\left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)\left(\sqrt{\frac{a+b}{2}}\right), \end{aligned} \tag{3.20}$$

we have

$$\begin{aligned} \frac{\partial D_{AG-N_2G}(a,b)}{\partial a} &= \frac{1}{8} + \frac{b}{24\sqrt{ab}} - \frac{\sqrt{a+b}}{12\sqrt{2a}} - \frac{\sqrt{a}+\sqrt{b}}{12\sqrt{2(a+b)}}, \\ \frac{\partial D_{AG-N_2G}(a,b)}{\partial b} &= \frac{1}{8} + \frac{a}{24\sqrt{ab}} - \frac{\sqrt{a+b}}{12\sqrt{2b}} - \frac{\sqrt{a}+\sqrt{b}}{12\sqrt{2(a+b)}}, \end{aligned} \tag{3.21}$$

and then

$$\begin{aligned}\Lambda &= (\ln a - \ln b)\left(a\frac{\partial D_{AG-N_2G}(a,b)}{\partial a} - b\frac{\partial D_{AG-N_2G}(a,b)}{\partial b}\right)\\ &= (\ln a - \ln b)\left(\frac{a-b}{8} - \frac{\sqrt{a+b}\left(\sqrt{a}-\sqrt{b}\right)}{12\sqrt{2}} - \frac{(a-b)\left(\sqrt{a}+\sqrt{b}\right)}{12\sqrt{2(a+b)}}\right)\\ &= \frac{(a-b)(\ln a - \ln b)}{8}\left(1 - \frac{2}{3}\left(\frac{\sqrt{a+b}}{\sqrt{2}\left(\sqrt{a}+\sqrt{b}\right)} + \frac{\sqrt{a}+\sqrt{b}}{\sqrt{2}\sqrt{a+b}}\right)\right).\end{aligned} \tag{3.22}$$

From (2.4) we have $\Lambda \geq 0$, and, consequently, by Lemma 2.4, we obtain that D_{AG-N_2G} is Schur-geometrically convex in R_+^2.

(8) In order to prove that the function $D_{AN_2-AG}(a,b)$ is Schur-geometrically convex in R_+^2 it is enough to notice that

$$D_{AN_2-AG}(a,b) = M_{AN_2}(a,b) - \frac{1}{4}M_{AG}(a,b) = 3D_{AG-N_2G}(a,b). \tag{3.23}$$

(9) For

$$\begin{aligned}D_{SN_2-SA}(a,b) &= \frac{4}{5}M_{SN_2}(a,b) - M_{SA}(a,b)\\ &= \frac{a+b}{2} - \frac{1}{5}\sqrt{\frac{a^2+b^2}{2}} - \frac{1}{5}\left(\sqrt{a}+\sqrt{b}\right)\sqrt{2(a+b)},\end{aligned} \tag{3.24}$$

we have

$$\begin{aligned}\frac{\partial D_{SN_2-SA}(a,b)}{\partial a} &= \frac{1}{2} - \frac{a}{5\sqrt{2(a^2+b^2)}} - \frac{1}{5}\sqrt{\frac{a+b}{2a}} - \frac{\sqrt{a}+\sqrt{b}}{5\sqrt{2(a+b)}},\\ \frac{\partial D_{SN_2-SA}(a,b)}{\partial b} &= \frac{1}{2} - \frac{b}{5\sqrt{2(a^2+b^2)}} - \frac{1}{5}\sqrt{\frac{a+b}{2b}} - \frac{\sqrt{a}+\sqrt{b}}{5\sqrt{2(a+b)}},\end{aligned} \tag{3.25}$$

and then

$$\begin{aligned}\Lambda &= (\ln a - \ln b)\left(\frac{\partial D_{SN_2-SA}(a,b)}{\partial a} - \frac{\partial D_{SN_2-SA}(a,b)}{\partial b}\right)\\ &= (\ln a - \ln b)\left(\frac{a-b}{2} - \frac{a^2-b^2}{5\sqrt{2(a^2+b^2)}} - \frac{1}{5}\left(\sqrt{\frac{a(a+b)}{2}} - \sqrt{\frac{b(a+b)}{2}}\right) - \frac{\left(\sqrt{a}+\sqrt{b}\right)(a-b)}{5\sqrt{2(a+b)}}\right)\\ &= \frac{(a-b)(\ln a - \ln b)}{5\sqrt{2}}\left(\frac{5}{\sqrt{2}} - \frac{a+b}{\sqrt{a^2+b^2}} - \frac{\sqrt{a+b}}{\sqrt{a}+\sqrt{b}} - \frac{\sqrt{a}+\sqrt{b}}{\sqrt{a+b}}\right).\end{aligned} \tag{3.26}$$

From (2.2) and (2.4) we obtain that

$$\frac{5}{\sqrt{2}} - \frac{a+b}{\sqrt{a^2+b^2}} - \frac{\sqrt{a+b}}{\sqrt{a}+\sqrt{b}} - \frac{\sqrt{a}+\sqrt{b}}{\sqrt{a+b}} \geq \frac{5}{\sqrt{2}} - \sqrt{2} - \frac{3}{\sqrt{2}} = 0, \tag{3.27}$$

so $\Lambda \geq 0$, which proves that the function $D_{SN_2-SA}(a,b)$ is Schur-geometrically convex in R_+^2.

(10) One can easily check that

$$D_{AN_AN_2-SN_2}(a,b) = 4D_{SN_2-SA}(a,b), \tag{3.28}$$

and, consequently, the function $D_{AN_2-SN_2}$ is Schur-geometrically convex in R_+^2.

(11) To prove that the function

$$D_{SN_1-SH}(a,b) = 2M_{SN_1}(a,b) - M_{SH}(a,b) = \sqrt{\frac{a^2+b^2}{2}} - \frac{a+b}{2} - \sqrt{ab} + \frac{2ab}{a+b} \tag{3.29}$$

is Schur-geometrically convex in R_+^2 it is enough to notice that

$$D_{SN_1-SH}(a,b) = D_{SG-AH}(a,b). \tag{3.30}$$

(12) For

$$\begin{aligned}D_{SG-SN_1}(a,b) &= \frac{3}{2}M_{SG}(a,b) - 2M_{SN_1}(a,b)\\ &= \frac{1}{2}\left(a+b-\sqrt{ab}-\sqrt{\frac{a^2+b^2}{2}}\right),\end{aligned} \tag{3.31}$$

we have

$$\frac{\partial D_{SG-SN_1}(a,b)}{\partial a} = \frac{1}{2}\left(1 - \frac{b}{2\sqrt{ab}} - \frac{a}{\sqrt{2(a^2+b^2)}}\right),$$
$$\frac{\partial D_{SG-SN_1}(a,b)}{\partial b} = \frac{1}{2}\left(1 - \frac{a}{2\sqrt{ab}} - \frac{b}{\sqrt{2(a^2+b^2)}}\right), \tag{3.32}$$

and then

$$\Lambda = (\ln a - \ln b)\left(a\frac{\partial D_{SG-SN_1}(a,b)}{\partial a} - b\frac{\partial D_{SG-SN_1}(a,b)}{\partial b}\right)$$
$$= \frac{(a-b)(\ln a - \ln b)}{2}\left(1 - \frac{a+b}{\sqrt{2(a^2+b^2)}}\right). \tag{3.33}$$

By the inequality (2.2) we get that $\Lambda \geq 0$, which proves that D_{SG-SN_1} is Schur-geometrically convex in R_+^2.

(13) It is easy to check that

$$D_{SN_3-SA}(a,b) = \frac{1}{2}D_{AG-SG}(a,b), \tag{3.34}$$

which means that the function D_{SN_3-SA} is Schur-geometrically convex in R_+^2.

(14) To prove that the function $D_{SN_1-SN_3}$ is Schur-geometrically convex in R_+^2 it is enough to notice that

$$D_{SN_1-SN_3}(a,b) = \frac{1}{6}D_{AG-SG}(a,b). \tag{3.35}$$

The proof of Theorem I is complete. □

4. Applications

Applying Theorem I, Lemma 2.6, and Definition 2.2 one can easily prove the following.

Theorem II. *Let* $0 < a \leq b$. $1/2 \leq t \leq 1$ *or* $0 \leq t \leq 1/2$, $u = a^t b^{1-t}$ *and* $v = b^t a^{1-t}$. *Then*

$$\begin{aligned}
M_{SA}(a,b) &\leq \frac{1}{3}M_{SH}(a,b) - \left(\frac{1}{3}M_{SH}(u,v) - M_{SA}(u,v)\right) \leq \frac{1}{3}M_{SH}(a,b) \\
&\leq \frac{1}{2}M_{AH}(a,b) - \left(\frac{1}{2}M_{AH}(u,v) - \frac{1}{3}M_{SH}(u,v)\right) \leq \frac{1}{2}M_{AH}(a,b) \\
&\leq \frac{1}{2}M_{SG}(a,b) - \left(\frac{1}{2}M_{SG}(u,v) - \frac{1}{2}M_{AH}(u,v)\right) \leq \frac{1}{2}M_{SG}(a,b) \\
&\leq M_{AG}(a,b) - \left(M_{AG}(u,v) - \frac{1}{2}M_{SG}(u,v)\right) \leq M_{AG}(a,b),
\end{aligned} \tag{4.1}$$

$$\begin{aligned}
\frac{1}{8}M_{AH}(a,b) &\leq M_{N_2N_1}(a,b) - \left(M_{N_2N_1}(u,v) - \frac{1}{8}M_{AH}(u,v)\right) \leq M_{N_2N_1}(a,b) \\
&\leq \frac{1}{3}M_{N_2G}(a,b) - \left(\frac{1}{3}M_{N_2G}(u,v) - M_{N_2N_1}(u,v)\right) \leq \frac{1}{3}M_{N_2G}(a,b) \\
&\leq \frac{1}{4}M_{AG}(a,b) - \left(\frac{1}{4}M_{AG}(u,v) - \frac{1}{3}M_{N_2G}(u,v)\right) \leq \frac{1}{4}M_{AG}(a,b) \\
&\leq M_{AN_2}(a,b) - \left(M_{AN_2}(u,v) - \frac{1}{4}M_{AG}(u,v)\right) \leq M_{AN_2}(a,b),
\end{aligned} \tag{4.2}$$

$$\begin{aligned}
M_{SA}(a,b) &\leq \frac{4}{5}M_{SN_2}(a,b) - \left(\frac{4}{5}M_{SN_2}(u,v) - \frac{4}{5}M_{SN_2}(u,v)\right) \leq \frac{4}{5}M_{SN_2}(a,b) \\
&\leq 4M_{AN_2}(a,b) - \left(4M_{AN_2}(u,v) - \frac{4}{5}M_{SN_2}(u,v)\right) \leq 4M_{AN_2}(a,b),
\end{aligned} \tag{4.3}$$

$$\begin{aligned}
M_{SH}(a,b) &\leq 2M_{SN_1}(a,b) - (2M_{SN_1}(u,v) - M_{SH}(u,v)) \leq 2M_{SN_1}(a,b) \\
&\leq \frac{3}{2}M_{SG}(a,b) - \left(\frac{3}{2}M_{SG}(u,v) - \frac{3}{2}M_{SG}(u,v)\right) \leq \frac{3}{2}M_{SG}(a,b),
\end{aligned} \tag{4.4}$$

$$\begin{aligned}
M_{SA}(a,b) &\leq \frac{3}{4}M_{SN_3}(a,b) - \left(\frac{3}{4}M_{SN_3}(u,v) - M_{SA}(u,v)\right) \leq \frac{3}{4}M_{SN_3}(a,b) \\
&\leq \frac{2}{3}M_{SN_1}(a,b) - \left(\frac{2}{3}M_{SN_1}(u,v) - \frac{3}{4}M_{SN_3}(u,v)\right) \leq \frac{2}{3}M_{SN_1}(a,b).
\end{aligned} \tag{4.5}$$

Remark 4.1. Equation (4.1), (4.2), (4.3), (4.4), and (4.5) are a refinement of (1.5), (1.6), (1.7), (1.8), and (1.9), respectively.

Acknowledgments

The authors are grateful to the referees for their helpful comments and suggestions. The first author was supported in part by the Scientific Research Common Program of Beijing Municipal Commission of Education (KM201011417013).

References

[1] I. J. Taneja, "Refinement of inequalities among means," *Journal of Combinatorics, Information & System Sciences*, vol. 31, no. 1–4, pp. 343–364, 2006.

[2] I. J. Taneja, "On a Difference of Jensen Inequality and its Applications to Mean Divergence Measures," *RGMIA Research Report Collection*, vol. 7, article 16, no. 4, 2004, http://rgmia.vu.edu.au/.

[3] I. J. Taneja, "On symmetric and nonsymmetric divergence measures and their generalizations," *Advances in Imaging and Electron Physics*, vol. 138, pp. 177–250, 2005.

[4] A. W. Marshall and I. Olkin, *Inequalities: Theory of Majorization and Its Applications*, vol. 143 of *Mathematics in Science and Engineering*, Academic Press, New York, NY, USA, 1979.

[5] X. Zhang and Y. Chu, "The Schur geometrical convexity of integral arithmetic mean," *International Journal of Pure and Applied Mathematics*, vol. 41, no. 7, pp. 919–925, 2007.

[6] K. Guan, "Schur-convexity of the complete symmetric function," *Mathematical Inequalities & Applications*, vol. 9, no. 4, pp. 567–576, 2006.

[7] K. Guan, "Some properties of a class of symmetric functions," *Journal of Mathematical Analysis and Applications*, vol. 336, no. 1, pp. 70–80, 2007.

[8] C. Stepniak, "An effective characterization of Schur-convex functions with applications," *Journal of Convex Analysis*, vol. 14, no. 1, pp. 103–108, 2007.

[9] H.-N. Shi, "Schur-convex functions related to Hadamard-type inequalities," *Journal of Mathematical Inequalities*, vol. 1, no. 1, pp. 127–136, 2007.

[10] H.-N. Shi, D.-M. Li, and C. Gu, "The Schur-convexity of the mean of a convex function," *Applied Mathematics Letters*, vol. 22, no. 6, pp. 932–937, 2009.

[11] Y. Chu and X. Zhang, "Necessary and sufficient conditions such that extended mean values are Schur-convex or Schur-concave," *Journal of Mathematics of Kyoto University*, vol. 48, no. 1, pp. 229–238, 2008.

[12] N. Elezović and J. Pečarić, "A note on Schur-convex functions," *The Rocky Mountain Journal of Mathematics*, vol. 30, no. 3, pp. 853–856, 2000.

[13] J. Sándor, "The Schur-convexity of Stolarsky and Gini means," *Banach Journal of Mathematical Analysis*, vol. 1, no. 2, pp. 212–215, 2007.

[14] H.-N. Shi, S.-H. Wu, and F. Qi, "An alternative note on the Schur-convexity of the extended mean values," *Mathematical Inequalities & Applications*, vol. 9, no. 2, pp. 219–224, 2006.

[15] X. M. Zhang, *Geometrically Convex Functions*, An'hui University Press, Hefei, China, 2004.

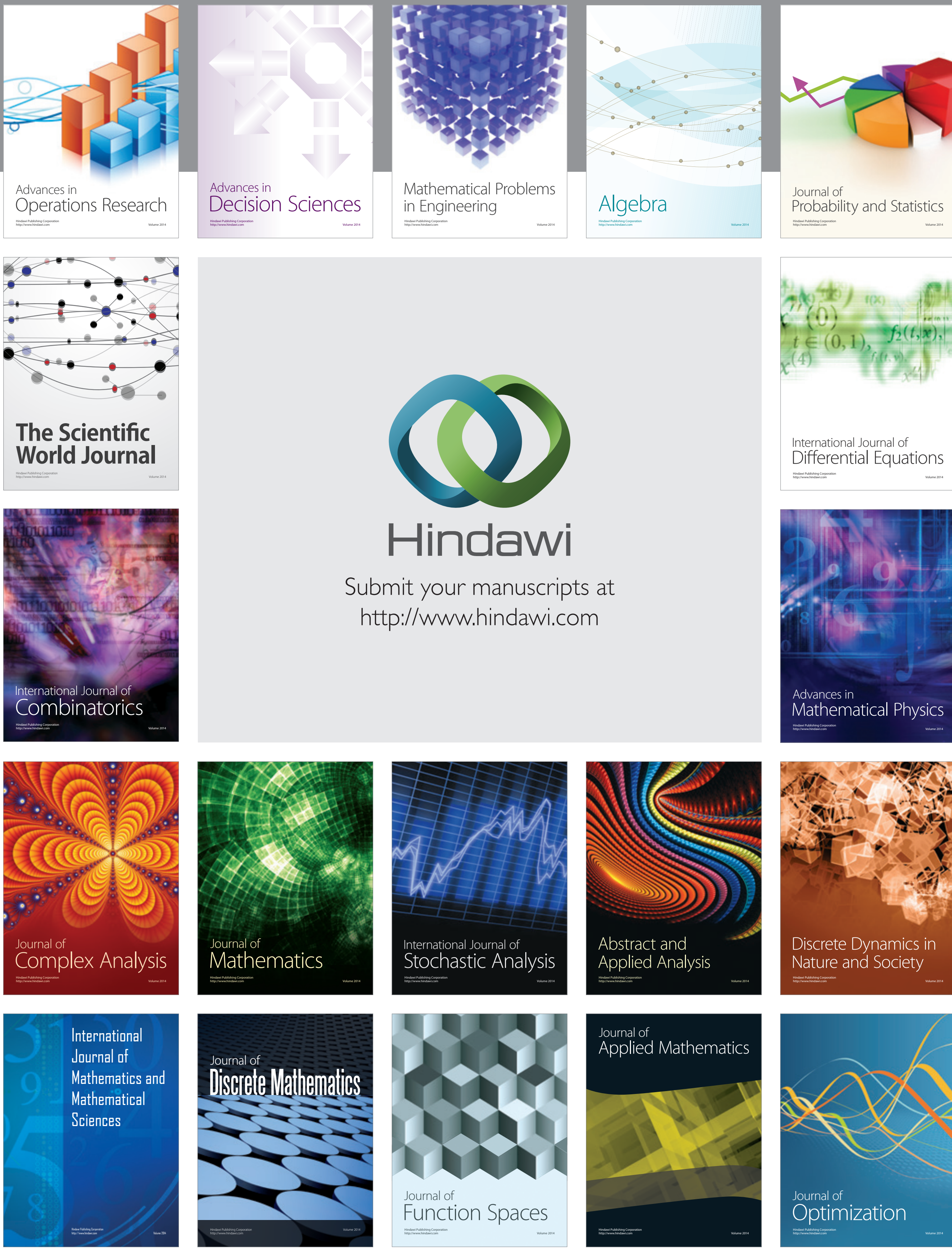
[16] H.-N. Shi, Y.-M. Jiang, and W.-D. Jiang, "Schur-convexity and Schur-geometrically concavity of Gini means," *Computers & Mathematics with Applications*, vol. 57, no. 2, pp. 266–274, 2009.

[17] Y. Chu, X. Zhang, and G. Wang, "The Schur geometrical convexity of the extended mean values," *Journal of Convex Analysis*, vol. 15, no. 4, pp. 707–718, 2008.

[18] K. Guan, "A class of symmetric functions for multiplicatively convex function," *Mathematical Inequalities & Applications*, vol. 10, no. 4, pp. 745–753, 2007.

[19] H.-N. Shi, M. Bencze, S.-H. Wu, and D.-M. Li, "Schur convexity of generalized Heronian means involving two parameters," *Journal of Inequalities and Applications*, vol. 2008, Article ID 879273, 9 pages, 2008.

[20] B. Y. Wang, *Foundations of Majorization Inequalities*, Beijing Normal University Press, Beijing, China, 1990.


Advances in Operations Research
Advances in Decision Sciences
Mathematical Problems in Engineering
Algebra
Journal of Probability and Statistics
The Scientific World Journal
International Journal of Differential Equations
Hindawi
Submit your manuscripts at
http://www.hindawi.com
International Journal of Combinatorics
Advances in Mathematical Physics
Journal of Complex Analysis
Journal of Mathematics
International Journal of Stochastic Analysis
Abstract and Applied Analysis
Discrete Dynamics in Nature and Society
International Journal of Mathematics and Mathematical Sciences
Journal of Discrete Mathematics
Journal of Function Spaces
Journal of Applied Mathematics
Journal of Optimization