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Research Article

Properties of Certain Subclass of Multivalent Functions with Negative Coefficients

Jingyu Yang[1,2] and Shuhai Li[1]

[1] *Department of Mathematics, Chifeng University, Inner Mongolia, Chifeng 024000, China*
[2] *School of Mathematical Sciences, Dalian University of Technology, Liaoning 116024, China*

Correspondence should be addressed to Shuhai Li, lishms66@sina.com

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Making use of a linear operator, which is defined by the Hadamard product, we introduce and study a subclass $Y^{+}_{a,c}(A,B;p,\lambda,\alpha)$ of the class $A^{+}(p)$. In this paper, we obtain the coefficient inequality, distortion theorem, radius of convexity and starlikeness, neighborhood property, modified convolution properties of this class. Furthermore, an application of fractional calculus operator is given. The results are presented here would provide extensions of some earlier works. Several new results are also obtained.

1. Introduction

Let $A^{+}(p)$ denote the class of functions

$$f(z) = z^{p} - \sum_{k=1}^{\infty} \left|a_{k+p}\right| z^{k+p} \quad (p \in N = \{1,2,\ldots\}) \tag{1.1}$$

which are analytic and p-valent in the open unit disc $U = \{z : |z| < 1\}$ on the complex plane C.

We denote by $S_{p}^{*}(A,B)$ and $K_{p}(A,B)$ $(-1 \le B < A \le 1)$ the subclass of p-valent starlike functions and the subclass of p-valent convex functions, respectively, that is, (see for details [1, 2])

$$S_{p}^{*}(A,B) = \left\{ f(z) \in A(p) : \frac{zf'(z)}{pf(z)} \prec \frac{1+Az}{1+Bz} \ (z \in U;\ -1 \le B < A \le 1) \right\},$$

$$K_p(A,B) = \left\{ f(z) \in A(p) : \frac{1}{p} + \frac{zf''(z)}{pf'(z)} \prec \frac{1+Az}{1+Bz} \ (z \in U;\ -1 \le B < A \le 1) \right\}. \tag{1.2}$$

A function $f \in A^+(p)$ is said to be analytic starlike of order ξ if it satisfies

$$\operatorname{Re}\left\{ \frac{zf'(z)}{pf(z)} \right\} > \xi \tag{1.3}$$

for some ξ $(0 \le \xi < 1)$ and for all $z \in U$.

Further, a function $f \in A^+(p)$ is said to be analytic convex of order ξ if it satisfies

$$\operatorname{Re}\left\{ \frac{1}{p} + \frac{zf''(z)}{pf'(z)} \right\} > \xi \tag{1.4}$$

for some $\xi(0 \le \xi < 1)$ and for all $z \in U$.

For $f(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} z^{k+p}$ and $g(z) = z^p + \sum_{k=1}^{\infty} b_{k+p} z^{k+p}$, the Hadamard product (or convolution) is defined by

$$(f * g)(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} b_{k+p} z^{k+p} = (g * f)(z). \tag{1.5}$$

The linear operator $L_p(a,c)$ is defined as follows (see Saitoh [3]):

$$L_p(a,c)f(z) = \varphi_p(a,c;z) * f(z) \quad (f(z) \in A(p)), \tag{1.6}$$

and $\varphi_p(a,c;z)$ is defined by

$$\varphi_p(a,c;z) = z^p + \sum_{k=1}^{\infty} \frac{(a)_k}{(c)_k} z^{k+p} \quad \left(a \in R;\ c \setminus z_0^-;\ z_0^- := \{0,-1,-2,\ldots\}\right), \tag{1.7}$$

where $(v)_k$ is the pochharmmer symbol defined (in terms of the Gamma function) by

$$(\nu)_n = \frac{\Gamma(\nu+n)}{\Gamma(\nu)} = \begin{cases} 1, & n = 0, \\ \nu(\nu+1)\cdots(\nu+n-1), & n \in N. \end{cases} \tag{1.8}$$

The operator $L_p(a,c)$ was studied recently by Srivastava and Patel [4]. It is easily verified from (1.6) and (1.7) that

$$z\left(L_p(a,c)f(z)\right)' = aL_p(a+1,c)f(z) - (a-p)L_p(a,c)f(z). \tag{1.9}$$

Moreover, for $f(z) \in A(p)$,

$$L_p(a,a)f(z) = f(z), \qquad L_p(p+1,p)f(z) = \frac{zf'(z)}{p},$$
$$L_p(n+p,1)f(z) = D^{n+p-1}f(z) \quad (n > -p), \tag{1.10}$$

where $D^{n+p-1}f(z)$ denotes the Ruscheweyh derivative of a function $f(z) \in A(z)$ of order $n+p-1$ (see [5]).

Aouf, Silverman and Srivastava [6] introduced the class of $P^{+}_{a,c}(A,B;p,\lambda)$ by use of linear operator $L_p(a,c)$, further investigated the properties of this class. In [7], SokóŁ investigated several properties of the linear Aouf-Silverman-Srivastava operator and furthermore obtained the corresponding characterizations of multivalent analytic functions which were studied by Aouf et al. in paper [6]. The properties of multivalent functions with negative coefficients were studied in [8–10]. References [11, 12] gave the results of the univalent function with negative coefficients.

In this paper, we will use operator $L_p(a,c)$ to define a new subclass of $A^{+}(p)$ as follows.

For $p \in N,\ a > 0,\ c > 0,\ \alpha \geq 0$ and for the parameters $\lambda,\ A,$ and B such that

$$-1 \leq B < A \leq 1, \qquad -1 \leq B < 0, \qquad 0 \leq \lambda < p, \tag{1.11}$$

we say that a function $f(z) \in A^{+}(p)$ is in the class $\Upsilon^{+}_{a,c}(A,B;p,\lambda,\alpha)$ if it satisfies the following subordination:

$$\frac{1}{p-\lambda}\left(\frac{(L_p(a,c)f(z))'}{z^{p-1}} - \alpha\left|\frac{(L_p(a,c)f(z))'}{pz^{p-1}} - 1\right| - \lambda\right) \prec \frac{1+Az}{1+Bz} \quad (z \in U), \tag{1.12}$$

or equivalently, if the following inequality holds true:

$$\left|\frac{(L_p(a,c)f(z))'/z^{p-1} - \alpha\left|(L_p(a,c)f(z))'/pz^{p-1} - 1\right| - p}{B\left((L_p(a,c)f(z))'/z^{p-1} - \alpha\left|(L_p(a,c)f(z))'/pz^{p-1} - 1\right|\right) - (pB + (A-B)(p-\lambda))}\right| < 1. \tag{1.13}$$

From the above definition, we can imply that the function class $P^{+}_{a,c}(A,B;p,\lambda)$ in [6] is the special case of $\Upsilon^{+}_{a,c}(A,B;p,\lambda,\alpha)$ in our present paper because $\Upsilon^{+}_{a,c}(A,B;p,\lambda,0) = P^{+}_{a,c}(A,B;p,\lambda)$.

Since $\Upsilon^{+}_{a,c}(A,B;p,\lambda,0) = P^{+}_{a,c}(A,B;p,\lambda)$, then, like [6], we have the following subclasses which were studied in many earlier works:

(1) $\Upsilon^{+}_{n+p,1}(-1,1;p,\lambda,0) = Q_{n+p-1}(\lambda)$ $(0 \leq \lambda < 1;\ n > -p;\ p \in N)$ (Aouf and Darwish [8]),

(2) $\Upsilon^{+}_{a,a}(A,B;p,0,0) = P^{*}(p,A,B)$ (Shukla and Dashrath [9]),

(3) $\Upsilon^{+}_{a,a}(1,-1;p,\lambda,0) = F_p(1,\lambda)$ $(0 \leq \lambda < p;\ p \in N)$ (Lee et al. [10]),

(4) $\Upsilon^{+}_{a,a}(-\beta,\beta;\ 1,\lambda,0) = P^{*}(\lambda,\beta)$ $(0 \leq \lambda < 1;\ 0 < \beta \leq 1)$ (Gupta and Jain [11]),

(5) $\Upsilon^{+}_{n+p,1}(-1,1;\ 1,\lambda,0) = Q_n(\lambda)$ $(n \in N_0 := N \cup \{0\};\ 0 \leq \lambda < 1)$ (Uralegaddi and Sarangi [12]).

The purpose of this paper is to give various properties of class $\Upsilon^{+}_{a,c}(A,B;p,\lambda,\alpha)$. We extend the results of basic paper [6].

2. Necessary and Sufficient Condition for $f(z) \in \Upsilon^{+}_{a,c}(A,B;p,\lambda,\alpha)$

Theorem 2.1. *Let the function* $f(z) \in A^{+}(p)$ *be given by* (1.1), *then* $f(z) \in \Upsilon^{+}_{a,c}(A,B;p,\lambda,\alpha)$ *if and only if*

$$\sum_{k=1}^{\infty} \phi_k(p,\alpha;B,a,c)\left|a_{k+p}\right| \leq (A-B), \tag{2.1}$$

where

$$\phi_k(p,\alpha;B,a,c) = \frac{(p+\alpha)(1-B)(k+p)(a)_k}{p(p-\lambda)(c)_k}. \tag{2.2}$$

Proof. For the sufficient condition, let $f(z) = z^p - \sum_{k=1}^{\infty} |a_{k+p}| z^{k+p}$, we have

$$\begin{aligned}
&\left|\frac{(L_p(a,c)f(z))'/z^{p-1} - \alpha\left|(L_p(a,c)f(z))'/pz^{p-1} - 1\right| - p}{B\Big((L_p(a,c)f(z))'/z^{p-1} - \alpha\left|(L_p(a,c)f(z))'/pz^{p-1}\right|\Big) - (pB + (A-B)(p-\lambda))}\right| \\
&= \left|\frac{p(L_p(a,c)f(z))' - \alpha e^{i\theta}\left|(L_p(a,c)f(z))' - pz^{p-1}\right| - p^2 z^{p-1}}{B\Big(p(L_p(a,c)f(z))' - \alpha e^{i\theta}\left|(L_p(a,c)f(z))' - pz^{p-1}\right|\Big) - p(pB + (A-B)(p-\lambda))\ z^{p-1}}\right|,
\end{aligned} \tag{2.3}$$

then

$$\begin{aligned}
&\left|p(L_p(a,c)f(z))' - \alpha e^{i\theta}\left|(L_p(a,c)f(z))' - pz^{p-1}\right| - p^2 z^{p-1}\right| \\
&\quad - \left|B\Big(p(L_p(a,c)f(z))' - \alpha e^{i\theta}\left|(L_p(a,c)f(z))' - pz^{p-1}\right|\Big) - p(pB + (A-B)(p-\lambda))z^{p-1}\right| \\
&= \left|-\sum_{k=1}^{\infty} p(k+p)\left|a_{k+p}\right|\frac{(a)_k}{(c)_k}z^{k+p-1} - \alpha e^{i\theta}\left|-\sum_{k=1}^{\infty}(k+p)\left|a_{k+p}\right|\frac{(a)_k}{(c)_k}z^{k+p-1}\right|\right| \\
&\quad - \left|-\sum_{k=1}^{\infty} Bp(k+p)\left|a_{k+p}\right|\frac{(a)_k}{(c)_k}z^{k+p-1} - \alpha B e^{i\theta}\left|-\sum_{k=1}^{\infty}(k+p)\left|a_{k+p}\right|\frac{(a)_k}{(c)_k}z^{k+p-1}\right|\right. \\
&\qquad \left. -p(A-B)(p-\lambda)z^{p-1}\right|
\end{aligned}$$

$$\leq \sum_{k=1}^{\infty} p(k+p)\left|a_{k+p}\right| \frac{(a)_k}{(c)_k}|z|^{k+p-1} + \alpha\left|e^{i\theta}\right| \sum_{k=1}^{\infty}(k+p)\left|a_{k+p}\right| \frac{(a)_k}{(c)_k}|z|^{k+p-1}$$
$$+ |B| \sum_{k=1}^{\infty} p(k+p)\left|a_{k+p}\right| \frac{(a)_k}{(c)_k}|z|^{k+p-1} + \alpha|B| \sum_{k=1}^{\infty}(k+p)\left|a_{k+p}\right| \frac{(a)_k}{(c)_k}|z|^{k+p-1}$$
$$- p(A-B)(p-\lambda)|z|^{p-1}$$
$$= (p+\alpha)\sum_{k=1}^{\infty}(k+p)(1-B)\left|a_{k+p}\right| \frac{(a)_k}{(c)_k} - p(A-B)(p-\lambda) \quad (z \in \partial U). \tag{2.4}$$

By the maximum modulus theorem, for any $z \in U$, we have

$$\begin{aligned}
&\left| p\left(L_p(a,c)f(z)\right)' - \alpha e^{i\theta}\left|\left(L_p(a,c)f(z)\right)' - pz^{p-1}\right| - p^2 z^{p-1} \right| \\
&\quad - \left| B\left(p\left(L_p(a,c)f(z)\right)' - \alpha e^{i\theta}\left|\left(L_p(a,c)f(z)\right)' - pz^{p-1}\right|\right) - p\left(pB + (A-B)(p-\lambda)\right)z^{p-1} \right| \\
&\leq (p+\alpha)\sum_{k=1}^{\infty}(k+p)(1-B)\left|a_{k+p}\right| \frac{(a)_k}{(c)_k} - p(A-B)(p-\lambda) \\
&= \sum_{k=1}^{\infty}\phi_k(p,\alpha;B,a,c)\left|a_{k+p}\right| - (A-B) \\
&\leq 0,
\end{aligned} \tag{2.5}$$

this implies $f(z) \in \Upsilon_{a,c}^{+}(A,B;p,\lambda,\alpha)$.

For necessary condition, let $f(z) \in \Upsilon_{a,c}^{+}(A,B;p,\lambda,\alpha)$ be given by (1.1), then

$$\frac{\left(L_p(a,c)f(z)\right)'}{z^{p-1}} = p - \sum_{k=1}^{\infty}(k+p)\left|a_{k+p}\right| \frac{(a)_k}{(c)_k} z^k, \tag{2.6}$$

from (1.13) and (2.6), we find that

$$\begin{aligned}
&\left| \frac{\left(L_p(a,c)f(z)\right)'/z^{p-1} - \alpha\left|\left(L_p(a,c)f(z)\right)'/pz^{p-1} - 1\right| - p}{B\left(\left(L_p(a,c)f(z)\right)'/z^{p-1} - \alpha\left|\left(L_p(a,c)f(z)\right)'/pz^{p-1} - 1\right|\right) - \left(pB + (A-B)(p-\lambda)\right)} \right| \\
&= \left| \frac{-\sum_{k=1}^{\infty}(k+p)\left|a_{k+p}\right|\left((a)_k/(c)_k\right)z^k - \alpha\left|-\sum_{k=1}^{\infty}\left((k+p)/p\right)\left|a_{k+p}\right|\left((a)_k/(c)_k\right)z^k\right|}{-(A-B)(p-\lambda) - B\sum_{k=1}^{\infty}(k+p)\left|a_{k+p}\right|\left((a)_k/(c)_k\right)z^k - \alpha B\left|-\sum_{k=1}^{\infty}\left((k+p)/p\right)\left|a_{k+p}\right|\left((a)_k/(c)_k\right)z^k\right|} \right| \\
&< 1.
\end{aligned} \tag{2.7}$$

Now, since $|\mathrm{Re} z| \leq |z|$ for all z, we have

$$\left|\mathrm{Re}\, \frac{-\sum_{k=1}^{\infty}(k+p)\left|a_{k+p}\right|((a)_k/(c)_k)z^k-\alpha\left|-\sum_{k=1}^{\infty}((k+p)/p)\right|a_{k+p}\left|((a)_k/(c)_k)z^k\right|}{-(A-B)(p-\lambda)-B\sum_{k=1}^{\infty}(k+p)\left|a_{k+p}\right|((a)_k/(c)_k)z^k-\alpha B\left|-\sum_{k=1}^{\infty}((k+p)/p)\right|a_{k+p}\left|((a)_k/(c)_k)z^k\right|}\right| < 1. \tag{2.8}$$

We choose value of z on the real axis so that the expression $(L_p(a,c)f(z))'/z^{p-1}$ is real.

Then, we have

$$\begin{aligned}
&\mathrm{Re}\, \frac{-\sum_{k=1}^{\infty}(k+p)\left|a_{k+p}\right|((a)_k/(c)_k)z^k-\alpha\left|-\sum_{k=1}^{\infty}((k+p)/p)\right|a_{k+p}\left|((a)_k/(c)_k)z^k\right|}{-(A-B)(p-\lambda)-B\sum_{k=1}^{\infty}(k+p)\left|a_{k+p}\right|((a)_k/(c)_k)z^k-\alpha B\left|-\sum_{k=1}^{\infty}((k+p)/p)\right|a_{k+p}\left|((a)_k/(c)_k)z^k\right|}\\
&= \frac{-\sum_{k=1}^{\infty}(k+p)\left|a_{k+p}\right|((a)_k/(c)_k)z^k-\alpha\left|-\sum_{k=1}^{\infty}((k+p)/p)\right|a_{k+p}\left|((a)_k/(c)_k)z^k\right|}{-(A-B)(p-\lambda)-B\sum_{k=1}^{\infty}(k+p)\left|a_{k+p}\right|((a)_k/(c)_k)z^k-\alpha B\left|-\sum_{k=1}^{\infty}((k+p)/p)\right|a_{k+p}\left|((a)_k/(c)_k)z^k\right|},
\end{aligned} \tag{2.9}$$

$$\begin{aligned}
&\left|\frac{-\sum_{k=1}^{\infty}(k+p)\left|a_{k+p}\right|((a)_k/(c)_k)z^k-\alpha\left|-\sum_{k=1}^{\infty}((k+p)/p)\right|a_{k+p}\left|((a)_k/(c)_k)z^k\right|}{-(A-B)(p-\lambda)-B\sum_{k=1}^{\infty}(k+p)\left|a_{k+p}\right|((a)_k/(c)_k)z^k-\alpha B\left|-\sum_{k=1}^{\infty}((k+p)/p)\right|a_{k+p}\left|((a)_k/(c)_k)z^k\right|}\right|\\
&= \left|\frac{-\sum_{k=1}^{\infty}(k+p)\left|a_{k+p}\right|((a)_k/(c)_k)z^k-\alpha\sum_{k=1}^{\infty}((k+p)/p)\left|a_{k+p}\right|((a)_k/(c)_k)|z|^k}{-(A-B)(p-\lambda)-B\sum_{k=1}^{\infty}(k+p)\left|a_{k+p}\right|((a)_k/(c)_k)z^k-\alpha B\sum_{k=1}^{\infty}((k+p)/p)\left|a_{k+p}\right|((a)_k/(c)_k)|z|^k}\right|\\
&< 1.
\end{aligned} \tag{2.10}$$

Letting $z \to 1^-$ throughout real values in (2.10), we get

$$\left|\frac{-\sum_{k=1}^{\infty}(k+p)\left|a_{k+p}\right|((a)_k/(c)_k)-\alpha\sum_{k=1}^{\infty}((k+p)/p)\left|a_{k+p}\right|((a)_k/(c)_k)}{-(A-B)(p-\lambda)-B\sum_{k=1}^{\infty}(k+p)\left|a_{k+p}\right|((a)_k/(c)_k)-\alpha B\sum_{k=1}^{\infty}((k+p)/p)\left|a_{k+p}\right|((a)_k/(c)_k)}\right| < 1. \tag{2.11}$$

So we have

$$\frac{\sum_{k=1}^{\infty}p(k+p)\left|a_{k+p}\right|((a)_k/(c)_k)+\alpha\sum_{k=1}^{\infty}(k+p)\left|a_{k+p}\right|((a)_k/(c)_k)}{(A-B)(p-\lambda)+B\sum_{k=1}^{\infty}p(k+p)\left|a_{k+p}\right|((a)_k/(c)_k)+\alpha B\sum_{k=1}^{\infty}(k+p)\left|a_{k+p}\right|((a)_k/(c)_k)} < 1, \tag{2.12}$$

it is

$$\sum_{k=1}^{\infty}(p+\alpha)(1-B)(k+p)\left|a_{k+p}\right|\frac{(a)_k}{(c)_k}\le p(A-B)(p-\lambda). \tag{2.13}$$

This completes the proof of the theorem. □

Corollary 2.2. *Let the function* $f(z)\in A^{+}(p)$ *be given by* (1.1). *If* $f(z)\in \Upsilon^{+}_{a,c}(A,B;p,\lambda,\alpha)$, *then*

$$\left|a_{k+p}\right|\le\frac{p(A-B)(p-\lambda)(c)_k}{(p+\alpha)(1-B)(k+p)(a)_k}\quad (k,p\in N). \tag{2.14}$$

The result is sharp for the function given by

$$f(z)=z^p-\sum_{k=1}^{\infty}\frac{p(A-B)(p-\lambda)(c)_k}{(p+\alpha)(1-B)(k+p)(a)_k}z^{k+p}. \tag{2.15}$$

3. Distortion Inequality of Class $\Upsilon^{+}_{a,c}(A,B;p,\lambda,\alpha)$

Theorem 3.1. *If a function* $f(z)$ *defined by* (1.1) *is in the class* $\Upsilon^{+}_{a,c}(A,B;p,\lambda,\alpha)$, *then*

$$\begin{aligned}&\left(\frac{p!}{(p-m)!}-\frac{cp(A-B)(p-\lambda)p!}{a(p+\alpha)(1-B)(p+1-m)!}|z|\right)|z|^{p-m}\le\left|f^{(m)}(z)\right|\\&\le\left(\frac{p!}{(p-m)!}+\frac{cp(A-B)(p-\lambda)p!}{a(p+\alpha)(1-B)(p+1-m)!}|z|\right)|z|^{p-m}\quad (z\in U;\ a\ge c>0;\ m\in N_0;\ p>m).\end{aligned} \tag{3.1}$$

The result is sharp for the function $f(z)$ *given by*

$$f(z)=z^p-\frac{c(A-B)(p-\lambda)}{a(1-B)(p+\alpha)}z^{1+p}\quad (p\in N). \tag{3.2}$$

Proof. Since $f(z)=z^p-\sum_{k=1}^{\infty}|a_{k+p}|z^{k+p}$, then

$$f^{(m)}(z)=\frac{p!}{(p-m)!}z^{p-m}-\sum_{k=1}^{\infty}\frac{(k+p)!}{(k+p-m)!}\left|a_{k+p}\right|z^{k+p-m}. \tag{3.3}$$

From $f(z)\in \Upsilon^{+}_{a,c}(A,B;p,\lambda,\alpha)$, using Theorem 2.1, we have

$$\frac{a(p+\alpha)(1-B)(1+p)}{cp(A-B)(p-\lambda)(p+1)!}\sum_{k=1}^{\infty}(k+p)!\left|a_{k+p}\right|\le\sum_{k=1}^{\infty}\frac{(p+\alpha)(k+p)(1-B)(a)_k}{p(A-B)(p-\lambda)(c)_k}\left|a_{k+p}\right|\le 1 \tag{3.4}$$

which readily yields

$$\sum_{k=1}^{\infty}(k+p)!\left|a_{k+p}\right| \le \frac{cp(A-B)(p-\lambda)p!}{a(p+\alpha)(1-B)}. \tag{3.5}$$

By (3.3) and (3.5), we can imply (3.1). □

4. Radius of Starlikeness and Convexity

Theorem 4.1. *Let the function* $f(z)$ *defined by* (1.1) *be in the class* $\Upsilon^{+}_{a,c}(A,B;p,\lambda,\alpha)$. *Then,* $f(z)$ *is starlike of* η $(0 \le \eta < 1)$ *in* $|z| < r_{\eta}(p,k,\alpha,A,B,a,c)$ *where*

$$r_{\eta}(p,k,\alpha,A,B,a,c) = \inf\left\{\frac{p(1-\eta)\phi_k(p,\alpha;B,a,c)}{(A-B)\left[k+p(1-\eta)\right]}\right\}^{1/k} \tag{4.1}$$

and $\phi_k(p,\alpha;B,a,c)$ *is defined by* (2.1).

Proof. We must show that

$$\left|\frac{zf'(z)}{pf(z)}-1\right| < 1-\eta \quad \text{for } |z| < r_{\eta}(p,k,\alpha,A,B,a,c). \tag{4.2}$$

Since

$$\left|\frac{zf'(z)}{pf(z)}-1\right| = \left|\frac{-\sum_{k=1}^{\infty}k\left|a_{k+p}\right|z^k}{p-\sum_{k=1}^{\infty}p\left|a_{k+p}\right|z^k}\right| \le \frac{\sum_{k=1}^{\infty}k\left|a_{k+p}\right||z|^k}{p-\sum_{k=1}^{\infty}p\left|a_{k+p}\right||z|^k}, \tag{4.3}$$

to prove (4.1), it is sufficient to prove

$$\frac{\sum_{k=1}^{\infty}k\left|a_{k+p}\right||z|^k}{p-\sum_{k=1}^{\infty}p\left|a_{k+p}\right||z|^k} < 1-\eta. \tag{4.4}$$

It is equivalent to

$$\sum_{k=1}^{\infty}\frac{\left[k+p(1-\eta)\right]\left|a_{k+p}\right|}{p(1-\eta)}|z|^k \le 1. \tag{4.5}$$

By Theorem 2.1, we have

$$\sum_{k=1}^{\infty}\frac{\phi_k(p,\alpha;B,a,c)}{A-B}\left|a_{k+p}\right| \le 1, \tag{4.6}$$

hence (4.5) will be true if

$$\frac{[k+p(1-\eta)]|a_{k+p}|}{p(1-\eta)}|z|^k \le \frac{\phi_k(p,\alpha;B,a,c)}{A-B}|a_{k+p}|. \tag{4.7}$$

It is equivalent to

$$|z| \le \left\{\frac{p(1-\eta)\phi_k(p,\alpha;B,a,c)}{(A-B)[k+p(1-\eta)]}\right\}^{1/k}. \tag{4.8}$$

This completes the proof. □

Theorem 4.2. *Let the function* $f(z)$ *defined by* (1.1) *be in the class* $\Upsilon^+_{a,c}(A,B;p,\lambda,\alpha)$. *Then,* $f(z)$ *is convex of* $\xi(0 \le \xi < 1)$ *in* $|z| < r_\xi(p,k,\alpha,A,B,a,c)$ *where*

$$r_\xi(p,k,\alpha,A,B,a,c) = \inf\left\{\frac{p^2(1-\xi)\phi_k(p,\alpha;B,a,c)}{(k+p)(A-B)[k+p(1-\xi)]}\right\}^{1/k} \tag{4.9}$$

and $\phi_k(p,\alpha;B,a,c)$ *is defined by* (2.1).

Proof. It sufficient to show that

$$\left|\frac{zf''(z)}{pf'(z)} - \frac{p-1}{p}\right| < 1-\xi \quad \text{for } |z| < r_\xi(p,k,\alpha,A,B,a,c). \tag{4.10}$$

Since

$$\left|\frac{zf''(z)}{pf'(z)} - \frac{p-1}{p}\right| = \left|\frac{-\sum_{k=1}^{\infty} k(k+p)|a_{k+p}|z^k}{p^2 - \sum_{k=1}^{\infty} p(k+p)|a_{k+p}|z^k}\right| \le \frac{\sum_{k=1}^{\infty} k(k+p)|a_{k+p}||z|^k}{p^2 - \sum_{k=1}^{\infty} p(k+p)|a_{k+p}||z|^k}, \tag{4.11}$$

to prove (4.9), it is sufficient to prove

$$\frac{\sum_{k=1}^{\infty} k(k+p)|a_{k+p}||z|^k}{p^2 - \sum_{k=1}^{\infty} p(k+p)|a_{k+p}||z|^k} < 1-\xi. \tag{4.12}$$

It is equivalent to

$$\sum_{k=1}^{\infty}\frac{(k+p)[k+p(1-\xi)]|a_{k+p}|}{p^2(1-\xi)}|z|^k \le 1. \tag{4.13}$$

By Theorem 2.1, we have

$$\sum_{k=1}^{\infty}\frac{\phi_k(p,\alpha;B,a,c)}{A-B}|a_{k+p}| \le 1, \tag{4.14}$$

hence (4.13) will be true if

$$\frac{(k+p)\left[k+p(1-\xi)\right]\left|a_{k+p}\right|}{p^2(1-\xi)}|z|^k \leq \frac{\phi_k(p,\alpha;B,a,c)}{A-B}\left|a_{k+p}\right|. \tag{4.15}$$

It is equivalent to

$$|z| \leq \left\{\frac{p^2(1-\xi)\phi_k(p,\alpha;B,a,c)}{(k+p)(A-B)\left[k+p(1-\xi)\right]}\right\}^{1/k}. \tag{4.16}$$

This completes the proof. □

5. δ-Neighborhood of $\Upsilon^{+}_{a,c}(A,B;p,\lambda,\alpha)$

Based on the earlier works by Aouf et al. [6], Altintas et al. [13–15], and Aouf [16], we introduce the δ-neighborhood of a function $f(z) \in A^{+}(p)$ of the form (1.1) and present the relationship between δ-neighborhood and corresponding function class.

Definition 5.1. For $\delta > 0$, $a > 0$, $c > 0$, the δ-neighborhood of a function $f(z) \in A^{+}(p)$ is defined as follows:

$$N^{+}_{\delta}(f) = \left\{g : g(z) = z^p - \sum_{k=1}^{\infty}\left|b_{k+p}\right|z^{k+p} \in A(p),\ \sum_{k=1}^{\infty}t_k\left|\left|b_{k+p}\right| - \left|a_{k+p}\right|\right| < \delta\right\}, \tag{5.1}$$

where

$$t_k = \frac{(p+\alpha)(1-B)(k+p)(a)_k}{p(A-B)(p-\lambda)(c)_k}. \tag{5.2}$$

Theorem 5.2. *Let the function $f(z)$ defined by* (1.1) *be in the class $\Upsilon^{+}_{a+1,c}(A,B;p,\lambda,\alpha)$. Then,*

$$N^{+}_{\delta}(f) \subset \Upsilon^{+}_{a,c}(A,B;p,\lambda,\alpha) \quad \left(\frac{1}{a+1}\right). \tag{5.3}$$

This result is the best possible in the sense that δ cannot be increased.

Proof. Let $f(z) \in \Upsilon^{+}_{a+1,c}(A,B;p,\lambda,\alpha)$, then by Theorem 2.1, we have

$$\sum_{k=1}^{\infty}\frac{(p+\alpha)(1-B)(k+p)(a+1)_k}{p(A-B)(p-\lambda)(c)_k}\left|a_{k+p}\right| \leq 1 \tag{5.4}$$

which is equivalent to

$$\sum_{k=1}^{\infty}\frac{(p+\alpha)(1-B)(k+p)(a)_k}{p(A-B)(p-\lambda)(c)_k}|a_{k+p}|\leq\frac{a}{a+1}. \tag{5.5}$$

For any

$$g(z)=z^p-\sum_{k=1}^{\infty}|b_{k+p}|z^{k+p}\in N_{\delta}^{+}(f), \tag{5.6}$$

we find from (5.1) that

$$\sum_{k=1}^{\infty}\frac{(p+\alpha)(1-B)(k+p)(a)_k}{p(A-B)(p-\lambda)(c)_k}\left||b_{k+p}|-|a_{k+p}|\right|\leq\delta. \tag{5.7}$$

By (5.5) and (5.7), we get

$$\begin{aligned}
&\sum_{k=1}^{\infty}\frac{(p+\alpha)(1-B)(k+p)(a)_k}{p(A-B)(p-\lambda)(c)_k}|b_{k+p}|\\
&\quad\leq\sum_{k=1}^{\infty}\frac{(p+\alpha)(1-B)(k+p)(a)_k}{p(A-B)(p-\lambda)(c)_k}|a_{k+p}|+\sum_{k=1}^{\infty}\frac{(p+\alpha)(1-B)(k+p)(a)_k}{p(A-B)(p-\lambda)(c)_k}\left||b_{k+p}|-|a_{k+p}|\right|\\
&\quad\leq\frac{a}{a+1}+\delta=1.
\end{aligned} \tag{5.8}$$

By Theorem 2.1, it implies that $g(z)\in\Upsilon_{a,c}^{+}(A,B;p,\lambda,\alpha)$.

To show the sharpness of the assertion of Theorem 5.2, we consider the functions $f(z)$ and $g(z)$ given by

$$\begin{aligned}
f(z)&=z^p-\frac{cp(A-B)(p-\lambda)}{(a+1)(p+\alpha)(1-B)(p+1)}z^{p+1}\in\Upsilon_{a+1,c}^{+}(A,B;p,\lambda,\alpha),\\
g(z)&=z^p-\left(\frac{cp(A-B)(p-\lambda)}{(a+1)(p+\alpha)(1-B)(p+1)}+\frac{cp(A-B)(p-\lambda)\delta'}{a(p+\alpha)(1-B)(1+p)}\right)z^{p+1},
\end{aligned} \tag{5.9}$$

where $\delta'>\delta=1/(a+1)$.

Since

$$\begin{aligned}
&\frac{(p+\alpha)(p+1)(1-B)(a)_1}{p(A-B)(p-\lambda)(c)_1}\left|\left|b_{1+p}\right|-\left|a_{1+p}\right|\right| \\
&\quad=\frac{(p+\alpha)(p+1)(1-B)(a)_1}{p(A-B)(p-\lambda)(c)_1}\cdot\frac{cp(A-B)(p-\lambda)\delta'}{a(p+\alpha)(1-B)(1+p)} \\
&\quad=\delta',
\end{aligned} \tag{5.10}$$

so $g(z)\in N^{+}_{\delta'}(f)$.

But

$$\begin{aligned}
&\frac{(p+\alpha)(p+1)(1-B)(a)_1}{p(A-B)(p-\lambda)(c)_1}\left[\frac{cp(A-B)(p-\lambda)}{(a+1)(p+\alpha)(p+1)(1-B)}+\frac{cp(A-B)(p-\lambda)\delta'}{a(p+\alpha)(1-B)(1+p)}\right] \\
&\quad=\frac{a}{a+1}+\delta' \\
&\quad>1,
\end{aligned} \tag{5.11}$$

by Theorem 2.1 $g(z)\overline{\in}\Upsilon^{+}_{a,c}(A,B;p,\lambda,\alpha)$. □

6. Properties Associated with Modified Hadamard Product

Following early works by Aouf et al. in [6], we provide the properties of modified Hadamard product of $\Upsilon^{+}_{a,c}(A,B;p,\lambda,\alpha)$.

For the function

$$f_j(z)=z^p-\sum_{k=1}^{\infty}\left|a_{k+p,j}\right|z^{k+p}\quad(j=1,2;\ p\in N), \tag{6.1}$$

the modified Hadamard product of the functions $f_1(z)$ and $f_2(z)$ was denoted by $(f_1\cdot f_2)(z)$ and defined as follows:

$$(f_1\cdot f_2)(z)=z^p-\sum_{k=1}^{\infty}\left|a_{k+p,1}\right|\left|a_{k+p,2}\right|z^{k+p}=(f_2\cdot f_1)(z). \tag{6.2}$$

Theorem 6.1. *Let $f_j(z)$ $(j=1,2)$ given by (6.1) be in the class $\Upsilon^{+}_{a,c}(A,B;p,\lambda,\alpha)$, then*

$$(f_1\cdot f_2)(z)\in\Upsilon^{+}_{a,c}(A,B;p,\gamma,\alpha), \tag{6.3}$$

where

$$\gamma:=p-\frac{cp(A-B)(p-\lambda)^2}{a(p+\alpha)(1-B)(1+p)}. \tag{6.4}$$

The result is sharp for the functions $f_j(z)$ $(j = 1, 2)$ *given by*

$$f_j(z) = z^p - \frac{cp(A-B)(p-\lambda)}{a(p+\alpha)(1-B)(1+p)} z^{p+1}. \tag{6.5}$$

Proof. By Theorem 2.1, we need to find the largest γ such that

$$\sum_{k=1}^{\infty} \frac{(p+\alpha)(1-B)(k+p)(a)_k}{p(A-B)(p-\gamma)(c)_k} \left|a_{k+p,1}\right| \cdot \left|a_{k+p,2}\right| \leq 1. \tag{6.6}$$

Since $f_j(z) \in \Upsilon^{+}_{a,c}(A, B; p, \lambda, \alpha)$, $(j = 1, 2)$, then we see that

$$\sum_{k=1}^{\infty} \frac{(p+\alpha)(1-B)(k+p)(a)_k}{p(A-B)(p-\lambda)(c)_k} \left|a_{k+p,j}\right| \leq 1 \quad (j = 1, 2). \tag{6.7}$$

By Cauchy-Schwartz inequality, we obtain

$$\sum_{k=1}^{\infty} \frac{(p+\alpha)(1-B)(k+p)(a)_k}{p(A-B)(p-\lambda)(c)_k} \sqrt{\left|a_{k+p,1}\right| \cdot \left|a_{k+p,2}\right|} \leq 1. \tag{6.8}$$

This implies that we only need to show that

$$\frac{\left|a_{k+p,1}\right| \cdot \left|a_{k+p,2}\right|}{p-\gamma} \leq \frac{\sqrt{\left|a_{k+p,1}\right| \cdot \left|a_{k+p,2}\right|}}{p-\lambda} \quad (k \in N) \tag{6.9}$$

or equivalently that

$$\sqrt{\left|a_{k+p,1}\right| \cdot \left|a_{k+p,2}\right|} \leq \frac{p-\gamma}{p-\lambda}. \tag{6.10}$$

By making use of the inequality (6.8), it is sufficient to prove that

$$\frac{p(A-B)(p-\lambda)(c)_k}{(p+\alpha)(1-B)(k+p)(a)_k} \leq \frac{p-\gamma}{p-\lambda} \quad (k \in N). \tag{6.11}$$

From (6.11), we have

$$\gamma \leq p - \frac{p(A-B)(p-\lambda)^2(c)_k}{(p+\alpha)(1-B)(k+p)(a)_k} \quad (k \in N). \tag{6.12}$$

Define the function $\Phi(k)$ by

$$\Phi(k) := p - \frac{p(A-B)(p-\lambda)^2(c)_k}{(p+\alpha)(1-B)(k+p)(a)_k} \quad (k \in N). \tag{6.13}$$

We see that $\Phi(k)$ is an increasing function of k. Therefore, we conclude that

$$\gamma \le \Phi(1) = p - \frac{p(A-B)(p-\lambda)^2(c)_1}{(p+\alpha)(1-B)(k+p)(a)_1}. \tag{6.14}$$

By using arguments similar to those in the proof of Theorem 6.1, we can derive the following result. □

Theorem 6.2. *Let $f_1(z)$ defined by* (6.1) *be in the class $\Upsilon^+_{a,c}(A,B;p,\lambda,\alpha)$, $f_2(z)$ defined by* (6.1) *be in the class $\Upsilon^+_{a,c}(A,B;p,\gamma,\alpha)$, then*

$$(f_1 \cdot f_2)(z) \in \Upsilon^+_{a,c}(A,B;p,\xi,\alpha), \tag{6.15}$$

where

$$\xi := p - \frac{cp(A-B)(p-\lambda)(p-\gamma)}{a(p+\alpha)(1-B)(1+p)}. \tag{6.16}$$

The result is sharp for the functions $f_j(z)$ $(j=1,2)$ given by

$$f_1(z) = z^p - \frac{cp(A-B)(p-\lambda)}{a(p+\alpha)(1-B)(1+p)} z^{p+1} \quad (p \in N),$$
$$f_2(z) = z^p - \frac{cp(A-B)(p-\gamma)}{a(p+\alpha)(1-B)(1+p)} z^{p+1} \quad (p \in N). \tag{6.17}$$

Theorem 6.3. *Let $f_j(z)$ $(j=1,2)$ defined by* (6.1) *be in the class $\Upsilon^+_{a,c}(A,B;p,\lambda,\alpha)$, then the function $h(z)$ defined by*

$$h(z) = z^p - \sum_{k=1}^{\infty} \left(|a_{k+p,1}|^2 + |a_{k+p,2}|^2 \right) \tag{6.18}$$

belongs to the class $\Upsilon^+_{a,c}(A,B;p,\chi,\alpha)$ where

$$\chi := p - \frac{2pc(A-B)(p-\lambda)}{a(p+\alpha)(1-B)(1+p)}. \tag{6.19}$$

This result is sharp for the functions given by (6.5).

Proof. By Theorem 2.1, we want to find the largest χ such that

$$\sum_{k=1}^{\infty}\frac{(p+\alpha)(1-B)(k+p)(a)_k}{p(A-B)(p-\chi)(c)_k}\left(\left|a_{k+p,1}\right|^2+\left|a_{k+p,2}\right|^2\right)\le 1. \tag{6.20}$$

Since $f_j(z)\in \Upsilon^{+}_{a,c}(A,B;p,\lambda,\alpha)$ $(j=1,2)$, we readily see that

$$\sum_{k=1}^{\infty}\frac{(p+\alpha)(1-B)(k+p)(a)_k}{p(A-B)(p-\chi)(c)_k}\left|a_{k+p,j}\right|\le 1 \quad (j=1,2). \tag{6.21}$$

From (6.21), we have

$$\sum_{k=1}^{\infty}\frac{(p+\alpha)^2(1-B)^2(k+p)^2}{p^2(A-B)^2(p-\chi)^2}\left(\frac{(a)_k}{(c)_k}\right)^2\left|a_{k+p,j}\right|^2\le 1 \quad (j=1,2), \tag{6.22}$$

then we have

$$\sum_{k=1}^{\infty}\frac{(p+\alpha)^2(1-B)^2(k+p)^2}{p^2(A-B)^2(p-\chi)^2}\left(\frac{(a)_k}{(c)_k}\right)^2\left(\left|a_{k+p,1}\right|^2+\left|a_{k+p,2}\right|^2\right)\le 2. \tag{6.23}$$

From (6.23), if we want to prove (6.20), it is sufficient to prove there exists the largest χ such that

$$\frac{1}{p-\chi}\le\frac{(p+\alpha)(k+p)(1-B)(a)_k}{2p(A-B)(p-\lambda)^2(c)_k} \quad (k\in N), \tag{6.24}$$

that is

$$\chi\le p-\frac{2p(A-B)(p-\lambda)^2(c)_k}{(p+\alpha)(k+p)(1-B)(a)_k} \quad (k\in N). \tag{6.25}$$

Now we define $\Psi(k)$ by

$$\Psi(k)=p-\frac{2p(A-B)(p-\lambda)^2(c)_k}{(p+\alpha)(k+p)(1-B)(a)_k} \quad (k\in N). \tag{6.26}$$

We see that $\Psi(k)$ is an increasing function of k. Therefore, we conclude that

$$\chi\le\Psi(1)=p-\frac{2pc(A-B)(p-\lambda)^2}{a(p+\alpha)(k+p)(1-B)} \tag{6.27}$$

which completes the proof of Theorem 6.1. □

7. Application of Fractional Calculus Operator

References [17–19] have studied the fractional calculus operators extensively. In this part, we only investigate the application of fractional calculus operator which was defined by [6] in the class of $\Upsilon^{+}_{a,c}(A,B;p,\lambda,\alpha)$.

Definition 7.1 (see [6]). The fractional integral of order μ is defined, for a function $f(z)$, by

$$D_z^{-\mu} f(z) = \frac{1}{\Gamma(\mu)} \int_0^z \frac{f(z)}{(z-\xi)^{1-\mu}} d\xi \quad (\mu > 0), \tag{7.1}$$

where the function $f(z)$ is analytic in a simply connected domain of the complex z-plane containing the origin and the multiplicity of $(z-\xi)^{\mu-1}$ is removed by requiring $\log(z-\xi)$ to be real when $z-\xi > 0$.

Definition 7.2 (see [6]). The fractional integral of order μ is defined, for a function $f(z)$, by

$$D_z^{\mu} f(z) = \frac{1}{\Gamma(1-\mu)} \int_0^z \frac{f(z)}{(z-\xi)^{\mu}} d\xi \quad (0 \le \mu < 1), \tag{7.2}$$

where the function $f(z)$ is constrained, the multiplicity of $(z-\xi)^{-\mu}$ is removed as in Definition 7.1.

In our investigation, we will use the operators $J_{\delta,p}$ defined by (cf, [20–22])

$$(J_{\delta,p} f)(z) := \frac{\delta+p}{z^p} \int_0^z t^{\delta-1} f(t) dt \quad (f \in A(p);\ \delta > -p;\ p \in N) \tag{7.3}$$

as well as D_z^{μ} for which it is well known that (see [23])

$$D_z^{\mu} z^{\rho} = \frac{\Gamma(\rho+1)}{\Gamma(\rho-\mu+1)} z^{\rho-\mu} \quad (\rho > -1;\ \mu \in R) \tag{7.4}$$

in terms of Gamma.

Lemma 7.3 (see Chen et al. [18]). *For a function $f(z) \in A(p)$,*

$$\begin{aligned} D_z^{\mu}\{(J_{\delta,p} f)(z)\} &= \frac{\Gamma(p+1)}{\Gamma(p-\mu+1)} z^{p-\mu} + \sum_{k=1}^{\infty} \frac{(\delta+p)\Gamma(k+p+1)}{(\delta+k+p)\Gamma(k+p-\mu+1)} a_{k+p} z^{k+p-\mu}, \\ J_{\delta,p}\left(D_z^{\mu}\{f(z)\}\right) &= \frac{(\delta+p)\Gamma(p+1)}{(\delta+p-\mu)\Gamma(p-\mu+1)} z^{p-\mu} \\ &\quad + \sum_{k=1}^{\infty} \frac{(\delta+p)\Gamma(k+p+1)}{(\delta+k+p-\mu)\Gamma(k+p-\mu+1)} a_{k+p} z^{k+p-\mu} \end{aligned} \tag{7.5}$$

$(\mu \in R;\ \delta > -p;\ p \in N)$ *provided that no zeros appear in the denominators in* (7.5).

Remark 7.4. Throughout this section, we assume further that $a \geq c > 0$.

Theorem 7.5. *Let function defined by* (1.1) *be in the class* $\Upsilon^{+}_{a,c}(A,B;p,\lambda,\alpha)$*, then*

$$\left|D_z^{-\mu}\{(J_{\delta,p}f)(z)\}\right| \geq \frac{\Gamma(p+1)}{\Gamma(p+\mu+1)}|z|^{p+\mu}\left(1-\frac{cp(\delta+p)(A-B)(p-\lambda)}{a(p+\alpha)(p+\delta+1)(p+\mu+1)(1-B)}|z|\right), \tag{7.6}$$

$$\left|D_z^{-\mu}\{(J_{\delta,p}f)(z)\}\right| \leq \frac{\Gamma(p+1)}{\Gamma(p+\mu+1)}|z|^{p+\mu}\left(1+\frac{cp(\delta+p)(A-B)(p-\lambda)}{a(p+\alpha)(p+\delta+1)(p+\mu+1)(1-B)}|z|\right) \tag{7.7}$$

$\mu > 0;\ \delta > -p;\ p \in N;\ z \in U$*, each of the assertions is sharp.*

Proof. Since

$$D_z^{-\mu}\{(J_{\delta,p}f)(z)\} = \frac{\Gamma(p+1)}{\Gamma(p-\mu+1)}z^{p+\mu} - \sum_{k=1}^{\infty}\frac{(\delta+p)\Gamma(k+p+1)}{(\delta+k+p)\Gamma(k+p+\mu+1)}\left|a_{k+p}\right|z^{k+p+\mu}, \tag{7.8}$$

then

$$\left|D_z^{-\mu}\{(J_{\delta,p}f)(z)\}\right| = \frac{\Gamma(p+1)}{\Gamma(p-\mu+1)}|z^{\mu}|\left|z^{p} - \sum_{k=1}^{\infty}\frac{(\delta+p)\Gamma(k+p+1)\Gamma(p+\mu+1)}{(\delta+k+p)\Gamma(k+p+\mu+1)\Gamma(p+1)}\left|a_{k+p}\right|z^{k+p}\right|. \tag{7.9}$$

Let $G(z)$ is defined by

$$G(z) = z^{p} - \sum_{k=1}^{\infty}\frac{(\delta+p)\Gamma(k+p+1)\Gamma(p+\mu+1)}{(\delta+k+p)\Gamma(k+p+\mu+1)\Gamma(p+1)}\left|a_{k+p}\right|z^{k+p} = z^{p} - \sum_{k=1}^{\infty}Q(k)\left|a_{k+p}\right|z^{k+p}, \tag{7.10}$$

where

$$Q(k) = \frac{(\delta+p)\Gamma(k+p+1)\Gamma(p+\mu+1)}{(\delta+k+p)\Gamma(k+p+\mu+1)\Gamma(p+1)} \quad (k,p \in N;\ \mu > 0). \tag{7.11}$$

Since $Q(k)$ is a decreasing function of k when $\mu > 0$, we get

$$0 < Q(k) \leq Q(1) = \frac{(\delta+p)(p+1)}{(\delta+k+p)(1+p+\mu)}, \tag{7.12}$$

since $f(z) \in \Upsilon^{+}_{a,c}(A,B;p,\lambda,\alpha)$, by Theorem 2.1, we get

$$\frac{a(p+\alpha)(1-B)(p+1)}{cp(A-B)(p-\lambda)}\sum_{k=1}^{\infty}\left|a_{k+p}\right| \leq \sum_{k=1}^{\infty}\frac{(p+\alpha)(k+p)(1-B)(a)_k}{p(A-B)(p-\lambda)(c)_k}\left|a_{k+p}\right| \leq 1. \tag{7.13}$$

It is that

$$\sum_{k=1}^{\infty}\left|a_{k+p}\right| \leq \frac{cp(A-B)(p-\lambda)}{a(\alpha+p)(1-B)(p+1)}, \tag{7.14}$$

then

$$\begin{aligned}
|G(z)| &= \left|z^p - \sum_{k=1}^{\infty}Q(k)\left|a_{k+p}\right|z^{k+p}\right| \geq |z|^p - Q(1)|z|^{p+1}\sum_{k=1}^{\infty}\left|a_{k+p}\right| \\
&\geq |z|^p - \frac{cp(p+\delta)(A-B)(p-\lambda)}{a(p+\alpha)(\delta+k+p)(p+1+\mu)(1-B)}|z|^{p+1}, \\
|G(z)| &= \left|z^p - \sum_{k=1}^{\infty}Q(k)\left|a_{k+p}\right|z^{k+p}\right| \leq |z|^p + Q(1)|z|^{p+1}\sum_{k=1}^{\infty}\left|a_{k+p}\right| \\
&\leq |z|^p + \frac{cp(p+\delta)(A-B)(p-\lambda)}{a(p+\alpha)(\delta+k+p)(p+1+\mu)(1-B)}|z|^{p+1}.
\end{aligned} \tag{7.15}$$

From (7.15), we obtain (7.6) and (7.7), respectively.

Equalities in (7.6) and (7.7) are attained for the function $f(z)$ given by

$$D_z^{-\mu}\{(J_{\delta,p}f)(z)\} = \frac{\Gamma(p+1)}{\Gamma(p-\mu+1)}z^{p+\mu}\left(1 - \frac{cp(\delta+p)(A-B)(p-\lambda)}{a(p+\alpha)(\delta+k+p)(p+\mu+1)(1-B)}z\right). \tag{7.16}$$

□

Theorem 7.6. *Let function defined by* (1.1) *be in the class* $\Upsilon^{+}_{a,c}(A,B;p,\lambda,\alpha)$, *then*

$$\left|D_z^{\mu}\{(J_{\delta,p}f)(z)\}\right| \geq \frac{\Gamma(p+1)}{\Gamma(p-\mu+1)}|z|^{p-\mu}\left(1 - \frac{cp(\delta+p)(A-B)(p-\lambda)}{a(p+\alpha)(p+\delta+1)(p-\mu+1)(1-B)}|z|\right), \tag{7.17}$$

$$\left|D_z^{\mu}\{(J_{\delta,p}f)(z)\}\right| \leq \frac{\Gamma(p+1)}{\Gamma(p-\mu+1)}|z|^{p-\mu}\left(1 + \frac{cp(\delta+p)(A-B)(p-\lambda)}{a(p+\alpha)(p+\delta+1)(p-\mu+1)(1-B)}|z|\right), \tag{7.18}$$

$\mu > 0;\ \delta > -p;\ p \in N;\ z \in U$, *each of the assertions is sharp.*

Proof. Since

$$D_z^{\mu}\{(J_{\delta,p}f)(z)\} = \frac{\Gamma(p+1)}{\Gamma(p-\mu+1)}z^{p-\mu} - \sum_{k=1}^{\infty}\frac{(\delta+p)\Gamma(k+p+1)}{(\delta+k+p)\Gamma(k+p-\mu+1)}|a_{k+p}|z^{k+p-\mu}, \tag{7.19}$$

then

$$\left|D_z^{\mu}\{(J_{\delta,p}f)(z)\}\right| = \frac{\Gamma(p+1)}{\Gamma(p-\mu+1)}|z^{-\mu}|\left|z^p - \sum_{k=1}^{\infty}\frac{(\delta+p)\Gamma(k+p+1)\Gamma(p-\mu+1)}{(\delta+k+p)\Gamma(k+p-\mu+1)\Gamma(p+1)}|a_{k+p}|z^{k+p}\right|. \tag{7.20}$$

Let $H(z)$ be defined by

$$\begin{aligned} H(z) &= z^p - \sum_{k=1}^{\infty}\frac{(\delta+p)\Gamma(k+p+1)\Gamma(p-\mu+1)}{(\delta+k+p)\Gamma(k+p-\mu+1)\Gamma(p+1)}|a_{k+p}|z^{k+p} \\ &= z^p - \sum_{k=1}^{\infty}E(k)(k+p)|a_{k+p}|z^{k+p}, \end{aligned} \tag{7.21}$$

where

$$E(k) = \frac{(\delta+p)\Gamma(k+p)\Gamma(p-\mu+1)}{(\delta+k+p)\Gamma(k+p-\mu+1)\Gamma(p+1)} \quad (k,p \in N;\ \mu > 0). \tag{7.22}$$

Since $E(k)$ is a decreasing function of k when $0 \le \mu < 1$, then

$$0 < E(k) \le E(1) = \frac{(\delta+p)}{(\delta+1+p)(1+p-\mu)}, \tag{7.23}$$

since $f(z) \in \Upsilon_{a,c}^{+}(A,B;p,\lambda,\alpha)$, by Theorem 2.1, we get

$$\frac{a(p+\alpha)(1-B)}{cp(A-B)(p-\lambda)}\sum_{k=1}^{\infty}(k+p)|a_{k+p}| \le \sum_{k=1}^{\infty}\frac{(p+\alpha)(k+p)(1-B)(a)_k}{p(A-B)(p-\lambda)(c)_k}|a_{k+p}| \le 1. \tag{7.24}$$

It is that

$$\sum_{k=1}^{\infty}(k+p)|a_{k+p}| \le \frac{cp(A-B)(p-\lambda)}{a(\alpha+p)(1-B)}, \tag{7.25}$$

then

$$
\begin{aligned}
|H(z)| &= \left| z^p - \sum_{k=1}^{\infty} E(k)(k+p)\left|a_{k+p}\right| z^{k+p} \right| \geq |z|^p - E(1)|z|^{p+1} \sum_{k=1}^{\infty}(k+p)\left|a_{k+p}\right| \\
&\geq |z|^p - \frac{cp(p+\delta)(A-B)(p-\lambda)}{a(p+\alpha)(\delta+k+p)(p+1-\mu)(1-B)}|z|^{p+1}, \\
|H(z)| &= \left| z^p - \sum_{k=1}^{\infty} E(k)(k+p)\left|a_{k+p}\right| z^{k+p} \right| \leq |z|^p + E(1)|z|^{p+1} \sum_{k=1}^{\infty}(k+p)\left|a_{k+p}\right| \\
&\leq |z|^p + \frac{cp(p+\delta)(A-B)(p-\lambda)}{a(p+\alpha)(\delta+k+p)(p+1-\mu)(1-B)}|z|^{p+1}.
\end{aligned} \tag{7.26}
$$

From (7.26), we obtain (7.17) and (7.18), respectively.

Equalities in (7.17) and (7.18) are attained for the function $f(z)$ given by

$$
D_z^{\mu}\{(J_{\delta,p}f)(z)\} = \frac{\Gamma(p+1)}{\Gamma(p-\mu+1)} z^{p-\mu}\left(1 - \frac{cp(\delta+p)(A-B)(p-\lambda)}{a(p+\alpha)(\delta+k+p)(p-\mu+1)(1-B)}z\right) \tag{7.27}
$$

or equivalently by

$$
(J_{\delta,p}f)(z) = z^p - \frac{cp(\delta+p)(A-B)(p-\lambda)}{a(p+\alpha)(\delta+k+p)(p-\mu+1)(1-B)} z^{p+1}. \tag{7.28}
$$

□

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References

[1] M. K. Aouf, H. M. Hossen, and H. M. Srivastava, "Some families of multivalent functions," *Computers & Mathematics with Applications*, vol. 39, no. 7-8, pp. 39–48, 2000.
[2] S. Owa, "Some properties of certain multivalent functions," *Applied Mathematics Letters*, vol. 4, no. 5, pp. 79–83, 1991.
[3] H. Saitoh, "A linear operator and its applications of first order differential subordinations," *Mathematica Japonica*, vol. 44, no. 1, pp. 31–38, 1996.
[4] H. M. Srivastava and J. Patel, "Some subclasses of multivalent functions involving a certain linear operator," *Journal of Mathematical Analysis and Applications*, vol. 310, no. 1, pp. 209–228, 2005.
[5] V. Kumar and S. L. Shukla, "Multivalent functions defined by Ruscheweyh derivatives. I, II," *Indian Journal of Pure and Applied Mathematics*, vol. 15, no. 11, pp. 1216–1238, 1984.
[6] M. K. Aouf, H. Silverman, and H. M. Srivastava, "Some families of linear operators associated with certain subclasses of multivalent functions," *Computers & Mathematics with Applications*, vol. 55, no. 3, pp. 535–549, 2008.

[7] J. Sokół, “Classes of multivalent functions associated with a convolution operator,” *Computers & Mathematics with Applications*, vol. 60, no. 5, pp. 1343–1350, 2010.
[8] M. K. Auof and H. E. Darwish, “Some classes of multivalent functions with negative coefficients. I,” *Honam Mathematical Journal*, vol. 16, no. 1, pp. 119–135, 1994.
[9] S. L. Shukla and Dashrath, “On certain classes of multivalent functions with negative coefficients,” *Soochow Journal of Mathematics*, vol. 8, pp. 179–188, 1982.
[10] S. K. Lee, S. Owa, and H. M. Srivastava, “Basic properties and characterizations of a certain class of analytic functions with negative coefficients,” *Utilitas Mathematica*, vol. 36, pp. 121–128, 1989.
[11] V. P. Gupta and P. K. Jain, “Certain classes of univalent functions with negative coefficients. II,” *Bulletin of the Australian Mathematical Society*, vol. 15, no. 3, pp. 467–473, 1976.
[12] B. A. Uralegaddi and S. M. Sarangi, “Some classes of univalent functions with negative coefficients,” *Analele Ştiinţifice ale Universităţii “Al. I. Cuza” din Iaşi*, vol. 34, no. 1, pp. 7–11, 1988.
[13] O. Altintas and S. Owa, “Neighborhoods of certain analytic functions with negative coefficients,” *International Journal of Mathematics and Mathematical Sciences*, vol. 19, no. 4, pp. 797–800, 1996.
[14] O. Altintaş, Ö. Özkan, and H. M. Srivastava, “Neighborhoods of a class of analytic functions with negative coefficients,” *Applied Mathematics Letters*, vol. 13, no. 3, pp. 63–67, 2000.
[15] O. Altintaş, Ö. Özkan, and H. M. Srivastava, “Neighborhoods of a certain family of multivalent functions with negative coefficients,” *Computers & Mathematics with Applications*, vol. 47, no. 10-11, pp. 1667–1672, 2004.
[16] M. K. Aouf, “Neighborhoods of certain classes of analytic functions with negative coefficients,” *International Journal of Mathematics and Mathematical Sciences*, vol. 2006, Article ID 38258, 6 pages, 2006.
[17] *Univalent Functions, Fractional Calculus, and Their Applications*, Ellis Horwood Series: Mathematics, and Its Applications, Horwood, Chichester, UK.
[18] M.-P. Chen, H. Irmak, and H. M. Srivastava, “Some families of multivalently analytic functions with negative coefficients,” *Journal of Mathematical Analysis and Applications*, vol. 214, no. 2, pp. 674–690, 1997.
[19] H. M. Srivastava and M. K. Aouf, “A certain fractional derivative operator and its applications to a new class of analytic and multivalent functions with negative coefficients. II,” *Journal of Mathematical Analysis and Applications*, vol. 192, no. 3, pp. 673–688, 1995.
[20] R. J. Libera, “Some classes of regular univalent functions,” *Proceedings of the American Mathematical Society*, vol. 16, pp. 755–758, 1965.
[21] A. E. Livingston, “On the radius of univalence of certain analytic functions,” *Proceedings of the American Mathematical Society*, vol. 17, pp. 352–357, 1966.
[22] H. M. Srivastava and S. Owa, Eds., *Current Topics in Analytic Function Theory*, World Scientific Publishing, River Edge, NJ, USA, 1992.
[23] S. Owa, “On distortion theorem, I,” *Kyungpook Mathematical Journal*, vol. 18, pp. 55–59, 1978.