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Research Article

On *GCR*-Lightlike Product of Indefinite Cosymplectic Manifolds

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We define *GCR*-lightlike submanifolds of indefinite cosymplectic manifolds and give an example. Then, we study mixed geodesic *GCR*-lightlike submanifolds of indefinite cosymplectic manifolds and obtain some characterization theorems for a *GCR*-lightlike submanifold to be a *GCR*-lightlike product.

1. Introduction

To fill the gaps in the general theory of submanifolds, Duggal and Bejancu [1] introduced lightlike (degenerate) geometry of submanifolds. Since the geometry of *CR*-submanifolds has potential for applications in mathematical physics, particularly in general relativity, and the geometry of lightlike submanifolds has extensive uses in mathematical physics and relativity, Duggal and Bejancu [1] clubbed these two topics and introduced the theory of *CR*-lightlike submanifolds of indefinite Kaehler manifolds and then Duggal and Sahin [2], introduced the theory of *CR*-lightlike submanifolds of indefinite Sasakian manifolds, which were further studied by Kumar et al. [3]. But *CR*-lightlike submanifolds do not include the complex and real subcases contrary to the classical theory of *CR*-submanifolds [4]. Thus, later on, Duggal and Sahin [5] introduced a new class of submanifolds, generalized-Cauchy-Riemann-(*GCR*-) lightlike submanifolds of indefinite Kaehler manifolds and then of indefinite Sasakian manifolds in [6]. This class of submanifolds acts as an umbrella of invariant, screen real, contact *CR*-lightlike subcases and real hypersurfaces. Therefore, the study of *GCR*-lightlike submanifolds is the topic of main discussion in the present scenario. In [7], the present authors studied totally contact umbilical GCR-lightlike submanifolds of indefinite Sasakian manifolds.

In present paper, after defining GCR-lightlike submanifolds of indefinite cosymplectic manifolds, we study mixed geodesic GCR-lightlike submanifolds of indefinite cosymplectic manifolds. In [8, 9], Kumar et al. obtained some necessary and sufficient conditions for a GCR-lightlike submanifold of indefinite Kaehler and Sasakian manifolds to be a GCR-lightlike product, respectively. Thus, in this paper, we obtain some characterization theorems for a GCR-lightlike submanifold of indefinite cosymplectic manifold to be a GCR-lightlike product.

2. Lightlike Submanifolds

Let V be a real m-dimensional vector space with a symmetric bilinear mapping $g : V \times V \to \Re$. The mapping g is called degenerate on V if there exists a vector $\xi \neq 0$ of V such that

$$g(\xi, v) = 0, \quad \forall v \in V, \tag{2.1}$$

otherwise g is called nondegenerate. It is important to note that a non-degenerate symmetric bilinear form on V may induce either a non-degenerate or a degenerate symmetric bilinear form on a subspace of V. Let W be a subspace of V and $g \mid w$ degenerate; then W is called a degenerate (lightlike) subspace of V.

Let $(\overline{M}, \overline{g})$ be a real $(m+n)$-dimensional semi-Riemannian manifold of constant index q such that $m, n \geq 1$, $1 \leq q \leq m+n-1$, and let (M, g) be an m-dimensional submanifold of $\overline{M}$ and g the induced metric of $\overline{g}$ on M. Thus, if $\overline{g}$ is degenerate on the tangent bundle TM of M, then M is called a lightlike (degenerate) submanifold of $\overline{M}$ (for detail see [1]). For a degenerate metric g on M, $TM^{\perp}$ is also a degenerate n-dimensional subspace of $T_x\overline{M}$. Thus, both T_xM and $T_xM^{\perp}$ are degenerate orthogonal subspaces but no longer complementary. In this case, there exists a subspace $\operatorname{Rad} T_xM = T_xM \cap T_xM^{\perp}$, which is known as radical (null) subspace. If the mapping $\operatorname{Rad} TM : x \in M \to \operatorname{Rad} T_xM$ defines a smooth distribution on M of rank $r > 0$, then the submanifold M of $\overline{M}$ is called an r-lightlike submanifold and $\operatorname{Rad} TM$ is called the radical distribution on M. Then, there exists a non-degenerate screen distribution $S(TM)$ which is a complementary vector subbundle to $\operatorname{Rad} TM$ in TM. Therefore,

$$TM = \operatorname{Rad} TM \perp S(TM), \tag{2.2}$$

where $\perp$ denotes orthogonal direct sum. Let $S(TM^{\perp})$, called screen transversal vector bundle, be a non-degenerate complementary vector subbundle to $\operatorname{Rad} TM$ in $TM^{\perp}$. Let $\operatorname{tr}(TM)$ and $\operatorname{ltr}(TM)$ be complementary (but not orthogonal) vector bundles to TM in $T\overline{M}|_M$ and to $\operatorname{Rad} TM$ in $S(TM^{\perp})^{\perp}$, called transversal vector bundle and lightlike transversal vector bundle of M, respectively. Then, we have

$$\operatorname{tr}(TM) = \operatorname{ltr}(TM) \perp S\left(TM^{\perp}\right), \tag{2.3}$$

$$T\overline{M}|_M = TM \oplus \operatorname{tr}(TM) = (\operatorname{Rad} TM \oplus \operatorname{ltr}(TM)) \perp S(TM) \perp S\left(TM^{\perp}\right). \tag{2.4}$$

Let u be a local coordinate neighborhood of M and consider the local quasiorthonormal fields of frames of $\overline{M}$ along M on u as $\{\xi_1,\ldots,\xi_r,W_{r+1},\ldots,W_n,N_1,\ldots,N_r,X_{r+1},\ldots,X_m\}$, where $\{\xi_1,\ldots,\xi_r\}$ and $\{N_1,\ldots,N_r\}$ are local lightlike bases of $\Gamma(\mathrm{Rad}\,TM|_u)$ and $\Gamma(\mathrm{ltr}(TM)|_u)$ and $\{W_{r+1},\ldots,W_n\}$ and $\{X_{r+1},\ldots,X_m\}$ are local orthonormal bases of $\Gamma(S(TM^{\perp})|_u)$ and $\Gamma(S(TM)|_u)$, respectively. For these quasiorthonormal fields of frames, we have the following theorem.

Theorem 2.1 (see [1]). *Let $(M,g,S(TM),S(TM^{\perp}))$ be an r-lightlike submanifold of a semi-Riemannian manifold $(\overline{M},\overline{g})$. Then there, exist a complementary vector bundle* ltr*(TM) of* Rad *TM in $S(TM^{\perp})^{\perp}$ and a basis of $\Gamma(\mathrm{ltr}\,(TM)|_u)$ consisting of smooth section $\{N_i\}$ of $S(TM^{\perp})^{\perp}|_u$, where u is a coordinate neighborhood of M, such that*

$$\overline{g}(N_i,\xi_j)=\delta_{ij},\quad \overline{g}(N_i,N_j)=0,\quad \textit{for any } i,j\in\{1,2,\ldots,r\},\tag{2.5}$$

where $\{\xi_1,\ldots,\xi_r\}$ is a lightlike basis of $\Gamma(\mathrm{Rad}\,(TM))$.

Let $\overline{\nabla}$ be the Levi-Civita connection on $\overline{M}$. Then, according to decomposition (2.4), the Gauss and Weingarten formulas are given by

$$\overline{\nabla}_XY=\nabla_XY+h(X,Y),\qquad \overline{\nabla}_XU=-A_UX+\nabla^{\perp}_XU,\tag{2.6}$$

for any $X,Y\in\Gamma(TM)$ and $U\in\Gamma(\mathrm{tr}(TM))$, where $\{\nabla_XY,A_UX\}$ and $\{h(X,Y),\nabla^{\perp}_XU\}$ belong to $\Gamma(TM)$ and $\Gamma(\mathrm{tr}(TM))$, respectively. Here ∇ is a torsion-free linear connection on M, h is a symmetric bilinear form on $\Gamma(TM)$ that is called second fundamental form, and A_U is a linear operator on M, known as shape operator.

According to (2.3), considering the projection morphisms L and S of $\mathrm{tr}(TM)$ on $\mathrm{ltr}(TM)$ and $S(TM^{\perp})$, respectively, then (2.6) gives

$$\overline{\nabla}_XY=\nabla_XY+h^l(X,Y)+h^s(X,Y),\qquad \overline{\nabla}_XU=-A_UX+D^l_XU+D^s_XU,\tag{2.7}$$

where we put $h^l(X,Y)=L(h(X,Y))$, $h^s(X,Y)=S(h(X,Y))$, $D^l_XU=L(\nabla^{\perp}_XU)$, $D^s_XU=S(\nabla^{\perp}_XU)$.

As h^l and h^s are $\Gamma(\mathrm{ltr}(TM))$-valued and $\Gamma(S(TM^{\perp}))$-valued, respectively, they are called the lightlike second fundamental form and the screen second fundamental form on M. In particular,

$$\overline{\nabla}_XN=-A_NX+\nabla^l_XN+D^s(X,N),\qquad \overline{\nabla}_XW=-A_WX+\nabla^s_XW+D^l(X,W),\tag{2.8}$$

where $X\in\Gamma(TM)$, $N\in\Gamma(\mathrm{ltr}(TM))$, and $W\in\Gamma(S(TM^{\perp}))$. By using (2.3)-(2.4) and (2.7)-(2.8), we obtain

$$\overline{g}(h^s(X,Y),W)+\overline{g}\Big(Y,D^l(X,W)\Big)=g(A_WX,Y),\tag{2.9}$$

$$\overline{g}\Big(h^l(X,Y),\xi\Big)+\overline{g}\Big(Y,h^l(X,\xi)\Big)+g(Y,\nabla_X\xi)=0,\tag{2.10}$$

for any $\xi\in\Gamma(\mathrm{Rad}\,TM)$, $W\in\Gamma(S(TM^{\perp}))$, and $N,N'\in\Gamma(\mathrm{ltr}(TM))$.

Let P be the projection morphism of TM on $S(TM)$. Then, using (2.2), we can induce some new geometric objects on the screen distribution $S(TM)$ on M as

$$\nabla_X PY = \nabla^*_X PY + h^*(X,Y), \qquad \nabla_X \xi = -A^*_\xi X + \nabla^{*t}_X \xi, \tag{2.11}$$

for any $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(\operatorname{Rad} TM)$, where $\{\nabla^*_X PY, A^*_\xi X\}$ and $\{h^*(X,Y), \nabla^{*t}_X \xi\}$ belong to $\Gamma(S(TM))$ and $\Gamma(\operatorname{Rad} TM)$, respectively. ∇^* and ∇^{*t} are linear connections on complementary distributions $S(TM)$ and $\operatorname{Rad} TM$, respectively. Then, using (2.7), (2.8), and (2.11), we have

$$\overline{g}\left(h^l(X,PY),\xi\right) = g\left(A^*_\xi X, PY\right), \qquad \overline{g}(h^*(X,PY),N) = g(A_N X, PY). \tag{2.12}$$

Next, an odd-dimensional semi-Riemannian manifold $\overline{M}$ is said to be an indefinite almost contact metric manifold if there exist structure tensors $(\phi, V, \eta, \overline{g})$, where ϕ is a $(1,1)$ tensor field, V is a vector field called structure vector field, η is a 1-form, and $\overline{g}$ is the semi-Riemannian metric on $\overline{M}$ satisfying (see [10])

$$\begin{gathered}\overline{g}(\phi X, \phi Y) = \overline{g}(X,Y) - \eta(X)\eta(Y), \qquad \overline{g}(X,V) = \eta(X), \\ \phi^2 X = -X + \eta(X)V, \qquad \eta\circ\phi = 0, \qquad \phi V = 0, \qquad \eta(V) = 1,\end{gathered} \tag{2.13}$$

for any $X, Y \in \Gamma(TM)$.

An indefinite almost contact metric manifold $\overline{M}$ is called an indefinite cosymplectic manifold if (see [11])

$$\overline{\nabla}_X \phi = 0, \tag{2.14}$$

$$\overline{\nabla}_X V = 0. \tag{2.15}$$

3. Generalized Cauchy-Riemann Lightlike Submanifolds

Calin [12] proved that if the characteristic vector field V is tangent to $(M, g, S(TM))$, then it belongs to $S(TM)$. We assume that the characteristic vector V is tangent to M throughout this paper. Thus, we define the generalized Cauchy-Riemann lightlike submanifolds of an indefinite cosymplectic manifold as follows.

Definition 3.1. Let $(M, g, S(TM), S(TM^\perp))$ be a real lightlike submanifold of an indefinite cosymplectic manifold $(\overline{M}, \overline{g})$ such that the structure vector field V is tangent to M; then M is called a generalized-Cauchy-Riemann- *(GCR-)* lightlike submanifold if the following conditions are satisfied:

(A) there exist two subbundles D_1 and D_2 of $\operatorname{Rad}(TM)$ such that

$$\operatorname{Rad}(TM) = D_1 \oplus D_2, \qquad \phi(D_1) = D_1, \qquad \phi(D_2) \subset S(TM), \tag{3.1}$$

(B) there exist two subbundles D_0 and $\overline{D}$ of $S(TM)$ such that

$$S(TM) = \left\{\phi D_2 \oplus \overline{D}\right\} \perp D_0 \perp V, \qquad \phi\left(\overline{D}\right) = L \perp S, \tag{3.2}$$

where D_0 is invariant nondegenerate distribution on M, $\{V\}$ is one-dimensional distribution spanned by V, and L and S are vector subbundles of $\operatorname{ltr}(TM)$ and $S(TM)^{\perp}$, respectively.

Therefore, the tangent bundle TM of M is decomposed as

$$TM = \left\{D \oplus \overline{D} \oplus \{V\}\right\}, \qquad D = \operatorname{Rad}(TM) \oplus D_0 \oplus \phi(D_2). \tag{3.3}$$

A contact GCR-lightlike submanifold is said to be proper if $D_0 \neq \{0\}, D_1 \neq \{0\}, D_2 \neq \{0\}$, and $L \neq \{0\}$. Hence, from the definition of GCR-lightlike submanifolds, we have that

(a) condition (A) implies that $\dim(\operatorname{Rad} TM) \geq 3$,

(b) condition (B) implies that $\dim(D) \geq 2s \geq 6$ and $\dim(D_2) = \dim(S)$, and thus $\dim(M) \geq 9$ and $\dim(\overline{M}) \geq 13$.

(c) any proper 9-dimensional contact GCR-lightlike submanifold is 3-lightlike,

(d) (a) and contact distribution ($\eta = 0$) imply that index $(\overline{M}) \geq 4$.

The following proposition shows that the class of GCR-lightlike submanifolds is an umbrella of invariant, contact CR and contact SCR-lightlike submanifolds.

Proposition 3.2. *A GCR-lightlike submanifold M of an indefinite cosymplectic manifold $\overline{M}$ is contact CR-submanifold (resp., contact SCR-lightlike submanifold) if and only if $D_1 = \{0\}$ (resp., $D_2 = \{0\}$).*

Proof. Let M be a contact CR-lightlike submanifold; then $\phi\operatorname{Rad} TM$ is a distribution on M such that $\operatorname{Rad} TM \bigcap \phi\operatorname{Rad} TM = \{0\}$. Therefore, $D_2 = \operatorname{Rad} TM$ and $D_1 = \{0\}$. Since $\operatorname{ltr}(TM) \bigcap \phi(\operatorname{ltr}(TM)) = \{0\}$, this implies that $\phi(\operatorname{ltr}(TM)) \subset S(TM)$. Conversely, suppose that M is a GCR-lightlike submanifold of an indefinite Cosymplectic manifold such that $D_1 = \{0\}$. Then, from (3.1), we have $D_2 = \operatorname{Rad}(TM)$, and therefore $\operatorname{Rad} TM \bigcap \phi\operatorname{Rad} TM = \{0\}$. Hence, $\phi\operatorname{Rad} TM$ is a vector subbundle of $S(TM)$. This implies that M is a contact CR-lightlike submanifold of an indefinite cosymplectic manifold. Similarly the other assertion follows.

The following construction helps in understanding the example of GCR-lightlike submanifold. Let $(R_q^{2m+1}, \phi_0, V, \eta, \overline{g})$ be with its usual Cosymplectic structure and given by

$$\eta = dz, \qquad V = \partial z,$$

$$\overline{g} = \eta \otimes \eta - \sum_{i=1}^{q/2}\left(dx^i \otimes dx^i + dy^i \otimes dy^i\right) + \sum_{i=q+1}^{m}\left(dx^i \otimes dx^i + dy^i \otimes dy^i\right),$$
$$\phi_0(X_1, X_2, \ldots, X_{m-1}, X_m, Y_1, Y_2, \ldots, Y_{m-1}, Y_m, Z) = (-X_2, X_1, \ldots, -X_m, X_{m-1}, -Y_2, Y_1, \ldots, -Y_m, Y_{m-1}, 0), \tag{3.4}$$

where $(x^i; y^i; z)$ are the Cartesian coordinates. □

Example 3.3. Let $\overline{M} = (R_4^{13}, \overline{g})$ be a semi-Euclidean space and M a 9-dimensional submanifold of $\overline{M}$ that is given by

$$x^4 = x^1 \cos\theta - y^1 \sin\theta, \qquad y^4 = x^1 \sin\theta + y^1 \cos\theta,$$
$$x^2 = y^3, \qquad x^5 = \sqrt{1 + (y^5)^2}, \tag{3.5}$$

where $\overline{g}$ is of signature $(-,-,+,+,+,+,-,-,+,+,+,+,+)$ with respect to the canonical basis $\{\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial x_5, \partial x_6, \partial y_1, \partial y_2, \partial y_3, \partial y_4, \partial y_5, \partial y_6, \partial z\}$. Then, the local frame of TM is given by

$$\xi_1 = \partial x_1 + \cos\theta \partial x_4 + \sin\theta \partial y_4, \qquad \xi_2 = -\sin\theta \partial x_4 + \partial y_1 + \cos\theta \partial y_4,$$
$$\xi_3 = \partial x_2 + \partial y_3,$$
$$X_1 = \partial x_3 - \partial y_2, \qquad X_2 = \partial x_6, \qquad X_3 = \partial y_6, \tag{3.6}$$
$$X_4 = y^5 \partial x_5 + x^5 \partial y_5, \qquad X_5 = \partial x_3 + \partial y_2, \qquad X_6 = V = \partial z.$$

Hence, M is a 3-lightlike as $\operatorname{Rad} TM = \operatorname{span}\{\xi_1, \xi_2, \xi_3\}$. Also, $\phi_0 \xi_1 = -\xi_2$ and $\phi_0 \xi_3 = X_1$; these imply that $D_1 = \operatorname{span}\{\xi_1, \xi_2\}$ and $D_2 = \operatorname{span}\{\xi_3\}$, respectively. Since $\phi_0 X_2 = -X_3$, $D_0 = \operatorname{span}\{X_2, X_3\}$. By straightforward calculations, we obtain

$$S\left(TM^{\perp}\right) = \operatorname{span}\left\{W = x^5 \partial x_5 - y^5 \partial y_5\right\}, \tag{3.7}$$

where $\phi_0(W) = X_4$; this implies that $S = S(TM^{\perp})$. Moreover, the lightlike transversal bundle $\operatorname{ltr}(TM)$ is spanned by

$$N_1 = \frac{1}{2}(-\partial x_1 + \cos\theta \partial x_4 + \sin\theta \partial y_4), \qquad N_2 = \frac{1}{2}(-\sin\theta \partial x_4 - \partial y_1 + \cos\theta \partial y_4),$$
$$N_3 = \frac{1}{2}(-\partial x_2 + \partial y_3), \tag{3.8}$$

where $\phi_0(N_1) = -N_2$ and $\phi_0(N_3) = X_5$. Hence, $L = \operatorname{span}\{N_3\}$. Therefore, $\overline{D} = \operatorname{span}\{\phi_0(N_3), \phi_0(W)\}$. Thus, M is a GCR-lightlike submanifold of R_4^{13}.

Let Q, P_1, P_2 be the projection morphism on D, $\phi S = M_2$, $\phi L = M_1$, respectively; therefore

$$X = QX + V + P_1 X + P_2 X, \tag{3.9}$$

for $X \in \Gamma(TM)$. Applying ϕ to (3.9), we obtain

$$\phi X = fX + \omega P_1 X + \omega P_2 X, \tag{3.10}$$

where $fX \in \Gamma(D)$, $\omega P_1 X \in \Gamma(L)$, and $\omega P_2 X \in \Gamma(S)$, or, we can write (3.10) as

$$\phi X = fX + \omega X, \tag{3.11}$$

where fX and ωX are the tangential and transversal components of ϕX, respectively.

Similarly,

$$\phi U = BU + CU, \quad U \in \Gamma(\mathrm{tr}(TM)), \tag{3.12}$$

where BU and CU are the sections of TM and $\mathrm{tr}(TM)$, respectively. Differentiating (3.10) and using (2.8)–(2.10) and (3.12), we have

$$\begin{aligned} D^s(X, \omega P_2 Y) &= -\nabla^s_X \omega P_1 Y + \omega P_1 \nabla_X Y - h^s(X, fY) + Ch^s(X, Y), \\ D^l(X, \omega P_1 Y) &= -\nabla^l_X \omega P_2 Y + \omega P_2 \nabla_X Y - h^l(X, fY) + Ch^l(X, Y), \end{aligned} \tag{3.13}$$

for all $X, Y \in \Gamma(TM)$. By using, cosymplectic property of $\overline{\nabla}$ with (2.7), we have the following lemmas.

Lemma 3.4. *Let M be a GCR-lightlike submanifold of an indefinite cosymplectic manifold $\overline{M}$; then one has*

$$(\nabla_X f)Y = A_{\omega Y} X + Bh(X, Y), \qquad (\nabla^t_X \omega)Y = Ch(X, Y) - h(X, fY), \tag{3.14}$$

where $X, Y \in \Gamma(TM)$ and

$$(\nabla_X f)Y = \nabla_X fY - f\nabla_X Y, \qquad (\nabla^t_X \omega)Y = \nabla^t_X \omega Y - \omega \nabla_X Y. \tag{3.15}$$

Lemma 3.5. *Let M be a GCR-lightlike submanifold of an indefinite cosymplectic manifold $\overline{M}$; then one has*

$$(\nabla_X B)U = A_{CU} X - f A_U X, \qquad (\nabla^t_X C)U = -\omega A_U X - h(X, BU), \tag{3.16}$$

where $X \in \Gamma(TM)$ and $U \in \Gamma(\mathrm{tr}(TM))$ and

$$(\nabla_X B)U = \nabla_X BU - B\nabla^t_X U, \qquad (\nabla^t_X C)U = \nabla^t_X CU - C\nabla^t_X U. \tag{3.17}$$

4. Mixed Geodesic *GCR*-Lightlike Submanifolds

Definition 4.1. A GCR-lightlike submanifold of an indefinite cosymplectic manifold is called mixed geodesic GCR-lightlike submanifold if its second fundamental form h satisfies $h(X, Y) = 0$, for any $X \in \Gamma(D \oplus V)$ and $Y \in \Gamma(\overline{D})$.

Definition 4.2. A GCR-lightlike submanifold of an indefinite cosymplectic manifold is called $\overline{D}$ geodesic GCR-lightlike submanifold if its second fundamental form h satisfies $h(X, Y) = 0$, for any $X, Y \in \Gamma(\overline{D})$.

Theorem 4.3. *Let M be a GCR-lightlike submanifold of an indefinite cosymplectic manifold $\overline{M}$. Then, M is mixed geodesic if and only if $A^*_\xi X$ and $A_W X \notin \Gamma(M_2 \perp \phi D_2)$, for any $X \in \Gamma(D \oplus V)$, $W \in \Gamma(S(TM^\perp))$ and $\xi \in \Gamma(\mathrm{Rad}\,(TM))$.*

Proof. Using, definition of GCR-lightlike submanifolds, M is mixed geodesic if and only if $\overline{g}(h(X,Y),W) = \overline{g}(h(X,Y),\xi) = 0$, for $X \in \Gamma(D \oplus V), Y \in \Gamma(\overline{D}), W \in \Gamma(S(TM^{\perp}))$, and $\xi \in \Gamma(\text{Rad}(TM))$. Using (2.8) and (2.11), we get

$$\begin{aligned} \overline{g}(h(X,Y),W) &= \overline{g}\left(\overline{\nabla}_X Y, W\right) = -g\left(Y, \overline{\nabla}_X W\right) = g(Y, A_W X), \\ \overline{g}(h(X,Y),\xi) &= \overline{g}\left(\overline{\nabla}_X Y, \xi\right) = -g(Y, \nabla_X \xi) = g\left(Y, A_\xi^* X\right). \end{aligned} \tag{4.1}$$

Therefore, from (4.1), the proof is complete. □

Theorem 4.4. *Let M be a GCR-lightlike submanifold of an indefinite cosymplectic manifold $\overline{M}$. Then, M is $\overline{D}$ geodesic if and only if $A_\xi^* X$ and $A_W X \notin \Gamma(M_2 \perp \phi D_2)$, for any $X \in \Gamma(\overline{D}), \xi \in \Gamma$ Rad(TM), and $W \in \Gamma(S(TM^{\perp}))$.*

Proof. The proof is similar to the proof of Theorem 4.3. □

Lemma 4.5. *Let M be a mixed geodesic GCR-lightlike submanifold of an indefinite cosymplectic manifold $\overline{M}$. Then $A_\xi^* X \in \Gamma(\phi D_2)$, for any $X \in \Gamma(\overline{D})$, $\xi \in \Gamma(D_2)$.*

Proof. For $X \in \Gamma(\overline{D})$ and $\xi \in \Gamma(D_2)$, using (2.7) we have

$$h(\phi\xi, X) = \overline{\nabla}_X \phi\xi - \nabla_X \phi\xi = \phi\nabla_X \xi + \phi h(X,\xi) - \nabla_X \phi\xi. \tag{4.2}$$

Since M is mixed geodesic, we obtain $\phi\nabla_X\xi = \nabla_X\phi\xi$. Here, using (2.11), we get $\phi(-A_\xi^* X + \nabla_X^{*t}\xi) = \nabla_X^* \phi\xi + h^*(X,\phi\xi)$, and then, by virtue of (3.11), we obtain $-f A_\xi^* X - \omega A_\xi^* X + \phi(\nabla_X^{*t}\xi) = \nabla_X^*\phi\xi + h^*(X,\phi\xi)$. Comparing the transversal components, we get $\omega A_\xi^* X = 0$; this implies that

$$A_\xi^* X \in \Gamma\left(D_0 \oplus \{V\} \perp \phi(D_2)\right). \tag{4.3}$$

If $A_\xi^* X \in D_0$, then the nondegeneracy of D_0 implies that there must exist a $Z_0 \in D_0$ such that $\overline{g}(A_\xi^* X, Z_0) \neq 0$. But using the hypothesis that M is a mixed geodesic with (2.7) and (2.11), we get

$$\overline{g}\left(A_\xi^* X, Z_0\right) = -\overline{g}(\nabla_X \xi, Z_0) = \overline{g}\left(\xi, \overline{\nabla}_X Z_0\right) = \overline{g}(\xi, \nabla_X Z_0 + h(X, Z_0)) = 0. \tag{4.4}$$

Therefore,

$$A_\xi^* X \notin \Gamma(D_0). \tag{4.5}$$

Also using (2.13), and (2.15), we get

$$\overline{g}\left(A_\xi^* X, V\right) = -\overline{g}(\nabla_X \xi, V) = \overline{g}\left(\xi, \overline{\nabla}_X V\right) = 0. \tag{4.6}$$

Therefore,

$$A^*_\xi X \notin \{V\}. \tag{4.7}$$

Hence, from (4.3), (4.5), and (4.7), the result follows. □

Corollary 4.6. *Let M be a mixed geodesic GCR-lightlike submanifold of an indefinite cosymplectic manifold $\overline{M}$. Then, $\overline{g}(h^l(X,Y),\xi) = 0$, for any $X \in \Gamma(\overline{D})$, $Y \in \Gamma(M_2)$ and $\xi \in \Gamma(D_2)$.*

Proof. The result follows from (2.12) and Lemma 4.5. □

Theorem 4.7. *Let M be a mixed geodesic GCR-lightlike submanifold of an indefinite cosymplectic manifold $\overline{M}$. Then, $A_U X \in \Gamma(D \oplus \{V\})$ and $\nabla^t_X U \in \Gamma(L \perp S)$, for any $X \in \Gamma(D \oplus \{V\})$ and $U \in \Gamma(L \perp S)$.*

Proof. Since M is mixed geodesic GCR-lightlike submanifold $h(X,Y) = 0$ for any $X \in \Gamma(D \oplus \{V\})$, $Y \in \Gamma(\overline{D})$, and thus (2.6) implies that

$$0 = \overline{\nabla}_X Y - \nabla_X Y. \tag{4.8}$$

Since $\overline{D}$ is an anti-invariant distribution there exists a vector field $U \in \Gamma(L \perp S)$ such that $\phi U = Y$. Thus, from (2.8), (2.14), (3.11), and (3.12), we get

$$\begin{aligned} 0 &= \overline{\nabla}_X \phi U - \nabla_X Y = \phi\left(-A_U X + \nabla^t_X U\right) - \nabla_X Y \\ &= -f A_U X - \omega A_U X + B\nabla^t_X U + C\nabla^t_X U - \nabla_X Y. \end{aligned} \tag{4.9}$$

Comparing the transversal components, we get $\omega A_U X = C\nabla^t_X U$. Since $\omega A_U X \in \Gamma(L \perp S)$ and $C\nabla^t_X U \in \Gamma(L \perp S)^\perp$, this implies that $\omega A_U X = 0$ and $C\nabla^t_X U = 0$. Hence, $A_U X \in \Gamma(D \oplus \{V\})$ and $\nabla^t_X U \in \Gamma(L \perp S)$. □

5. *GCR*-Lightlike Product

Definition 5.1. GCR-lightlike submanifold M of an indefinite cosymplectic manifold $\overline{M}$ is called GCR-lightlike product if both the distributions $D \oplus \{V\}$ and $\overline{D}$ define totally geodesic foliation in M.

Theorem 5.2. *Let M be a GCR-lightlike submanifold of an indefinite cosymplectic manifold $\overline{M}$. Then, the distribution $D \oplus \{V\}$ define a totally geodesic foliation in M if and only if $Bh(X,\phi Y) = 0$, for any $X, Y \in D \oplus \{V\}$.*

Proof. Since $\overline{D} = \phi(L \perp S)$, $D \oplus \{V\}$ defines a totally geodesic foliation in M if and only if $g(\nabla_X Y, \phi\xi) = g(\nabla_X Y, \phi W) = 0$, for any $X, Y \in \Gamma(D \oplus \{V\})$, $\xi \in \Gamma(D_2)$, and $W \in \Gamma(S)$. Using (2.7) and (2.14), we have

$$g(\nabla_X Y, \phi\xi) = -\overline{g}\left(\overline{\nabla}_X \phi Y, \xi\right) = -\overline{g}\left(h^l(X, fY), \xi\right), \tag{5.1}$$

$$g(\nabla_X Y, \phi W) = -\overline{g}\left(\overline{\nabla}_X \phi Y, W\right) = -\overline{g}(h^s(X, fY), W). \tag{5.2}$$

Hence, from (5.1) and (5.2), the assertion follows. □

Theorem 5.3. *Let M be a GCR-lightlike submanifold of an indefinite cosymplectic manifold $\overline{M}$. Then, the distribution $\overline{D}$ defines a totally geodesic foliation in M if and only if $A_N X$ has no component in $\phi S \perp \phi D_2$ and $A_{\omega Y} X$ has no component in $D_2 \perp D_0$, for any $X, Y \in \Gamma(\overline{D})$ and $N \in \Gamma(\mathrm{ltr}\,(TM))$.*

Proof. From the definition of a GCR-lightlike submanifold, we know that $\overline{D}$ defines a totally geodesic foliation in M if and only if

$$g(\nabla_X Y, N) = g(\nabla_X Y, \phi N_1) = g(\nabla_X Y, V) = g(\nabla_X Y, \phi Z) = 0, \tag{5.3}$$

for $X, Y \in \Gamma(\overline{D}), N \in \Gamma(\mathrm{ltr}(TM)), Z \in \Gamma(D_0)$ and $N_1 \in \Gamma(L)$. Using (2.7) and (2.8), we have

$$g(\nabla_X Y, N) = \overline{g}\left(\overline{\nabla}_X Y, N\right) = -\overline{g}\left(Y, \overline{\nabla}_X N\right) = g(Y, A_N X). \tag{5.4}$$

Using (2.7), (2.15), and (2.14), we obtain

$$g(\nabla_X Y, \phi N_1) = -g\left(\phi\overline{\nabla}_X Y, N_1\right) = -g\left(\overline{\nabla}_X \omega Y, N_1\right) = g(A_{\omega Y} X, N_1), \tag{5.5}$$

$$g(\nabla_X Y, \phi Z) = -g\left(\phi\overline{\nabla}_X Y, Z\right) = -g\left(\overline{\nabla}_X \omega Y, Z\right) = g(A_{\omega Y} X, Z), \tag{5.6}$$

$$g(\nabla_X Y, V) = g\left(\overline{\nabla}_X Y, V\right) = -g\left(Y, \overline{\nabla}_X V\right) = 0. \tag{5.7}$$

Thus, from (5.4)–(5.7), the result follows. □

Theorem 5.4. *Let M be a GCR-lightlike submanifold of an indefinite cosymplectic manifold $\overline{M}$. If $(\nabla_X f)Y = 0$, then M is a GCR lightlike product.*

Proof. Let $X, Y \in \Gamma(\overline{D})$; therefore $fY = 0$. Then using (3.15) with the hypothesis, we get $f\nabla_X Y = 0$. Therefore the distribution $\overline{D}$ defines a totally geodesic foliation. Next, let $X, Y \in D \oplus \{V\}$; therefore $\omega Y = 0$. Then using (3.14), we get $Bh(X, Y) = 0$. Therefore, $D \oplus \{V\}$ defines a totally geodesic foliation in M. Hence, M is a GCR lightlike product. □

Definition 5.5. A lightlike submanifold M of a semi-Riemannian manifold is said to be an irrotational submanifold if $\overline{\nabla}_X \xi \in \Gamma(TM)$, for any $X \in \Gamma(TM)$ and $\xi \in \Gamma\,\mathrm{Rad}(TM)$. Thus, M is an irrotational lightlike submanifold if and only if $h^l(X, \xi) = 0$ and $h^s(X, \xi) = 0$.

Theorem 5.6. *Let M be an irrotational GCR-lightlike submanifold of an indefinite cosymplectic manifold $\overline{M}$. Then, M is a GCR lightlike product if the following conditions are satisfied:*

(A) $\overline{\nabla}_X U \in \Gamma(S(TM^{\perp}))$, *for all* $X \in \Gamma(TM)$, *and* $U \in \Gamma(\mathrm{tr}(TM))$,

(B) $A^{*}_{\xi}Y \in \Gamma(\phi(S))$, *for all* $Y \in \Gamma(D)$.

Proof. Let (A) hold; then, using (2.8), we get $A_N X = 0$, $A_W X = 0$, $D^l(X, W) = 0$, and $\nabla^l_X N = 0$ for $X \in \Gamma(TM)$. These equations imply that the distribution $\overline{D}$ defines a totally geodesic foliation in M, and, with (2.9), we get $\overline{g}(h^s(X,Y), W) = 0$. Hence, the non degeneracy of $S(TM^{\perp})$ implies that $h^s(X,Y) = 0$. Therefore, $h^s(X,Y)$ has no component in S. Finally, from (2.10) and the hypothesis that M is irrotational, we have $\overline{g}(h^l(X,Y), \xi) = \overline{g}(Y, A^{*}_{\xi}X)$, for $X \in \Gamma(TM)$ and $Y \in \Gamma(D)$. Assume that (B) holds; then $h^l(X,Y) = 0$. Therefore, $h^l(X,Y)$ has no component in L. Thus, the distribution $D \oplus \{V\}$ defines a totally geodesic foliation in M. Hence, M is a GCR lightlike product. □

Definition 5.7 (see [13]). If the second fundamental form h of a submanifold, tangent to characteristic vector field V, of a Sasakian manifold $\overline{M}$ is of the form

$$h(X,Y) = \{g(X,Y) - \eta(X)\eta(Y)\}\alpha + \eta(X)h(Y,V) + \eta(Y)h(X,V), \tag{5.8}$$

for any $X, Y \in \Gamma(TM)$, where α is a vector field transversal to M, then M is called a totally contact umbilical submanifold of a Sasakian manifold.

Theorem 5.8. *Let M be a totally contact umbilical GCR-lightlike submanifold of an indefinite cosymplectic manifold $\overline{M}$. Then, M is a GCR-lightlike product if $Bh(X,Y) = 0$, for any $X, Y \in \Gamma(TM)$.*

Proof. Let $X, Y \in \Gamma(D \oplus \{V\})$; then the hypothesis that $Bh(X,Y) = 0$ implies that the distribution $D \oplus \{V\}$ defines a totally geodesic foliation in M.

If we assume that $X, Y \in \Gamma(\overline{D})$, then, using (3.14), we have $-f\nabla_X Y = A_{\omega Y}X + Bh(X,Y)$, and taking inner product with $Z \in \Gamma(D_0)$ and using (2.6) and (2.14), we obtain

$$-g(f\nabla_X Y, Z) = g(A_{\omega Y}X + Bh(X,Y), Z) = g\left(\overline{\nabla}_X Y, \phi Z\right) = -g(Y, \nabla_X Z'), \tag{5.9}$$

where $\phi Z = Z' \in \Gamma(D_0)$. For any $X \in \Gamma(\overline{D})$ from (3.14), we have $\omega P \nabla_X Z = h(X, fZ) - Ch(X,Z)$. Therefore, using the hypothesis with (5.8), we get $\omega P \nabla_X Z = 0$; this implies that $\nabla_X Z \in \Gamma(D)$, and thus (5.9) becomes $g(f\nabla_X Y, Z) = 0$. Then, the nondegeneracy of the distribution D_0 implies that the distribution $\overline{D}$ defines a totally geodesic foliation in M. Hence, the assertion follows. □

Theorem 5.9. *Let M be a totally geodesic GCR-lightlike submanifold of an indefinite cosymplectic manifold $\overline{M}$. Suppose that there exists a transversal vector bundle of M which is parallel along $\overline{D}$ with respect to Levi-Civita connection on M, that is, $\overline{\nabla}_X U \in \Gamma(\mathrm{tr}(TM))$, for any $U \in \Gamma(\mathrm{tr}(TM))$, $X \in \Gamma(\overline{D})$. Then, M is a GCR-lightlike product.*

Proof. Since M is a totally geodesic GCR-lightlike $Bh(X,Y) = 0$, for $X, Y \in \Gamma(D \oplus \{V\})$; this implies $D \oplus \{V\}$ defines a totally geodesic foliation in M.

Next $\overline{\nabla}_X U \in \Gamma(\mathrm{tr}(TM))$ implies $A_U X = 0$, and hence, by Theorem 5.3, the distribution $\overline{D}$ defines a totally geodesic foliation in M. Hence, the result follows. □

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References

[1] K. L. Duggal and A. Bejancu, *Lightlike Submanifolds of Semi-Riemannian Manifolds and Applications*, vol. 364 of *Mathematics and its Applications*, Kluwer Academic, Dordrecht, The Netherlands, 1996.

[2] K. L. Duggal and B. Sahin, "Lightlike submanifolds of indefinite Sasakian manifolds," *International Journal of Mathematics and Mathematical Sciences*, vol. 2007, Article ID 57585, 21 pages, 2007.

[3] R. Kumar, R. Rani, and R. K. Nagaich, "On contact *CR*-lightlike submanifolds of indefinite Sasakian manifolds," *Vietnam Journal of Mathematics*, vol. 38, no. 4, pp. 485–498, 2010.

[4] A. Bejancu, "*CR* submanifolds of a Kaehler manifold. I," *Proceedings of the American Mathematical Society*, vol. 69, no. 1, pp. 135–142, 1978.

[5] K. L. Duggal and B. Sahin, "Generalized Cauchy-Riemann lightlike submanifolds of Kaehler manifolds," *Acta Mathematica Hungarica*, vol. 112, no. 1-2, pp. 107–130, 2006.

[6] K. L. Duggal and B. Sahin, "Generalized Cauchy-Riemann lightlike submanifolds of indefinite Sasakian manifolds," *Acta Mathematica Hungarica*, vol. 122, no. 1-2, pp. 45–58, 2009.

[7] V. Jain, R. Kumar, and R. K. Nagaich, "Totally contact umbilical *GCR*-lightlike submanifolds indefinite Sasakian manifolds," *Vietnam Journal of Mathematics*, vol. 39, no. 1, pp. 91–103, 2011.

[8] R. Kumar, S. Kumar, and R. K. Nagaich, "*GCR*-lightlike product of indefinite Kaehler manifolds," *ISRN Geometry*, vol. 2011, Article ID 531281, 13 pages, 2011.

[9] R. Kumar, V. Jain, and R. K. Nagaich, "*GCR*-lightlike product of indefinite Sasakian manifolds," *Advances in Mathematical Physics*, vol. 2011, Article ID 983069, 13 pages, 2011.

[10] R. Kumar, R. Rani, and R. K. Nagaich, "On sectional curvatures of (ϵ)-Sasakian manifolds," *International Journal of Mathematics and Mathematical Sciences*, vol. 2007, Article ID 93562, 8 pages, 2007.

[11] D. E. Blair, *Riemannian Geometry of Contact and Symplectic Manifolds*, vol. 203 of *Progress in Mathematics*, Birkhäuser, Boston, Mass, USA, 2002.

[12] C. Calin, "On the existence of degenerate hypersurfaces in Sasakian manifolds," *Arab Journal of Mathematical Sciences*, vol. 5, no. 1, pp. 21–27, 1999.

[13] K. Yano and M. Kon, *Structures on Manifolds*, vol. 3 of *Series in Pure Mathematics*, World Scientific, Singapore, 1984.