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Research Article

Translative Packing of Unit Squares into Squares

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Every collection of n (arbitrary-oriented) unit squares admits a translative packing into any square of side length $\sqrt{2.5 \cdot n}$.

1. Introduction

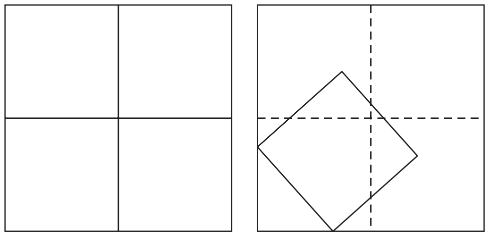
Let i be a positive integer, let $0 \le \alpha_i < \pi/2$, and let a rectangular coordinate system in the plane be given. One of the coordinate system's axes is called x-axis. Denote by $S(\alpha_i)$ a square in the plane with sides of unit length and with the angle between the x-axis and a side of $S(\alpha_i)$ equal to α_i. Furthermore, by $I(s)$ denote a square with side length s and with sides parallel to the coordinate axes.

We say that a collection of n unit squares $S(\alpha_1), \dots, S(\alpha_n)$ admits a *packing* into a set C if there are rigid motions $\sigma_1, \dots, \sigma_n$ such that the squares $\sigma_i S(\alpha_i)$ are subsets of C and that they have mutually disjoint interiors. A packing is *translative* if only translations are allowed as the rigid motions.

For example, two unit squares can be packed into $I(2)$, but they cannot be packed into $I(2 - \epsilon)$ for any $\epsilon > 0$. Three and four unit squares can be packed into $I(2)$ as well (see Figure 1(a)). Obviously, two, three, or four squares $S(0)$ can be translatively packed into $I(2)$. If either $\alpha_1 \ne 0$ or $\alpha_2 \ne 0$, then two squares $S(\alpha_1)$ and $S(\alpha_2)$ cannot be translatively packed into $I(2)$. The reason is that for every $\alpha \ne 0$, the interior of any square $S(\alpha)$ translatively packed into $I(2)$ covers the center of $I(2)$ (see Figure 1(b)).

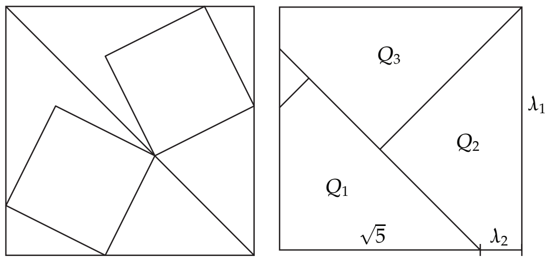
The problem of packing of unit squares into squares (with possibility of rigid motions) is a well-known problem (e.g., see [1–3]). The best packings are known for several values of n. Furthermore, for many values of n, there are good packings that seem to be optimal.

In this paper, we propose the problem of translative packing of squares. Denote by s_n the smallest number s such that any collection of n unit squares $S(\alpha_1), \dots, S(\alpha_n)$ admits



(a) (b)

Figure 1



(a) (b)

Figure 2

a translative packing into $I(s)$. The problem is to find s_n for $n = 1, 2, 3, \ldots$. Obviously, $s_n > \sqrt{n}$. By [4, Theorem 7], we deduce that $\lim_{n\to\infty} s_n/\sqrt{n} = 1$. We show that

$$s_n \leq \sqrt{2.5 \cdot n}. \tag{1.1}$$

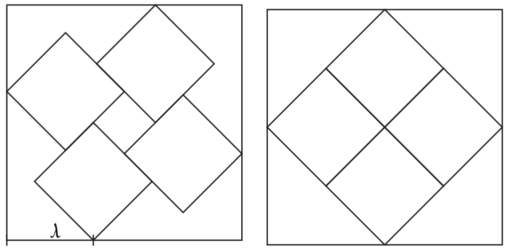
2. Packing into Squares

Example 2.1. We have $s_1 = \sqrt{2}$. Each unit square can be translatively packed into $I(\sqrt{2})$, but it is impossible to translatively pack $S(\pi/4)$ into $I(\sqrt{2} - \epsilon)$ for any $\epsilon > 0$.

Example 2.2. We have $s_2 = \sqrt{5}$ (see [5]). Here, we only recall that two squares: $S(\arctan 1/2)$ and $S(\arctan 2)$ cannot be translatively packed into $I(\sqrt{5} - \epsilon)$ for any $\epsilon > 0$ (see Figure 2(a)).

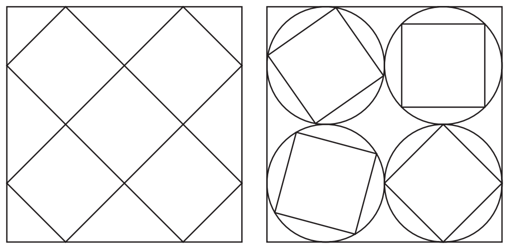
Example 2.3. We have $s_4 = 2\sqrt{2}$. Four squares $S(\pi/4)$ admit a translative packing into $I(2\sqrt{2})$ (see Figure 3, where $\sqrt{2}/2 \leq \lambda \leq 3\sqrt{2}/2$). In Figure 3(b) and Figure 4(a), we illustrate the cases when $\lambda = \sqrt{2}$ and $\lambda = \sqrt{2}/2$, respectively. By these three pictures, we conclude that four squares $S(\pi/4)$ cannot be translatively packed into $I(2\sqrt{2} - \epsilon)$, for any $\epsilon > 0$. Consequently, $s_4 \geq 2\sqrt{2}$. On the other hand, four circles of radius $\sqrt{2}/2$ can be packed into $I(2\sqrt{2})$ (see Figure 4(b)). Since any square $S(\alpha_i)$ can be translatively packed into a circle of radius $\sqrt{2}/2$, it follows that $s_4 \leq 2\sqrt{2}$.

Lemma 2.4 (see [5]). *Every unit square can be translatively packed into any isosceles right triangle with legs of length $\sqrt{5}$.*



(a) (b)

Figure 3



(a) (b)

Figure 4

Theorem 2.5. *If $n \geq 3$, then $s_n \leq ((\sqrt{10}+\sqrt{5})/2\sqrt{3}) \cdot \sqrt{n}$.*

Proof. Let $S(\alpha_1), S(\alpha_2)$, and $S(\alpha_3)$ be unit squares and put

$$\lambda_1 = \frac{1}{2}\left(\sqrt{10}+\sqrt{5}\right). \tag{2.1}$$

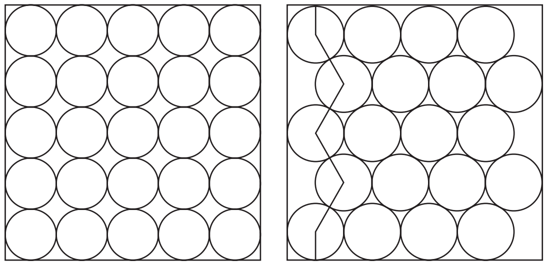
Three congruent quadrangles Q_1, Q_2, and Q_3, presented in Figure 2(b), of side lengths $\lambda_1, \sqrt{5}, \sqrt{2.5}$, and $\lambda_2 = \lambda_1 - \sqrt{5}$, are contained in $I(\lambda_1)$. Since the length of the diagonal of $S(\alpha_i)$ is smaller than $\sqrt{2.5}$, by Lemma 2.4 we deduce that $S(\alpha_i)$ can be translatively packed into Q_i for $i = 1,2,3$. Consequently, the squares $S(\alpha_1), S(\alpha_2)$, and $S(\alpha_3)$ can be translatively packed into $I(s)$ and

$$s_3 \leq \lambda_1 = \frac{\sqrt{10}+\sqrt{5}}{2\sqrt{3}} \cdot \sqrt{3}. \tag{2.2}$$

Now assume that $4 \leq n \leq 16$.

Denote by m_n the smallest number s such that n circles of unit radius can be packed into $I(s)$. The problem of minimizing the side of a square into which n congruent circles can be packed is a well-known question. The values of m_n are known, among others, for $n \leq 16$ (see Table 2.2.1 in [6] or [7]). We know that

$$\begin{gathered} m_4 = 4, \quad m_5 < 4.83, \quad m_6 < 5.33, \quad m_7 < 5.74, \quad m_8 < m_9 = 6, \\ m_{10} < 6.75, \quad m_{11} < 7.03, \quad m_{12} < m_{13} < 7.47, \quad m_{14} < m_{15} < m_{16} = 8. \end{gathered} \tag{2.3}$$



(a) (b)

Figure 5

Since each unit square is contained in a circle of radius $\sqrt{2}/2$, it follows that n unit squares can be translatively packed into $I(\sqrt{2}m_n/2)$. It is easy to verify that

$$s_n \leq \frac{\sqrt{2}}{2}m_n < \frac{\sqrt{10}+\sqrt{5}}{2\sqrt{3}} \cdot \sqrt{n}, \tag{2.4}$$

for $n = 4, 5, \ldots, 16$.

Finally, assume that $n > 16$. We use two lattice arrangements of circles.

There exists an integer $m \geq 4$ such that either

$$m^2 < n \leq m^2 + m \quad \text{or} \quad m^2 + m < n \leq (m+1)^2. \tag{2.5}$$

Obviously, m^2 circles of radius $\sqrt{2}/2$ can be packed into $I(\sqrt{2}m)$ (see Figure 5(a)). Moreover, m^2+m circles of radius $\sqrt{2}/2$ can be packed into $I(\sqrt{2}m+\sqrt{2}/2)$ (see Figure 5(b)); it is easy to check that

$$\left(\sqrt{3}m+2\right) \cdot \frac{\sqrt{2}}{2} < \sqrt{2}m + \frac{\sqrt{2}}{2}, \tag{2.6}$$

provided $m \geq 4$.

If $m^2 + 1 \leq n \leq m^2 + m$, then

$$s_n \leq \sqrt{2}m + \frac{\sqrt{2}}{2} \leq \sqrt{2} \cdot \sqrt{n-1} + \frac{\sqrt{2}}{2} < \frac{\sqrt{10}+\sqrt{5}}{2\sqrt{3}} \cdot \sqrt{n}. \tag{2.7}$$

If $m^2 + m + 1 \leq n \leq (m+1)^2$, then $s_n \leq \sqrt{2}(m+1)$. Since $m^2 + m + 1 \leq n$, it follows that $m \leq (1/2)\sqrt{4n-3} - 1/2$. Thus,

$$s_n \leq \sqrt{2}\left(\frac{1}{2}\sqrt{4n-3} + \frac{1}{2}\right) < \frac{\sqrt{10}+\sqrt{5}}{2\sqrt{3}} \cdot \sqrt{n}. \tag{2.8}$$

□

By Theorem 2.5, Examples 2.1 and 2.2, we conclude the following result.

Corollary 2.6. *Every collection of n unit squares admits a translative packing into any square of area $2.5\,n$, that is, $s_n \le \sqrt{2.5 \cdot n}$. Furthermore, $s_2 = \sqrt{5}$.*

For $n \ge 1980$, the following upper bound is better than the bound presented in Theorem 2.5.

Lemma 2.7. *Let n be a positive integer, and let k be the greatest integer not over $\sqrt[4]{n}$. Then*

$$s_n \le \left(1 + \frac{\pi}{2k}\right)\left[\sqrt{2}(1+k) + \sqrt{n + 2k^2 - 4k + 2}\right]. \tag{2.9}$$

Proof. Assume that $S(\alpha_1), \ldots, S(\alpha_n)$ is a collection of n unit squares. Let k be the greatest integer not over $\sqrt[4]{n}$, let $\eta = \pi/2k$, and put

$$\zeta = (1+\eta)\left[\sqrt{2}(1+k) + \sqrt{n + 2k^2 - 4k + 2}\right]. \tag{2.10}$$

For each $i \in \{1, \ldots, n\}$, there exists $j \in \{1, 2, \ldots, k\}$ such that

$$(j-1)\eta \le \alpha_i < j\eta. \tag{2.11}$$

Put $\varphi = \alpha_i - (j-1)\eta$. We have $0 \le \varphi < \eta$. Moreover, let λ_3 and λ_4 denote the lengths of the segments presented in Figure 6(a). Since

$$\lambda_3 + \lambda_4 = \cos\varphi + \sin\varphi \le 1 + \varphi < 1 + \eta, \tag{2.12}$$

it follows that $S(\alpha_i)$ is contained in a square P_i with side length $1+\eta$ and with the angle between the x-axis and a side of P_i equal to $(j-1)\eta$. We say that P_i is a *j-square*.

To prove Lemma 2.7, it suffices to show that $P_1, \ldots, P_n$ can be translatively packed into $I(\zeta)$. Denote by A_j the total area of the j-squares, for $j = 1, \ldots, k$. Obviously,

$$\sum_{j=1}^{k} A_j = n(1+\eta)^2. \tag{2.13}$$

Put

$$h_j = \frac{A_j}{\zeta - 2\sqrt{2}(1+\eta)} + 2\sqrt{2}(1+\eta), \tag{2.14}$$

for $j = 1, \ldots, k$. Observe that

$$\sum_{j=1}^{k} h_j = \frac{A_1 + \cdots + A_k}{\zeta - 2\sqrt{2}(1+\eta)} + 2k\sqrt{2}(1+\eta). \tag{2.15}$$

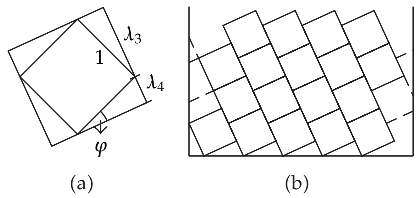


Figure 6

Thus,

$$\sum_{j=1}^{k} h_j = \frac{n(1+\eta)^2}{\zeta - 2\sqrt{2}(1+\eta)} + 2k\sqrt{2}(1+\eta). \tag{2.16}$$

The equation

$$\frac{n(1+\eta)^2}{x - 2\sqrt{2}(1+\eta)} + 2k\sqrt{2}(1+\eta) = x \tag{2.17}$$

is equivalent to

$$x^2 - x \cdot 2\sqrt{2}(1+\eta)(k+1) + (8k-n)(1+\eta)^2 = 0. \tag{2.18}$$

It is easy to verify that $x = \zeta$ is a solution of this equation. Consequently,

$$\sum_{j=1}^{k} h_j = \frac{n(1+\eta)^2}{\zeta - 2\sqrt{2}(1+\eta)} + 2k\sqrt{2}(1+\eta) = \zeta. \tag{2.19}$$

We divide $I(\zeta)$ into k rectangles $R_1, \ldots, R_k$, where R_j is a rectangle of side lengths ζ and h_j. Since the diagonal of each P_i equals $\sqrt{2}(1+\eta)$ and

$$A_j = \left[\zeta - 2\sqrt{2}(1+\eta)\right]\left[h_j - 2\sqrt{2}(1+\eta)\right], \tag{2.20}$$

it follows that all j-squares admit a translative packing into R_j for $j = 1, \ldots, k$ (see Figure 6(b)). Hence, $P_1, \ldots, P_n$, and consequently $S(\alpha_1), \ldots, S(\alpha_n)$ can be translatively packed into $I(\zeta)$. This implies that $s_n \le \zeta$. □

Theorem 2.8. *Let n be a positive integer. Then*

$$s_n \le \sqrt{n} + \left(\sqrt{2} + \frac{\pi}{2}\right)\sqrt[4]{n} + O(1), \tag{2.21}$$

as $n \to \infty$.

Proof. Let k be the greatest integer not over $\sqrt[4]{n}$. Since

$$\sqrt{n + 2k^2 - 4k + 2} < \sqrt{n + 2k^2} \le \sqrt{n + 2\sqrt{n}} < \sqrt{n} + 1, \tag{2.22}$$

by Lemma 2.7, it follows that, for $n > 1$,

$$\begin{aligned} s_n &< \left(1 + \frac{\pi}{2\sqrt[4]{n} - 2}\right)\left[\sqrt{2}(1 + \sqrt[4]{n}) + \sqrt{n} + 1\right] \\ &= \sqrt{n} + \left(\sqrt{2} + \frac{\pi}{2}\right)\sqrt[4]{n} + \left(\sqrt{2} + 1\right)\left(\frac{\pi}{2} + 1\right) + \frac{\pi + \pi\sqrt{2}}{\sqrt[4]{n} - 1}. \end{aligned} \tag{2.23}$$

□

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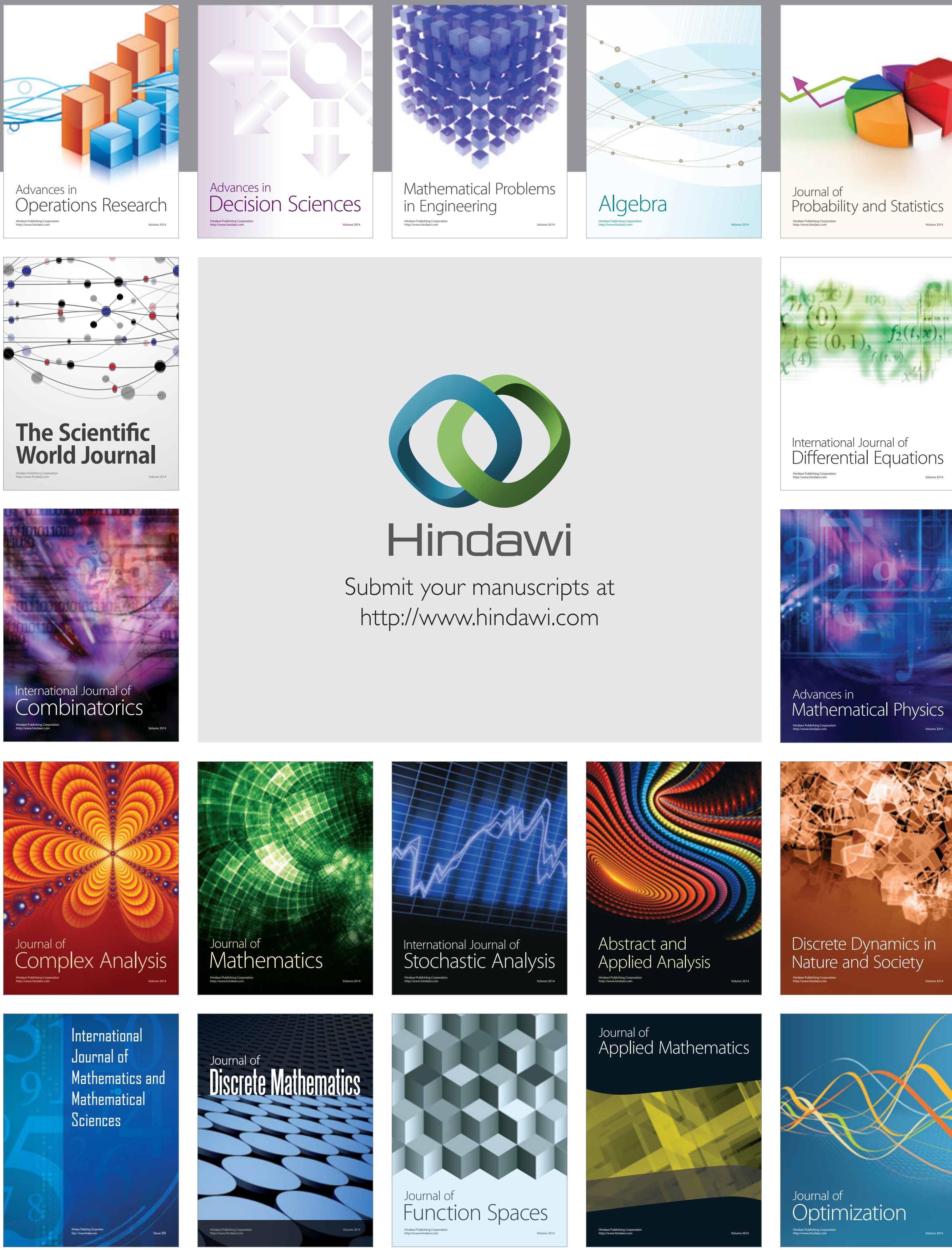
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