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Research Article

Localization and Perturbations of Roots to Systems of Polynomial Equations

Michael Gil'

Department of Mathematics, Ben Gurion University of the Negev, P.O. Box 653, Beer-Sheva 84105, Israel

Correspondence should be addressed to Michael Gil', gilmi@bezeqint.net

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We establish estimates for the sums of absolute values of solutions of a zero-dimensional polynomial system. By these estimates, inequalities for the counting function of the roots are derived. In addition, bounds for the roots of perturbed systems are suggested.

1. Introduction and Statements of the Main Results

Let us consider the system:

$$f(x,y) = g(x,y) = 0, \tag{1.1}$$

where

$$\begin{aligned} f(x,y) &= \sum_{j=0}^{m_1}\sum_{k=0}^{n_1} a_{jk}x^{m_1-j}y^{n_1-k} \quad (a_{m_1n_1} \neq 0), \\ g(x,y) &= \sum_{j=0}^{m_2}\sum_{k=0}^{n_2} b_{jk}x^{m_2-j}y^{n_2-k} \quad (b_{m_2n_2} \neq 0). \end{aligned} \tag{1.2}$$

The coefficients a_{jk}, b_{jk} are complex numbers.

The classical Bezout and Bernstein theorems give us bounds for the total number of solutions of a polynomial system, compared to [1, 2]. But for many applications, it is very important to know the number of solutions in a given domain. In the present paper we establish estimates for sums of absolute values of the roots of (1.1). By these estimates, bounds for the number of solutions in a given disk are suggested. In addition, we discuss perturbations of system (1.1). Besides, bounds for the roots of a perturbed system are suggested.

We use the approach based on the resultant formulations, which has a long history; the literature devoted to this approach is very rich, compared to [1, 3, 4]. We combine it with the recent estimates for the eigenvalues of matrices and zeros of polynomials. The problem of solving polynomial systems and systems of transcedental equations continues to attract the attention of many specialists despite its long history. It is still one of the burning problems of algebra, because of the absence of its complete solution, compared to the very interesting recent investigations [2, 5–8] and references therein. Of course we could not survey the whole subject here.

A pair of complex numbers $(\tilde{x}, \tilde{y})$ is a solution of (1.1) if $f(\tilde{x}, \tilde{y}) = g(\tilde{x}, \tilde{y}) = 0$. Besides $\tilde{x}$ will be called an X-root coordinate (corresponding to $\tilde{y}$) and $\tilde{y}$ a Y-root coordinate (corresponding to $\tilde{x}$). All the considered roots are counted with their multiplicities.

Put

$$a_j(y) = \sum_{k=0}^{n_1} a_{jk} y^{n_1-k} \quad (j = 0, \ldots, m_1), \qquad b_j(y) = \sum_{k=0}^{n_2} b_{jk} y^{n_2-k} \quad (j = 0, \ldots, m_2). \tag{1.3}$$

Then

$$\begin{aligned} f(x,y) &= \sum_{j=0}^{m_1} a_j(y) x^{m_1-j}, \\ g(x,y) &= \sum_{j=0}^{m_2} b_j(y) x^{m_2-j}. \end{aligned} \tag{1.4}$$

With $m = m_1 + m_2$ introduce the $m \times m$ Sylvester matrix

$$S(y) = \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_{m_1-1} & a_{m_1} & 0 & 0 & \cdots & 0 \\ 0 & a_0 & a_1 & \cdots & a_{m_1-2} & a_{m_1-1} & a_{m_1} & 0 & \cdots & 0 \\ . & . & . & \cdots & . & . & . & . & \cdots & . \\ 0 & 0 & 0 & \cdots & a_0 & a_1 & a_2 & a_3 & \cdots & a_{m_1} \\ b_0 & b_1 & b_2 & \cdots & b_{m_2-1} & b_{m_2} & 0 & 0 & \cdots & 0 \\ 0 & b_0 & b_1 & \cdots & b_{m_2-2} & b_{m_2-1} & b_{m_2} & 0 & \cdots & 0 \\ . & . & . & . & . & . & . & . & \cdots & . \\ 0 & 0 & 0 & \cdots & b_0 & b_1 & b_2 & b_3 & \cdots & b_{m_2} \end{pmatrix}, \tag{1.5}$$

with $a_k = a_k(y)$ and $b_k = b_k(y)$. Put $R(y) = \det S(y)$ and consider the expansion:

$$R(y) = \sum_{k=0}^{n} R_k y^{n-k}, \quad \text{where } n := \deg R(y). \tag{1.6}$$

Furthermore, denote

$$\theta(R) := \left[\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{R(e^{it})}{R_0}\right|^2 dt - 1\right]^{1/2}. \tag{1.7}$$

Clearly,

$$\theta(R) \leq \sup_{|y|=1}\left|\frac{R(y)}{R_0}\right|, \qquad R_0 = \lim_{y\to\infty}\frac{R(y)}{y^n}, \; R_0 = \frac{1}{2\pi}\int_0^{2\pi} e^{-int}R\left(e^{it}\right)dt. \tag{1.8}$$

Thanks to the Hadamard inequality, we have

$$\left|R(y)\right|^2 \leq \left(\sum_{k=0}^{m_1}\left|a_k(y)\right|^2\right)^{m_2}\left(\sum_{k=0}^{m_2}\left|b_k(y)\right|^2\right)^{m_1}. \tag{1.9}$$

Assume that

$$R_n \neq 0. \tag{1.10}$$

Theorem 1.1. *The Y-root coordinates y_k of* (1.1) *(if, they exist), taken with the multiplicities and ordered in the decreasing way: $|y_k| \geq |y_{k+1}|$, satisfy the estimates:*

$$\sum_{k=1}^{j}\left|y_k\right| < \theta(R+1) + j \quad \left(j = 1, 2, \ldots, n\right). \tag{1.11}$$

If, in addition, condition (1.10) *holds, then*

$$\sum_{k=1}^{j}\frac{1}{\left|\tilde{y}_k\right|} < \frac{R_0(\theta(R)+1)}{R_n} + j \quad \left(j = 1, 2, \ldots, n\right), \tag{1.12}$$

where $\tilde{y}_k$ are the Y-root coordinates of (1.1) *taken with the multiplicities and ordered in the increasing way: $|\tilde{y}_k| \leq |\tilde{y}_{k+1}|$.*

This theorem and the next one are proved in the next section. Note that another bound for $\sum_{k=1}^{j}|y_k|$ is derived in [9, Theorem 11.9.1]; besides, in the mentioned theorem $a_j(\cdot)$ and $b_j(\cdot)$ have the sense different from the one accepted in this paper.

From (1.12) it follows that

$$\min_k y_k > \frac{R_n}{R_0(\theta(R)+1) + R_n}. \tag{1.13}$$

So the disc $|y| \leq R_n/(R_0(\theta(R)+1) + R_n)$ is zero free.

To estimate the X-root coordinates, assume that

$$\inf_{|y|\le\theta(R)+1} |a_0(y)| > 0 \tag{1.14}$$

and put

$$\psi_f = \sup_{|y|=\theta(R)+1} \frac{1}{|a_0(y)|}\left[\sum_{j=1}^{m_1} |a_j(y)|^2\right]^{1/2}. \tag{1.15}$$

Theorem 1.2. *Let condition* (1.14) *holds. Then the X-root coordinates $x_k(y_0)$ of* (1.1) *corresponding to a Y-root coordinate y_0 (if they exist), taken with the multiplicities and ordered in the decreasing way, satisfy the estimates:*

$$\sum_{k=1}^{j} |x_k(y_0)| < \psi_f + j \quad (j = 1, 2, \ldots, \min\{m_1, m_2\}). \tag{1.16}$$

In Theorem 1.2 one can replace f by g.

Furthermore, since y_k are ordered in the decreasing way, by Theorem 1.1 we get $j|y_j| < j + \theta(R)$ and

$$|y_j| < r_j := 1 + \frac{\theta(R)}{j} \quad (j = 1, \ldots, n). \tag{1.17}$$

Thus (1.1) has in the disc $\{z \in \mathbb{C} : |z| \le r_j\}$ no more than $n-j$ Y-root coordinates. If we denote by $\nu_Y(r)$ the number of Y-root coordinates of (1.1) in $\Omega_r := \{z \in \mathbb{C} : |z| \le r\}$ for a positive number r, then we get

Corollary 1.3. *Under condition* (1.10)*, the inequality $\nu_Y(r) \le n - j + 1$ is valid for any*

$$r \le 1 + \frac{\theta(R)}{j}. \tag{1.18}$$

Similarly, by Theorem 1.2, we get the inequality:

$$|x_j(y_0)| \le 1 + \frac{\psi_f}{j} \quad (j = 1, 2, \ldots, \min\{m_1, m_2\}). \tag{1.19}$$

Denote by $\nu_X(y_0, r)$ the number of X-root coordinates of (1.1) in Ω_r, corresponding to a Y coordinate y_0.

Corollary 1.4. *Under conditions* (1.14)*, for any Y-root coordinate y_0, the inequality:*

$$\nu_X(y_0, r) \le \min\{m_1, m_2\} - j + 1 \quad (j = 1, 2, \ldots, \min\{m_1, m_2\}) \tag{1.20}$$

is valid, provided

$$r \le 1 + \frac{\psi_f}{j}. \tag{1.21}$$

In this corollary also one can replace f by g.

2. Proofs of Theorems 1.1 and 1.2

First, we need the following result.

Lemma 2.1. *Let $P(z) := z^n + c_1 z^{n-1} + \cdots + c_n$ be a polynomial with complex coefficients. Then its roots $z_k(P)$ ordered in the decreasing way satisfy the inequalities:*

$$\sum_{k=1}^{j} |z_k(P)| \le \left[\frac{1}{2\pi}\int_0^{2\pi} \left|P\left(e^{it}\right)\right|^2 dt - 1\right]^{1/2} + j \quad (j = 1, \dots, n). \tag{2.1}$$

Proof. As it is proved in [9, Theorem 4.3.1] (see also [10]),

$$\sum_{k=1}^{j} |z_k(P)| \le \left[\sum_{k=1}^{n} |c_k|^2\right]^{1/2} + j. \tag{2.2}$$

But thanks to the Parseval equality, we have

$$\sum_{k=1}^{n} |c_k|^2 + 1 = \frac{1}{2\pi}\int_0^{2\pi} \left|P\left(e^{it}\right)\right|^2 dt. \tag{2.3}$$

Hence the required result follows. □

Proof of Theorem 1.1. The bound (1.11) follows from the previous lemma with $P(y) = R(y)/R_0$. To derive bound (1.12) note that

$$R(y) = R_n y^n W\left(\frac{1}{y}\right), \quad \text{where } W(z) = \frac{1}{R_n}\sum_{k=0}^{n} R_k z^k = \sum_{k=0}^{n} d_k z^{n-k}, \tag{2.4}$$

with $d_k = R_{n-k}/R_n$. So any zero $z(W)$ of $W(z)$ is equal to $1/\tilde{y}$, where $\tilde{y}$ is a zero of $R(y)$. By the previous lemma

$$\sum_{k=1}^{j} |z_k(W)| < \left[\frac{1}{2\pi}\int_0^{2\pi} \left|W\left(e^{it}\right)\right|^2 dt - 1\right]^{1/2} + j \quad (j = 1, \dots, n). \tag{2.5}$$

Thus,

$$\sum_{k=1}^{j}\frac{1}{|\tilde{y}_k|} < \left[\frac{1}{2\pi}\int_0^{2\pi}\left|W\left(e^{it}\right)\right|^2 dt - 1\right]^{1/2} + j \quad (j = 1, \ldots, n). \tag{2.6}$$

But $W(z) = z^n R(1/z)/R_n$. Thus, $|W(e^{it})| = (1/R_n)|R(e^{-it})|$. This proves the required result. □

Proof of Theorem 1.2. Due to Theorem 1.1, for any fixed Y-root coordinate y_0 we have the inequality:

$$|y_0| \leq \theta(R) + 1. \tag{2.7}$$

We seek the zeros of the polynomial:

$$f(x, y_0) = \sum_{j=0}^{m_1} a_j(y_0) x^{m_1 - j}. \tag{2.8}$$

Besides, due to (1.14) and (2.7), $a_0(y_0) \neq 0$. Put

$$Q(x) = \frac{f(x, y_0)}{a_0(y_0)}, \qquad \theta_Q := \frac{1}{|a_0(y_0)|}\left[\sum_{j=1}^{m_1}|a_j(y_0)|^2\right]^{1/2}. \tag{2.9}$$

Clearly,

$$Q(x) = x^{m_1} + \frac{1}{a_0(y_0)}\sum_{j=1}^{m_1} a_j(y_0) x^{m_1 - j}. \tag{2.10}$$

Due to the above mentioned [9, Theorem 4.3.1] we have

$$\sum_{k=1}^{j}|x_k(y_0)| < \theta_Q + j \quad (j = 1, 2, \ldots, n). \tag{2.11}$$

But according to (2.7), $\theta_Q \leq \psi_f$. This proves the theorem. □

3. Perturbations of Roots

Together with (1.1), let us consider the coupled system:

$$\hat{f}(x, y) = \hat{g}(x, y) = 0, \tag{3.1}$$

where

$$\widehat{f}(x,y) = \sum_{j=0}^{m_1}\sum_{k=0}^{n_1}\widehat{a}_{jk}x^{m_1-j}y^{n_1-k},$$
$$\widehat{g}(x,y) = \sum_{j=0}^{m_2}\sum_{k=0}^{n_2}\widehat{b}_{jk}x^{m_2-j}y^{n_2-k}. \tag{3.2}$$

Here $\widehat{a}_{jk}, \widehat{b}_{jk}$ are complex coefficients. Put

$$\widehat{a}_j(y) = \sum_{k=1}^{n_1}\widehat{a}_{jk}y^{n_1-k} \quad (j = 0,\dots,m_1),$$
$$\widehat{b}_j(y) = \sum_{k=0}^{n_2}\widehat{b}_{jk}y^{n_2-k} \quad (j = 0,\dots,m_2). \tag{3.3}$$

Let $\widehat{S}(y)$ be the Sylvester matrix defined as above with $\widehat{a}_j(y)$ instead of $a_j(y)$, and $\widehat{b}_j(y)$ instead of $b_j(y)$, and put $\widehat{R}(y) = \det\widehat{S}(y)$. It is assumed that

$$\deg\widehat{R}(y) = \deg R(y) = n. \tag{3.4}$$

Consider the expansion:

$$\widehat{R}(y) = \sum_{k=0}^{n}\widehat{R}_k y^{n-k}. \tag{3.5}$$

Due to (3.4), $\widehat{R}_0 \neq 0$. Denote

$$q\left(R,\widehat{R}\right) := \left[\frac{1}{2\pi}\int_0^{2\pi}\left|\frac{R(e^{it})}{R_0} - \frac{\widehat{R}(e^{it})}{\widehat{R}_0}\right|^2 dt\right]^{1/2},$$
$$\eta(R) = \left[\theta^2(R) + n - 1\right]^{1/2}. \tag{3.6}$$

Clearly,

$$q\left(R,\widehat{R}\right) \leq \max_{|z|=1}\left|\frac{R(y)}{R_0} - \frac{\widehat{R}(y)}{\widehat{R}_0}\right|. \tag{3.7}$$

Theorem 3.1. *Under condition* (3.4), *for any Y-root coordinate $\widehat{y}_0$ of* (3.1), *there is a Y-root coordinate y_0 of* (1.1), *such that*

$$\left|y_0 - \widehat{y}_0\right| \leq \mu_R, \tag{3.8}$$

where μ_R is the unique positive root of the equation:

$$y^n = q\left(R, \widehat{R}\right)\sum_{k=0}^{n-1}\frac{\eta^k(R)y^{n-k-1}}{\sqrt{k!}}. \tag{3.9}$$

To prove Theorem 3.1, for a finite integer n, consider the polynomials:

$$P(\lambda) = \sum_{k=0}^{n} c_k\lambda^{n-k}, \quad \widehat{P}(\lambda) = \sum_{k=0}^{n}\widehat{c}_k\lambda^{n-k} \quad (c_0 = \widehat{c}_0 = 1), \tag{3.10}$$

with complex coefficients. Put

$$\begin{aligned} q_0 &= \left[\sum_{k=1}^{n}|c_k - \widehat{c}_k|^2\right]^{1/2}, \\ \eta(P) &= \left[\sum_{k=1}^{n}|c_k|^2 + n - 1\right]^{1/2}. \end{aligned} \tag{3.11}$$

Lemma 3.2. *For any root $z(\widehat{P})$ of $\widehat{P}(y)$, there is a root $z(P)$ of $P(y)$, such that $|z(P) - z(\widehat{P})| \le r(q_0)$, where $r(q_0)$ is the unique positive root of the equation*

$$y^n = q_0\sum_{k=0}^{n-1}\frac{\eta^k(P)y^{n-k-1}}{\sqrt{k!}}. \tag{3.12}$$

This result is due to Theorem 4.9.1 from the book [9] and inequality (9.2) on page 103 of that book.

By the Parseval equality we have

$$\begin{aligned} \sum_{k=0}^{n}|c_k|^2 &= \frac{1}{2\pi}\int_0^{2\pi}\left|P\left(e^{it}\right)\right|^2 dt, \\ q_0^2 = \frac{1}{2\pi}\int_0^{2\pi}\left|P\left(e^{it}\right) - \widetilde{P}\left(e^{it}\right)\right|^2 dt &\le \max_{|z|=1}\left|P(y) - \widetilde{P}(y)\right|^2. \end{aligned} \tag{3.13}$$

Thus

$$\eta^2(P) = \frac{1}{2\pi}\int_0^{2\pi}\left|P\left(e^{it}\right)\right|^2 dt + n - 2 \le \max_{|z|=1}\left|P(y)\right|^2 + n - 2. \tag{3.14}$$

The assertion of Theorem 3.1 now follows from in the previous lemma with $P(y) = R(y)/R_0$ and$\widetilde{P}(y) = \widehat{R}(y)/\widehat{R}_0$.

Further more, denote

$$p_R := q\left(R, \widehat{R}\right)\sum_{k=0}^{n-1}\frac{\eta^k(R)}{\sqrt{k!}},$$
$$\delta_R := \begin{cases} \sqrt[n]{p_R} & \text{if } p_R \le 1, \\ p_R & \text{if } p_R > 1. \end{cases} \tag{3.15}$$

Due to Lemma 1.6.1 from [11], the inequality $\mu_R \le \delta_R$ is valid. Now Theorem 3.1 implies the following.

Corollary 3.3. *Under condition* (3.4)*, for any* Y*-root coordinate* $\widehat{y}_0$ *of* (3.1)*, there is a* Y*-root coordinate* y_0 *of* (1.1)*, such that* $|y_0 - \widehat{y}_0| \le \delta_R$*.*

Similarly, one can consider perturbations of the X-root root coordinates.

To evaluate the quantity $R(y) - \widehat{R}(y)$ one can use the following result: let A and B two complex $n \times n$-matrices. Then

$$|\det(A) - \det(B)| \le \frac{N_2(A-B)}{n^{n/2}}\left(1 + \frac{1}{2}(N_2(A+B) + N_2(A-B))\right)^n, \tag{3.16}$$

where $N_2^2(A) = \text{Trace}A^*A$ is the Hilbert-Schmidt norm and A^* is the adjoint to A. For the proof of this inequality see [12]. Taking $B = \widehat{S}(y)$, $A = S(y)$ we get a bound for $|R(z) - \widehat{R}(z)|$.

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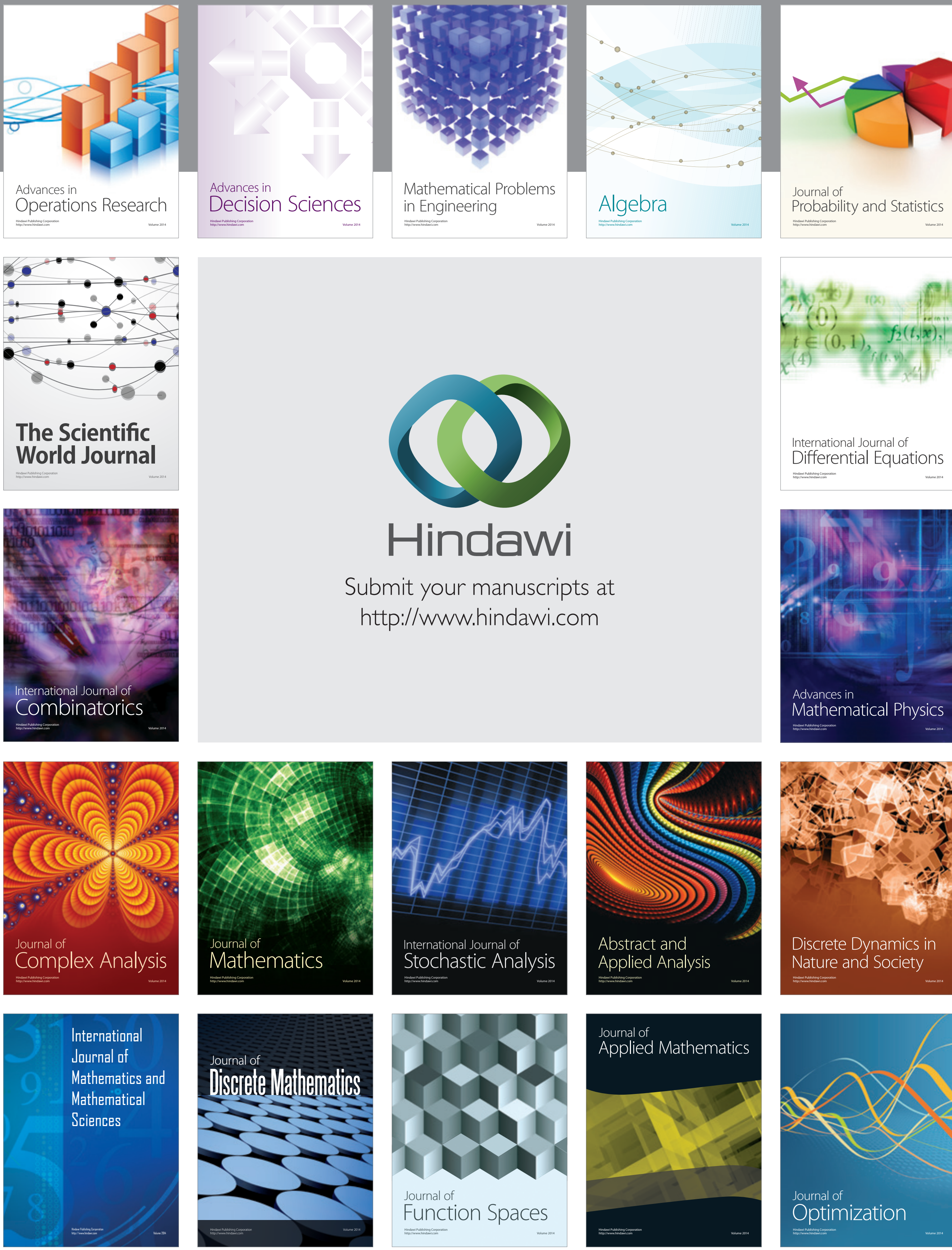

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