Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2012, Article ID 675094, 11 pages
doi:10.1155/2012/675094

Research Article

Property P and Some Fixed Point Results on (ψ, ϕ)-Weakly Contractive G-Metric Spaces

Mona Khandaqji,[1] Sharifa Al-Sharif,[2] and Mohammad Al-Khaleel[2]

[1] *Department of Mathematics, Hashemite University, P.O. Box 150459, Zarqa 13115, Jordan*
[2] *Department of Mathematics, Yarmouk University, P.O. Box 566, Irbid 21163, Jordan*

Correspondence should be addressed to Mona Khandaqji, mkhansun@yahoo.com

Received 22 March 2012; Accepted 6 June 2012

Academic Editor: Charles Chidume

We prove some fixed point results for (ψ, ϕ)-weakly contractive maps in G-metric spaces, we show that these maps satisfy property P. The results presented in this paper generalize several well-known comparable results in the literature.

1. Introduction

Metric fixed point theory is an important mathematical discipline because of its applications in areas such as variational and linear inequalities, optimization, and approximation theory. Generalizations of metric spaces were proposed by Gähler [1, 2] (called 2 metricspaces). In 2005, Mustafa and Sims [3] introduced a new structure of generalized metric spaces, which are called G-metric spaces as a generalization of metric space (X, d), to develop and introduce a new fixed point theory for various mappings in this new structure. Many papers dealing with fixed point theorems for mappings satisfying different contractive conditions on G-metric spaces can be found in [4–16]. Let T be a self-map of a complete metric space (X, d) with a nonempty fixed point set $F(T)$. Then T is said to satisfy property P if $F(T) = F(T^n)$ for each $n \in N$. However, the converse is false. For example, consider $X = [0, 1]$ and T defined by $Tx = 1 - x$. Then T has a unique fixed point at $x = 1/2$, but every even iterate of T is the identity map, which has every point of $[0, 1]$ as a fixed point. On the other hand, if $X = [0, \pi]$, $Tx = \cos x$, then every iterate of T has the same fixed point as T (see [17, 18]). Jeong and Rhoades [17] showed that maps satisfying many contractive conditions have property P. An interesting fact about maps satisfying property P is that they have no nontrivial periodic points. Papers dealing with property P are those in [17–19]. In this paper, we will prove some general fixed point theorems for (ψ, ϕ)-weakly contractive maps in G-metric spaces, and then we show that these maps satisfy property P.

Now we give first in what follows preliminaries and basic definitions which will be used throughout the paper.

2. Preliminaries

Consistent with Mustafa and Sims [3], the following definitions and results will be needed in the sequel.

Definition 2.1. Let X be a nonempty set and $G: X \times X \times X \rightarrow R^{+}$ satisfy the following properties:

(G1) $G\ (x, y, z) = 0$ if $x = y = z = 0$ (coincidence),

(G2) $0 < G\ (x, x, y)$, for all $x, y \in X$, where $x \neq y$,

(G3) $G\ (x, x, y) \leq G(x, y, z)$, for all $x, y, z \in X$, with $z \neq y$,

(G4) $G\ (x, y, z) = G(p\{x, y, z\})$, where p is a permutation of x, y, z (symmetry),

(G5) $G\ (x, y, z) \leq G(x, a, a) + G(a, y, z)$, for all $x, y, z, a \in X$ (rectangle inequality).

Then, the function G is called the G-metric on X, and the pair (X, G) is called the G-metric space.

Definition 2.2. A G-metric is said to be symmetric if $G(x, y, y) = G(y, x, x)$ for all $x, y \in X$.

Proposition 2.3. *Every G-metric space (X, G) will define a metric space (X, d_G) by $d_G(x, y) = G(x, y, y) + G(y, x, x)$, for all $x, y \in X$.*

Definition 2.4. Let (X, G) be a G-metric space, and (x_n) be a sequence of points in X. Then,

(i) a point $x \in X$ is said to be the limit of the sequence (x_n) if

$$G(x_n, x_m, x) \longrightarrow 0, \quad (\text{as } n, m \longrightarrow \infty), \tag{2.1}$$

and we say that the sequence (x_n) is G convergent to x (we say $x_n \overset{(G)}{\rightarrow} x$),

(ii) A sequence (x_n) is said to be G-Cauchy if

$$G(x_n, x_m, x_l) \longrightarrow 0, \quad (\text{as } n, m, l \longrightarrow \infty), \tag{2.2}$$

(iii) (X, G) is called a complete G-metric space if every G-Cauchy sequence in X is G converge in X.

Proposition 2.5. *Let (X, G) be a G-metric space, then the following are equivalent:*

(1) $x_n \overset{(G)}{\rightarrow} x$,

(2) $G(x_n, x_n, x) \rightarrow 0$ (*as* $n \rightarrow \infty$),

(3) $G(x_n, x, x) \rightarrow 0$ (*as* $n \rightarrow \infty$),

(4) $G(x_n, x_m, x) \rightarrow 0$ (*as* $n \rightarrow \infty$).

Proposition 2.6. *Let (X, G) be a G-metric space, then the following are equivalent:*

(1) (x_n) *is be G-Cauchy in* X,

(2) $G(x_n, x_m, x_m) \to 0$ (*as* $n, m \to \infty$).

Proposition 2.7. *Let (X, G) be a G-metric space. Then, for any $x, y, z, a \in X$, it follows that:*

(i) *If* $G(x, y, z) = 0$, *then* $x = y = z = 0$,

(ii) $G(x, y, z) \leq G(x, x, y) + G(x, x, z)$,

(iii) $G(x, x, y) \leq 2G(y, x, x)$,

(iv) $G(x, y, z) \leq G(x, a, z) + G(a, y, z)$,

(v) $G(x, y, z) \leq G(x, a, a) + G(y, a, a) + G(z, a, a)$.

3. Main Results

Throughout the paper, N denotes the set of all natural numbers.

Definition 3.1 (see [20]). A function $\psi : [0, \infty) \to [0, \infty)$ is called altering distance if the following properties are satisfied:

(1) ψ is continuous and increasing,

(2) $\psi(t) = 0$ if and only if $t = 0$.

The altering distance functions alter the metric distance between two points and enable us to deal with relatively new classes of fixed points problems.

Theorem 3.2. *Let (X, G) be a complete G-metric space. Let f be a self-map on X satisfying the following:*

$$\begin{aligned}
&\psi(G(fx, fy, fz)) \\
&\quad \leq \psi\left(\max\left\{\begin{array}{c} G(x,y,z), G(x,fx,fx), G(y,fy,fy), G(z,fz,fz), \\ \alpha G(fx,fx,y) + (1-\alpha)(G(fy,fy,z)), \\ \beta G(x,fx,fx) + (1-\beta)(G(y,fy,fy)), \end{array}\right\}\right) \\
&\qquad - \phi\left(\max\left\{\begin{array}{c} G(x,y,z), G(x,fx,fx), G(y,fy,fy), G(z,fz,fz), \\ \alpha G(fx,fx,y) + (1-\alpha)(G(fy,fy,z)), \\ \beta G(x,fx,fx) + (1-\beta)(G(y,fy,fy)), \end{array}\right\}\right),
\end{aligned} \tag{3.1}$$

for all $x, y, z \in X$, where $0 < \alpha, \beta < 1$, ψ is an altering distance function, and $\phi : [0, \infty) \to [0, \infty)$ is a continuous function with $\phi(t) = 0$ if and only if $t = 0$. Then, f has a unique fixed point (say u), where f is G continuous at u.

Proof. Fix $x_0 \in X$. Then construct a sequence $\{x_n\}$ by $x_{n+1} = fx_n = f^n x_0$. We may assume that $x_n \neq x_{n+1}$ for each $n \in N \cup \{0\}$. Since, if there exist $n \in N$ such that $x_n = x_{n+1}$, then x_n is a fixed point of f.

From (3.1), substituting $x \equiv x_{n-1}$, $y = z \equiv x_n$ then, for all $n \in N$,

$$\begin{aligned}
&\psi(G(x_n, x_{n+1}, x_{n+1})) \\
&\quad \leq \psi\left(\max\left\{\begin{array}{c} G(x_{n-1}, x_n, x_n), G(x_{n-1}, x_n, x_n), G(x_n, x_{n+1}, x_{n+1}), G(x_n, x_{n+1}, x_{n+1}), \\ \alpha G(x_n, x_n, x_n) + (1-\alpha)(G(x_{n+1}, x_{n+1}, x_n)), \\ \beta G(x_{n-1}, x_n, x_n) + (1-\beta)(G(x_{n+1}, x_{n+1}, x_n)) \end{array}\right\}\right) \\
&\qquad - \phi\left(\max\left\{\begin{array}{c} G(x_{n-1}, x_n, x_n), G(x_{n-1}, x_n, x_n), G(x_n, x_{n+1}, x_{n+1}), G(x_n, x_{n+1}, x_{n+1}), \\ \alpha G(x_n, x_n, x_n) + (1-\alpha)(G(x_{n+1}, x_{n+1}, x_n)), \\ \beta G(x_{n-1}, x_n, x_n) + (1-\beta)(G(x_{n+1}, x_{n+1}, x_n)) \end{array}\right\}\right) \\
&\quad \leq \psi\left(\max\left\{\begin{array}{c} G(x_{n-1}, x_n, x_n), G(x_n, x_{n+1}, x_{n+1}), \\ \beta G(x_{n-1}, x_n, x_n) + (1-\beta)(G(x_{n+1}, x_{n+1}, x_n)) \end{array}\right\}\right) \\
&\qquad - \phi\left(\max\left\{\begin{array}{c} G(x_{n-1}, x_n, x_n), G(x_n, x_{n+1}, x_{n+1}), \\ \beta G(x_{n-1}, x_n, x_n) + (1-\beta)(G(x_{n+1}, x_{n+1}, x_n)) \end{array}\right\}\right).
\end{aligned} \tag{3.2}$$

Let $M_n = \max\{G(x_{n-1}, x_n, x_n), G(x_n, x_{n+1}, x_{n+1})\}$. Then, (3.2) gives

$$\psi(G(x_n, x_{n+1}, x_{n+1})) \leq \psi(M_n) - \phi(M_n). \tag{3.3}$$

We have two cases, either $M_n = G(x_n, x_{n+1}, x_{n+1})$ or $M_n = G(x_{n-1}, x_n, x_n)$. Suppose that, for some $n \in N_0$, $M_n = G(x_n, x_{n+1}, x_{n+1})$. Then, we have

$$\psi(G(x_n, x_{n+1}, x_{n+1})) \leq \psi(G(x_n, x_{n+1}, x_{n+1})) - \phi(G(x_n, x_{n+1}, x_{n+1})). \tag{3.4}$$

Therefore, $\psi(G(x_n, x_{n+1}, x_{n+1})) = 0$. Hence $x_n = x_{n+1}$. This is a contradiction since the x_n's are distinct.

Thus, $M_n = G(x_n, x_{n+1}, x_{n+1})$, and (3.2) becomes

$$\begin{aligned}
\psi(G(x_n, x_{n+1}, x_{n+1})) &\leq \psi(G(x_{n-1}, x_n, x_n)) - \phi(G(x_{n-1}, x_n, x_n)) \\
&\leq \psi(G(x_{n-1}, x_n, x_n)).
\end{aligned} \tag{3.5}$$

But ψ is an increasing function. Thus, from (3.5), we get

$$G(x_n, x_{n+1}, x_{n+1}) \leq G(x_{n-1}, x_n, x_n), \quad \forall n \in N. \tag{3.6}$$

Therefore, $\{G(x_n, x_{n+1}, x_{n+1}), n \in N \cup \{0\}\}$ is a positive nonincreasing sequence. Hence there exists $r \geq 0$ such that

$$\lim_{n \to \infty} G(x_n, x_{n+1}, x_{n+1}) = r. \tag{3.7}$$

Letting $n \to \infty$, and using (3.5) and the continuity of ψ and ϕ, we get

$$\psi(r) \leq \psi(r) - \phi(r). \tag{3.8}$$

Hence, $\phi(r) = 0$, therefore $r = 0$, which implies that

$$\lim_{n\to\infty} G(x_n, x_{n+1}, x_{n+1}) = 0. \tag{3.9}$$

Consequently, for a given $\varepsilon > 0$, there is an integer n_0 such that

$$G(x_n, x_{n+1}, x_{n+1}) < \frac{\varepsilon}{2}, \quad \forall n > n_0. \tag{3.10}$$

For $m, n \in N$ with $m > n$, we claim that

$$G(x_n, x_m, x_m) < \frac{\varepsilon}{2}, \quad \forall m > n > n_0. \tag{3.11}$$

To show (3.11), we use induction on m. Inequality (3.11) holds for $m = n + 1$ from (3.10). Assume (3.11) holds for $m = k$, that is,

$$G(x_n, x_k, x_k) < \frac{\varepsilon}{2}, \quad \forall n > n_0. \tag{3.12}$$

For all $n > n_0$, take $m = k + 1$. Using $(G5)$ in Definition 2.1 and inequalities (3.10), (3.12), we get

$$\begin{aligned} G(x_n, x_{k+1}, x_{k+1}) &\leq G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{k+1}, x_{k+1}) \\ &\leq G(x_n, x_{n+1}, x_{n+1}) + G(x_n, x_k, x_k) < \varepsilon. \end{aligned} \tag{3.13}$$

By induction on m, we conclude that

$$G(x_n, x_m, x_m) < \frac{\varepsilon}{2}, \quad \forall m > n > n_0. \tag{3.14}$$

We conclude from Proposition 2.6 that $\{x_n\}$ is a G-Cauchy sequence in X. From the completeness of X, there exists u in X such that $x_n \stackrel{(G)}{\to} u$. For $n \in N$, we have

$$\begin{aligned}
\psi(G(fu,fu,x_n)) &= \psi(G(fu,fu,fx_{n-1})) \\
&\le \psi\left(\max\left\{\begin{matrix} G(u,u,x_{n-1}),G(u,fu,fu),G(u,fu,fu),G(x_{n-1},x_n,x_n), \\ \alpha G(fu,fu,u)+(1-\alpha)(G(fu,fu,x_{n-1})), \\ \beta G(u,fu,fu)+(1-\beta)(G(u,fu,fu)) \end{matrix}\right\}\right) \\
&\quad - \phi\left(\max\left\{\begin{matrix} G(u,u,x_{n-1}),G(u,fu,fu),G(u,fu,fu),G(x_{n-1},x_n,x_n), \\ \alpha G(fu,fu,u)+(1-\alpha)(G(fu,fu,x_{n-1})), \\ \beta G(u,fu,fu)+(1-\beta)(G(u,fu,fu)) \end{matrix}\right\}\right) \\
&\le \psi\left(\max\left\{\begin{matrix} G(u,u,x_{n-1}),G(u,fu,fu),G(x_{n-1},x_n,x_n), \\ \alpha G(fu,fu,u)+(1-\alpha)(G(fu,fu,x_{n-1})) \end{matrix}\right\}\right) \\
&\quad - \phi\left(\max\left\{\begin{matrix} G(u,u,x_{n-1}),G(u,fu,fu),G(x_{n-1},x_n,x_n), \\ \alpha G(fu,fu,u)+(1-\alpha)(G(fu,fu,x_{n-1})) \end{matrix}\right\}\right).
\end{aligned} \tag{3.15}$$

Letting $n \to \infty$, and using the fact that ψ is continuous and G is continuous on its variables, we get that $G(u,fu,fu) = 0$. Hence $fu = u$. So u is a fixed point of f. Now, to show uniqueness, let v be another fixed point of f with $v \neq u$. Therefore,

$$\begin{aligned}
\psi(G(u,u,v)) &= \psi(G(fu,fu,fv)) \\
&\le \psi\left(\max\left\{\begin{matrix} G(u,u,v),G(u,fu,fu),G(u,fu,fu),G(v,fv,fv), \\ \alpha G(fu,fu,u)+(1-\alpha)(G(fu,fu,v)), \\ \beta G(u,fu,fu)+(1-\beta)(G(v,fv,fv)) \end{matrix}\right\}\right) \\
&\quad - \phi\left(\max\left\{\begin{matrix} G(u,u,v),G(u,fu,fu),G(u,fu,fu),G(v,fv,fv), \\ \alpha G(fu,fu,u)+(1-\alpha)(G(fu,fu,v)), \\ \beta G(u,fu,fu)+(1-\beta)(G(v,fv,fv)) \end{matrix}\right\}\right) \\
&= \psi(\max\{G(u,u,v),(1-\alpha)G(u,u,v)\}) \\
&\quad - \phi(\max\{G(u,u,v),(1-\alpha)G(u,u,v)\}) \\
&= \psi(G(u,u,v)) - \phi(G(u,u,v)).
\end{aligned} \tag{3.16}$$

Hence,

$$\psi(G(u,u,v)) \le \psi(G(u,u,v)) - \phi(G(u,u,v)). \tag{3.17}$$

This implies that $\phi(G(u,u,v)) = 0$, then $G(u,u,v) = 0$ and $u = v$.

Now to show that f is G continuous at u, let $\{x_n\}$ be a sequence in X with limit u (i.e., $x_n \overset{(G)}{\to} u$). Using (3.1), we have

$$\begin{aligned}
\psi(G(fx_n,u,u)) &= \psi(G(fx_n,fu,fu)) \\
&\leq \psi\left(\max\left\{\begin{array}{c} G(x_n,u,u),G(x_n,fx_n,fx_n),G(u,fu,fu),G(u,fu,fu), \\ \alpha G(fx_n,fx_n,u)+(1-\alpha)(G(fu,fu,u)), \\ \beta G(x_n,fx_n,fx_n)+(1-\beta)(G(u,fu,fu)) \end{array}\right\}\right) \\
&\quad -\phi\left(\max\left\{\begin{array}{c} G(x_n,u,u),G(x_n,fx_n,fx_n),G(u,fu,fu),G(u,fu,fu), \\ \alpha G(fx_n,fx_n,u)+(1-\alpha)(G(fu,fu,u)), \\ \beta G(x_n,fx_n,fx_n)+(1-\beta)(G(u,fu,fu)) \end{array}\right\}\right) \\
&= \psi(\max\{G(x_n,u,u),\alpha G(fx_n,fx_n,u),\beta G(x_n,fx_n,fx_n)\}) \\
&\quad -\phi(\max\{G(x_n,u,u),\alpha G(fx_n,fx_n,u),\beta G(x_n,fx_n,fx_n)\}) \\
&\leq \psi(\max\{G(x_n,u,u),\alpha G(x_{n+1},x_{n+1},u),\beta G(x_n,x_{n+1},x_{n+1})\}) \\
&\leq \psi(\max\{G(x_n,u,u),\alpha G(x_{n+1},x_{n+1},u),\beta G(x_n,u,u)+\beta G(u,x_{n+1},x_{n+1})\}) \\
&\leq \psi(\max\{G(x_n,u,u),G(x_{n+1},x_{n+1},u),\beta G(x_n,u,u)+\beta G(u,x_{n+1},x_{n+1})\}).
\end{aligned} \tag{3.18}$$

But ψ is an increasing function, thus from (3.18), we get

$$G(fx_n,u,u) \leq \max\{G(x_n,u,u),G(x_{n+1},x_{n+1},u),\beta G(x_{n+1},u,u)+\beta G(u,x_{n+1},x_{n+1})\}. \tag{3.19}$$

Therefore, $\lim_{n\to\infty} G(fx_n,u,u)=0$. □

Corollary 3.3. *Let T be a self-map on a complete G-metric space X satisfying the following for all $x,y,z \in X$:*

$$G(fx,fy,fz) \leq \lambda \max\left\{\begin{array}{c} G(x,y,z),G(x,fx,fx),G(y,fy,fy),G(z,fz,fz), \\ \alpha G(fx,fx,y)+(1-\alpha)(G(fy,fy,z)), \\ \beta G(x,fx,fx)+(1-\beta)(G(y,fy,fy)) \end{array}\right\}, \tag{3.20}$$

where $0<\alpha,\beta,\lambda<1$, ψ is an altering distance function, and $\phi:[0,\infty)\to[0,\infty)$ is a continuous function with $\phi(t)=0$ if and only if $t=0$. Then f has a unique fixed point (say u), and f is G continuous at u.

Proof. We get the result by taking $\psi(t)=t$ and $\phi(t)=t-\lambda t$, then apply Theorem 3.2. □

Corollary 3.4. *Let (X,G) be a complete G-metric space. Let f be a self-map on X satisfying the following:*

$$G(fx,fy,fz) \leq \lambda \max \left\{ \begin{array}{c} G(x,y,z), G(x,fx,fx), G(y,fy,fy), G(z,fz,fz), \\ \frac{1}{2}(G(fx,fx,y) + (G(fy,fy,z))), \\ \frac{1}{2}(G(x,fx,fx) + G(y,fy,fy)) \end{array} \right\}, \tag{3.21}$$

for all $x,y,z \in X$ where $0 < \lambda < 1$, ψ is an altering distance function and, $\phi : [0,\infty) \to [0,\infty)$ is a continuous function with $\phi(t) = 0$ if and only if $t = 0$. Then f has a unique fixed point (say u), and f is G continuous at u.

Proof. We get the result by taking $\psi(t) = t$ and $\phi(t) = t - \lambda t$, $\alpha = \beta = 1/2$ in Theorem 3.2. □

Corollary 3.5. *Let (X,G) be a complete G-metric space. Let f be a self-map on X satisfying the following:*

$$G(fx,fy,fz) \leq \lambda \max \left\{ \begin{array}{c} G(x,y,z), G(x,fx,fx), G(y,fy,fy), G(z,fz,fz), \\ \frac{1}{3}G(fx,fx,y) + \frac{2}{3}G(fy,fy,z), \\ \frac{1}{3}G(fx,fx,x) + \frac{2}{3}G(fy,fy,y) \end{array} \right\}, \tag{3.22}$$

for all $x,y,z \in X$, where $0 < \lambda < 1$, ψ is an altering distance function, and $\phi : [0,\infty) \to [0,\infty)$ is a continuous function with $\phi(t) = 0$ if and only if $t = 0$. Then f has a unique fixed point (say u) and f is G continuous at u.

Proof. We get the result by taking $\psi(t) = t$ and $\phi(t) = t - \lambda t$, $\alpha = \beta = 1/3$ in Theorem 3.2. □

Theorem 3.6. *Under the condition of Theorem 3.2, f has property P.*

Proof. From Theorem 3.2, f has a fixed point. Therefore $F(f^n) \neq \varphi$ for each $n \in N$. Fix $n > 1$, and assume that $u \in F(f^n)$. We claim that $u \in F(f)$. To prove the claim, suppose that $u \neq fu$. Using (3.1), we have

$$\begin{aligned}
\psi(G(u,fu,fu)) &= \psi\left(G\left(f^n u, f^{n+1}u, f^{n+1}u\right)\right) = \psi\left(G\left(ff^{n-1}u, ff^n u, ff^n u\right)\right) \\
&\leq \psi\left(\max \left\{ \begin{array}{c} G(f^{n-1}u,u,u), G(u,fu,fu), \\ \alpha G(u,u,u) + (1-\alpha)(G(fu,fu,u)) \\ \beta G(f^{n-1}u,u,u) + (1-\beta)(G(u,fu,fu)) \end{array} \right\} \right) \\
&\quad - \phi\left(\max \left\{ \begin{array}{c} G(f^{n-1}u,u,u), G(u,fu,fu), \\ \alpha G(u,u,u) + (1-\alpha)(G(fu,fu,u)) \\ \beta G(f^{n-1}u,u,u) + (1-\beta)(G(u,fu,fu)) \end{array} \right\} \right)
\end{aligned}$$

$$
\begin{aligned}
&= \psi\Big(\max\Big\{G\Big(f^{n-1}u,u,u\Big),G(u,fu,fu)\Big\}\Big) \\
&\quad - \phi\Big(\max\Big\{G\Big(f^{n-1}u,u,u\Big),G(u,fu,fu)\Big\}\Big).
\end{aligned} \tag{3.23}
$$

Letting $M = \max\{G(f^{n-1}u,u,u),G(u,fu,fu)\}$, we deduce form (3.23),

$$
\psi(G(u,fu,fu)) \leq \psi(M) - \phi(M). \tag{3.24}
$$

If $M = G(u,fu,fu)$, then

$$
\psi(G(u,fu,fu)) \leq \psi(G(u,fu,fu)) - \phi(G(u,fu,fu)), \tag{3.25}
$$

hence, $\phi(G(u,fu,fu)) = 0$. By a property of ϕ, we deduce that $G(u,fu,fu) = 0$, therefore, $u = fu$. This is a contradiction. On the other hand, if $M = G(f^{n-1}u,u,u)$, then (3.1) gives that

$$
\begin{aligned}
\psi\Big(G\Big(f^{n}u,f^{n+1}u,f^{n+1}u\Big)\Big) &= \psi(G(u,fu,fu)) \\
&\leq \psi\Big(G\Big(f^{n-1}u,u,u\Big)\Big) - \phi\Big(G\Big(f^{n-1}u,u,u\Big)\Big) \\
&= \psi\Big(G\Big(f^{n-1}u,f^{n}u,f^{n}u\Big)\Big) - \phi\Big(G\Big(f^{n-1}u,f^{n}u,f^{n}u\Big)\Big) \\
&\leq \psi\Big(G\Big(f^{n-2}u,f^{n-1}u,f^{n-1}u\Big)\Big) \\
&\quad - \phi\Big(G\Big(f^{n-2}u,f^{n-1}u,f^{n-1}u\Big)\Big) - \phi\Big(G\Big(f^{n-1}u,f^{n}u,f^{n}u\Big)\Big) \\
&\leq \cdots \leq \psi(G(u,fu,fu)) - \sum_{k=0}^{n-1}\phi\Big(G\Big(f^{n-k-1}u,f^{n-k}u,f^{n-k}u\Big)\Big).
\end{aligned} \tag{3.26}
$$

Therefore,

$$
\sum_{k=0}^{n-1}\phi\Big(G\Big(f^{n-k-1}u,f^{n-k}u,f^{n-k}u\Big)\Big) = 0, \tag{3.27}
$$

which implies that $\phi(G(f^{n-k-1}u,f^{n-k}u,f^{n-k}u)) = 0$, for all $(0 \leq k \leq n-1)$. Thus, $\phi(G(u,fu,fu)) = 0$, and by a property of ϕ, we have $u = fu$. This is a contradiction.

Therefore, $u \in F(f)$, and f has property P.

$$
\text{Let } M_{\alpha,\beta}(x,y,z) = \max\left\{\begin{array}{c} G(x,y,z),G(x,fx,fx),G(y,fy,fy),G(z,fz,fz), \\ \alpha G(fx,fx,y) + (1-\alpha)(G(fy,fy,z)), \\ \beta G(x,fx,fx) + (1-\beta)(G(y,fy,fy)), \end{array}\right\}, \tag{3.28}
$$

where $\alpha,\beta \in (0,1]$. □

Example 3.7. Let $X = [0,1]$ and $G(x,y,z) = \max\{|x-y|, |y-z|, |z-x|\}$ be a G-metric on X. Define $f : X \to X$ by $f(x) = x/8$. We take $\psi(t) = t$ and $\phi(t) = 7/8t$, for $t \in [0,\infty)$ and $\alpha, \beta \in (0,1]$. So that

$$\psi(M_{\alpha,\beta}(x,y,z)) - \phi(M_{\alpha,\beta}(x,y,z)) = \frac{1}{8}M_{\alpha,\beta}(x,y,z). \tag{3.29}$$

We have

$$\begin{aligned} G(fx,fy,fz) &= \max\left\{\left|\frac{x}{8}-\frac{y}{8}\right|, \left|\frac{y}{8}-\frac{z}{8}\right|, \left|\frac{z}{8}-\frac{x}{8}\right|\right\} \\ &= \frac{1}{8}\max\{|x-y|, |y-z|, |z-x|\} \\ &= \frac{1}{8}G(x,y,z) \\ &\le \frac{1}{8}M_{\alpha,\beta}(x,y,z) \\ &= \psi(M_{\alpha,\beta}(x,y,z)) - \phi(M_{\alpha,\beta}(x,y,z)). \end{aligned} \tag{3.30}$$

4. Applications

Denote by Λ the set of functions $\lambda : [0,\infty) \to [0,\infty)$ satisfying the following hypotheses.

(1) λ is a Lebesgue integral mapping on each compact of $[0,\infty)$.

(2) For every $\varepsilon > 0$, we have $\int_0^t \lambda(s)ds > 0$.

It is an easy matter to see that the mapping $\psi : [0,\infty) \to [0,\infty)$, defined by $\psi(t) = \int_0^t \lambda(s)ds$, is an altering distance function. Now, we have the following result.

Theorem 4.1. *Let (X,G) be a complete G-metric space. Let f be a self-map on X satisfying the following:*

$$\int_0^{G(fx,fy,fz)} \lambda(s)ds \le \int_0^{M_{\alpha,\beta}(x,y,z)} \lambda(s)ds - \int_0^{M_{\alpha,\beta}(x,y,z)} \mu(s)ds, \tag{4.1}$$

where λ, $\mu \in \Lambda$ and $\alpha, \beta \in (0,1]$. Then f has a unique fixed point.

Proof. It follows from Theorem by taking $\psi(t) = \int_0^t \lambda(s)ds$ and $\phi(t) = \int_0^t \mu(s)ds$. □

Acknowledgment

The authors thank the referees for their valuable comments and suggestions.

References

[1] S. Gähler, "G-metrische Räume und ihre topologische Struktur," *Mathematische Nachrichten*, vol. 26, pp. 115–148, 1963.

[2] S. Gähler, "Zur geometric 2-metriche raume," *Revue Roumaine de Mathématiques Pures et Appliquées*, vol. 40, pp. 664–669, 1966.

[3] Z. Mustafa and B. Sims, "A new A pproach to Generalize Metric Spaces," *Journal of Nonlinear and Convex Analysis*, vol. 7, no. 2, pp. 289–297, 2006.

[4] Z. Mustafa, M. Khandagji, and W. Shatanawi, "Fixed point results on complete G-metric spaces," *Studia Scientiarum Mathematicarum Hungarica*, vol. 48, no. 3, pp. 304–319, 2011.

[5] Z. Mustafa and B. Sims, "Fixed point theorems for contractive mappings in complete G-metric spaces," *Fixed Point Theory and Applications*, vol. 2009, Article ID 917175, 10 pages, 2009.

[6] W. Shatanawi, "Fixed point theory for contractive mappings satisfying Φ-maps in G-metric spaces," *Fixed Point Theory and Applications*, vol. 2010, Article ID 181650, 9 pages, 2010.

[7] H. Aydi, B. Damjanović, B. Samet, and W. Shatanawi, "Coupled fixed point theorems for nonlinear contractions in partially ordered G-metric spaces," *Mathematical and Computer Modelling*, vol. 54, no. 9-10, pp. 2443–2450, 2011.

[8] B. S. Choudhury, "Unique fixed point theorem for weakly C-contractive mappings," *Kathmandu University Journal of Science, Engineering and Technology*, vol. 5, no. 1, pp. 6–13, 2009.

[9] W. Shatanawi, "Fixed point theorems for nonlinear weakly C-contractive mappings in metric spaces," *Mathematical and Computer Modelling*, vol. 54, no. 11-12, pp. 2816–2826, 2011.

[10] M. Abbas, T. Nazir, and S. Radenović, "Common fixed points of four maps in partially ordered metric spaces," *Applied Mathematics Letters*, vol. 24, no. 9, pp. 1520–1526, 2011.

[11] M. Abbas, T. Nazir, and S. Radenović, "Some periodic point results in generalized metric spaces," *Applied Mathematics and Computation*, vol. 217, no. 8, pp. 4094–4099, 2010.

[12] Z. Mustafa, H. Aydi, and E. Karapinar, "On common fixed points in G-metric spaces using (E, A) property," *Computers & Mathematics with Applications*. In press.

[13] N. Tahat, H. Aydi, E. Karapinar, and W. Shatanawi, "Common fixed point for single-valued and multi-valued maps satisfying a generalized contraction in G-metric spaces," *Fixed Point Theory and Applications*, vol. 2012, article 48, 2012.

[14] M. Abbas, T. Nazir, and D. Ðorić, "Common fixed point of mappings satisfying (E, A) property in generalized metric spaces," *Applied Mathematics and Computation*, vol. 218, no. 14, pp. 7665–7670, 2012.

[15] M. Abbas, S. H. Khan, and T. Nazir, "Common fixed points of R-weakly commuting maps in generalized metric space," *Fixed Point Theory and Applications*, vol. 2011, article 41, 2011.

[16] L. Gajić and Z. L. Crvenković, "A fixed point result for mappings with contractive iterate at a point in G-metric spaces," *Filomat*, vol. 25, no. 2, pp. 53–58, 2011.

[17] G. S. Jeong and B. E. Rhoades, "Maps for which $F(T) = F(T^n)$," in *Fixed Point Theory and Applications*, vol. 6, pp. 71–105, Nova Science Publishers, New York, NY, USA, 2007.

[18] G. S. Jeong and B. E. Rhoades, "More maps for which $F(T) = F(T^n)$," *Demonstratio Mathematica*, vol. 40, no. 3, pp. 671–680, 2007.

[19] B. E. Rhoades and M. Abbas, "Maps satisfying generalized contractive conditions of integral type for which $F(T) = F(T^n)$," *International Journal of Pure and Applied Mathematics*, vol. 45, no. 2, pp. 225–231, 2008.

[20] M. S. Khan, M. Swaleh, and S. Sessa, "Fixed point theorems by altering distances between the points," *Bulletin of the Australian Mathematical Society*, vol. 30, no. 1, pp. 1–9, 1984.