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*Research Article*

# Arithmetic Identities Involving Bernoulli and Euler Numbers

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The purpose of this paper is to give some arithmatic identities for the Bernoulli and Euler numbers. These identities are derived from the several $p$-adic integral equations on $\mathbb{Z}_p$.

## 1. Introduction

Let $p$ be a fixed odd prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ will denote the ring of $p$-adic rational integers, the field of $p$-adic rational numbers, and the completion of algebraic closure of $\mathbb{Q}_p$, respectively. The $p$-adic norm is normalized so that $|p|_p = 1/p$. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$.

Let $\mathrm{UD}(\mathbb{Z}_p)$ be the space of uniformly differentiable functions on $\mathbb{Z}_p$. For $f \in \mathrm{UD}(\mathbb{Z}_p)$, the bosonic $p$-adic integral on $\mathbb{Z}_p$ is defined by

$$I(f) = \int_{\mathbb{Z}_p} f(x)d\mu(x) = \lim_{N\to\infty} \sum_{x=0}^{p^N-1} f(x)\mu\left(x + p^N\mathbb{Z}_p\right) = \lim_{N\to\infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x), \tag{1.1}$$

and the fermionic $p$-adic integral on $\mathbb{Z}_p$ is defined by Kim as follows (see [1–8]):

$$I_{-1}(f) = \int_{\mathbb{Z}_p} f(x)d\mu_{-1}(x) = \lim_{N\to\infty} \sum_{x=0}^{p^N-1} f(x)(-1)^x. \tag{1.2}$$

The Euler polynomials, $E_n(x)$, are defined by the generating function as follows (see [1–16]):

$$F^E(t,x) = \frac{2}{e^t+1}e^{xt} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}. \tag{1.3}$$

In the special case, $x = 0$, $E_n(0) = E_n$ is called the $n$th Euler number.

By (1.3) and the definition of Euler numbers, we easily see that

$$E_n(x) = \sum_{l=0}^{n}\binom{n}{l}E_l x^{n-l} = (E+x)^n, \tag{1.4}$$

with the usual convention about replacing $E^l$ by $E_l$ (see [10]). Thus, by (1.3) and (1.4), we have

$$E_0 = 1, \qquad (E+1)^n + E_n = 2\delta_{0,n}, \tag{1.5}$$

where $\delta_{k,n}$ is the Kronecker symbol (see [9, 10, 17–19]).

From (1.2), we can also derive the following integral equation for the fermionic $p$-adic integral on $\mathbb{Z}_p$ as follows:

$$I_{-1}(f_1) = -I_{-1}(f) + 2f(0), \tag{1.6}$$

see [1, 2]. By (1.3) and (1.6), we get

$$\int_{\mathbb{Z}_p} e^{(x+y)t} d\mu_{-1}(y) = \frac{2}{e^t+1}e^{xt} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}. \tag{1.7}$$

Thus, by (1.7), we have

$$\int_{\mathbb{Z}_p} (x+y)^n d\mu_{-1}(y) = E_n(x), \tag{1.8}$$

see [1–8, 13–16].

The Bernoulli polynomials, $B_n(x)$, are defined by the generating function as follows:

$$F^B(t,x) = \frac{t}{e^t-1}e^{xt} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}, \tag{1.9}$$

see [18]. In the special case, $x = 0$, $B_n(0) = B_n$ is called the $n$th Bernoulli number. From (1.9) and the definition of Bernoulli numbers, we note that

$$B_n(x) = \sum_{l=0}^{n}\binom{n}{l}x^{n-l}B_l = (B+x)^n, \tag{1.10}$$

see [1–19], with the usual convention about replacing $B^l$ by $B_l$. By (1.9) and (1.10), we easily see that

$$B_0 = 1, \qquad (B+1)^n - B_n = \delta_{1,n}, \tag{1.11}$$

see [13].

From (1.1), we can derive the following integral equation on $\mathbb{Z}_p$:

$$I(f_1) = I(f) + f'(0), \tag{1.12}$$

where $f_1(x) = f(x+1)$ and $f'(0) = (df(x)/dx)|_{x=0}$.

By (1.12), we have

$$\int_{\mathbb{Z}_p} e^{(x+y)t} d\mu(y) = \frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}. \tag{1.13}$$

Thus, by (1.13), we can derive the following Witt's formula for the Bernoulli polynomials:

$$\int_{\mathbb{Z}_p} (x+y)^n d\mu(y) = B_n(x), \quad \text{for } n \in \mathbb{Z}_+. \tag{1.14}$$

In [19], it is known that for $k, m \in \mathbb{Z}_+$,

$$\sum_{j=1}^{\max\{k,m\}} \left[ \binom{k}{j} + (-1)^{j+1} \binom{m}{j} \right] \frac{B_{k+m+1-j}(x)}{k+m+1-j} = x^k (x-1)^m + \frac{(-1)^{m+1}}{(k+m+1)\binom{k+m}{k}}, \tag{1.15}$$

where $\binom{k}{j} = 0$ if $j < 0$ or $j > k$.

The purpose of this paper is to give some arithmetic identities involving Bernoulli and Euler numbers. To derive our identities, we use the properties of $p$-adic integral equations on $\mathbb{Z}_p$.

## 2. Arithmetic Identities for Bernoulli and Euler Numbers

Let us take the bosonic $p$-adic integral on $\mathbb{Z}_p$ in (1.15) as follows:

$$\begin{aligned} I_1 &= \int_{\mathbb{Z}_p} x^k (x-1)^m d\mu(x) + \frac{(-1)^{m+1}}{(k+m+1)\binom{k+m}{k}} \\ &= \sum_{l=0}^{m} \binom{m}{l} (-1)^l \int_{\mathbb{Z}_p} x^{k+m-l} d\mu(x) + \frac{(-1)^{m+1}}{(k+m+1)\binom{k+m}{k}} \\ &= \sum_{l=0}^{m} \binom{m}{l} (-1)^l B_{k+m-l} + \frac{(-1)^{m+1}}{(k+m+1)\binom{k+m}{k}}. \end{aligned} \tag{2.1}$$

On the other hand, we get

$$\begin{aligned} I_1 &= \sum_{j=1}^{\max\{k,m\}} \left[\binom{k}{j} + (-1)^{j+1}\binom{m}{j}\right] \frac{1}{k+m+1-j} \int_{\mathbb{Z}_p} B_{k+m+1-j}(x)\,d\mu(x) \\ &= \sum_{j=1}^{\max\{k,m\}} \left[\binom{k}{j} + (-1)^{j+1}\binom{m}{j}\right] \frac{1}{k+m+1-j} \\ &\quad \times \sum_{l=0}^{k+m+1-j} \binom{k+m+1-j}{l} B_{k+m+1-j-l} B_l. \end{aligned} \tag{2.2}$$

By (2.1) and (2.2), we get

$$\begin{aligned} &\sum_{j=1}^{\max\{k,m\}} \sum_{l=0}^{k+m+1-j} \frac{1}{k+m+1-j} \left[\binom{k}{j} + (-1)^{j+1}\binom{m}{j}\right] \\ &\quad \times \binom{k+m+1-j}{l} B_{k+m+1-j-l} B_l \\ &= \sum_{l=0}^{m} (-1)^l \binom{m}{l} B_{k+m-l} + \frac{(-1)^{m+1}}{(k+m+1)\binom{k+m}{k}}. \end{aligned} \tag{2.3}$$

Therefore, by (2.3), we obtain the following theorem.

**Theorem 2.1.** *For $k, m \in \mathbb{Z}_+$, one has*

$$\begin{aligned} &\sum_{j=1}^{\max\{k,m\}} \sum_{l=0}^{k+m+1-j} \frac{1}{k+m+1-j} \left[\binom{k}{j} + (-1)^{j+1}\binom{m}{j}\right] \\ &\quad \times \binom{k+m+1-j}{l} B_{k+m+1-j-l} B_l - \frac{(-1)^{m+1}}{(k+m+1)\binom{k+m}{k}} \\ &= \sum_{l=0}^{m} (-1)^l \binom{m}{l} B_{k+m-l}. \end{aligned} \tag{2.4}$$

Now we consider the fermionic $p$-adic integral on $\mathbb{Z}_p$ in (1.15) as follows:

$$\begin{aligned} I_2 &= \sum_{j=1}^{\max\{k,m\}} \left[\binom{k}{j} + (-1)^{j+1}\binom{m}{j}\right] \frac{1}{k+m+1-j} \sum_{l=0}^{k+m+1-j} \binom{k+m+1-j}{l} \\ &\quad \times B_{k+m+1-j-l} \int_{\mathbb{Z}_p} x^l \, d\mu_{-1}(x) \end{aligned}$$

$$= \sum_{j=1}^{\max\{k,m\}} \left[\binom{k}{j} + (-1)^{j+1}\binom{m}{j}\right] \frac{1}{k+m+1-j} \sum_{l=0}^{k+m+1-j} \binom{k+m+1-j}{l} \times B_{k+m+1-j-l} E_l. \tag{2.5}$$

On the other hand, we get

$$\begin{aligned} I_2 &= \sum_{l=0}^{m} (-1)^l \binom{m}{l} \int_{\mathbb{Z}_p} x^{m-l+k} d\mu_{-1}(x) + \frac{(-1)^{m+1}}{(k+m+1)\binom{k+m}{k}} \\ &= \sum_{l=0}^{m} (-1)^l \binom{m}{l} E_{k+m-l} + \frac{(-1)^{m+1}}{(k+m+1)\binom{k+m}{k}}. \end{aligned} \tag{2.6}$$

By (2.5) and (2.6), we get

$$\begin{aligned} &\sum_{j=1}^{\max\{k,m\}} \sum_{l=0}^{k+m+1-j} \frac{1}{k+m+1-j} \left[\binom{k}{j} + (-1)^{j+1}\binom{m}{j}\right] \binom{k+m+1-j}{l} \\ &\qquad \times B_{k+m+1-j-l} E_l \\ &= \sum_{l=0}^{m} (-1)^l \binom{m}{l} E_{k+m-l} + \frac{(-1)^{m+1}}{(k+m+1)\binom{k+m}{k}}. \end{aligned} \tag{2.7}$$

Therefore, by (2.7), we obtain the following theorem.

**Theorem 2.2.** *For $k, m \in \mathbb{Z}_+$, one has*

$$\begin{aligned} &\sum_{j=1}^{\max\{k,m\}} \sum_{l=0}^{k+m+1-j} \frac{1}{k+m+1-j} \left[\binom{k}{j} + (-1)^{j+1}\binom{m}{j}\right] \binom{k+m+1-j}{l} \\ &\qquad \times B_{k+m+1-j-l} E_l - \frac{(-1)^{m+1}}{(k+m+1)\binom{k+m}{k}} \\ &= \sum_{l=0}^{m} (-1)^l \binom{m}{l} E_{k+m-l}. \end{aligned} \tag{2.8}$$

Replacing $x$ by $(1-x)$ in (1.15), we have the identity:

$$\begin{aligned} &\sum_{j=1}^{\max\{k,m\}} \left[\binom{k}{j} + (-1)^{j+1}\binom{m}{j}\right] \frac{B_{k+m+1-j}(1-x)}{k+m+1-j} \\ &\qquad = (-1)^{k+m} x^m (1-x)^k + \frac{(-1)^{m+1}}{(k+m+1)\binom{k+m}{k}}. \end{aligned} \tag{2.9}$$

Let us take the bosonic $p$-adic integral on $\mathbb{Z}_p$ in (2.9) as follows:

$$\begin{aligned}
I_3 &= \sum_{j=1}^{\max\{k,m\}} \left[\binom{k}{j} + (-1)^{j+1}\binom{m}{j}\right] \frac{1}{k+m+1-j} \\
&\quad \times \sum_{l=0}^{k+m+1-j} \binom{k+m+1-j}{l} B_{k+m+1-j-l} \int_{\mathbb{Z}_p} (1-x)^l d\mu(x) \\
&= \sum_{j=1}^{\max\{k,m\}} \left[\binom{k}{j} + (-1)^{j+1}\binom{m}{j}\right] \frac{1}{k+m+1-j} \\
&\quad \times \sum_{l=0}^{k+m+1-j} \binom{k+m+1-j}{l} B_{k+m+1-j-l} B_l \\
&\quad + \sum_{j=1}^{\max\{k,m\}} \left[\binom{k}{j} + (-1)^{j+1}\binom{m}{j}\right] \frac{1}{k+m+1-j} \\
&\quad \times \sum_{l=0}^{k+m+1-j} \binom{k+m+1-j}{l} B_{k+m+1-j-l} l \\
&\quad + \sum_{j=1}^{\max\{k,m\}} \left[\binom{k}{j} + (-1)^{j+1}\binom{m}{j}\right] \frac{1}{k+m+1-j} \\
&\quad \times \sum_{l=0}^{k+m+1-j} \binom{k+m+1-j}{l} B_{k+m+1-j-l} \delta_{1,l} \\
&= \sum_{j=1}^{\max\{k,m\}} \sum_{l=0}^{k+m+1-j} \frac{1}{k+m+1-j} \left[\binom{k}{j} + (-1)^{j+1}\binom{m}{j}\right] \\
&\quad \times \binom{k+m+1-j}{l} B_{k+m+1-j-l} B_l \\
&\quad + \sum_{j=1}^{\max\{k,m\}} \left[\binom{k}{j} + (-1)^{j+1}\binom{m}{j}\right] \left(2B_{k+m-j} + \delta_{1,(k+m-j)}\right) \\
&= \sum_{j=1}^{\max\{k,m\}} \sum_{l=0}^{k+m+1-j} \frac{1}{k+m+1-j} \left[\binom{k}{j} + (-1)^{j+1}\binom{m}{j}\right] \\
&\quad \times \binom{k+m+1-j}{l} B_{k+m+1-j-l} B_l + 2 \sum_{j=1}^{\max(k,m)} \left[\binom{k}{j} + (-1)^{j+1}\binom{m}{j}\right] \\
&\quad \times B_{k+m-j} + \binom{k}{k+m-1} + (-1)^{k+m}\binom{m}{k+m-1}.
\end{aligned} \tag{2.10}$$

On the other hand, we see that

$$I_3 = (-1)^{k+m}\sum_{l=0}^{k}(-1)^l\binom{k}{l}B_{k+m-l} + \frac{(-1)^{m+1}}{(k+m+1)\binom{k+m}{k}}. \tag{2.11}$$

By (2.10) and (2.11), we get

$$\begin{aligned}
&\sum_{j=1}^{\max\{k,m\}}\sum_{l=0}^{k+m+1-j}\frac{1}{k+m+1-j}\left[\binom{k}{j}+(-1)^{j+1}\binom{m}{j}\right] \\
&\qquad \times \binom{k+m+1-j}{l}B_{k+m+1-j-l}B_l + 2\sum_{j=1}^{\max\{k,m\}}\left[\binom{k}{j}+(-1)^{j+1}\binom{m}{j}\right] \\
&\qquad \times B_{k+m-j} + \binom{k}{k+m-1} + (-1)^{k+m}\binom{m}{k+m-1} \\
&\quad = (-1)^{k+m}\sum_{l=0}^{k}(-1)^l\binom{k}{l}B_{k+m-l} + \frac{(-1)^{m+1}}{(k+m+1)\binom{k+m}{k}}.
\end{aligned} \tag{2.12}$$

Therefore, by (2.12), we obtain the following theorem.

**Theorem 2.3.** *For $k, m \in \mathbb{Z}_+$, one has*

$$\begin{aligned}
&\sum_{j=1}^{\max\{k,m\}}\sum_{l=0}^{k+m+1-j}\frac{1}{k+m+1-j}\left[\binom{k}{j}+(-1)^{j+1}\binom{m}{j}\right] \\
&\qquad \times \binom{k+m+1-j}{l}B_{k+m+1-j-l}B_l + 2\sum_{j=1}^{\max\{k,m\}}\left[\binom{k}{j}+(-1)^{j+1}\binom{m}{j}\right] \\
&\qquad \times B_{k+m-j} + \binom{k}{k+m-1} + (-1)^{k+m}\binom{m}{k+m-1} - \frac{(-1)^{m+1}}{(k+m+1)\binom{k+m}{k}} \\
&\quad = (-1)^{k+m}\sum_{l=0}^{k}(-1)^l\binom{k}{l}B_{k+m-l}.
\end{aligned} \tag{2.13}$$

We consider the fermionic $p$-adic integral on $\mathbb{Z}_p$ in (2.9) as follows:

$$\begin{aligned}
I_4 = &\sum_{j=1}^{\max\{k,m\}}\left[\binom{k}{j}+(-1)^{j+1}\binom{m}{j}\right]\frac{1}{k+m+1-j} \\
&\times \sum_{l=0}^{k+m+1-j}\binom{k+m+1-j}{l}B_{k+m+1-j-l}\int_{\mathbb{Z}_p}(1-x)^l d\mu_{-1}(x)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=1}^{\max\{k,m\}} \left[\binom{k}{j} + (-1)^{j+1}\binom{m}{j}\right] \frac{1}{k+m+1-j} \\
&\quad \times \sum_{l=0}^{k+m+1-j} \binom{k+m+1-j}{l} B_{k+m+1-j-l} E_l \\
&\quad + 2 \sum_{j=1}^{\max\{k,m\}} \left[\binom{k}{j} + (-1)^{j+1}\binom{m}{j}\right] \frac{1}{k+m+1-j} \\
&\quad \times \sum_{l=0}^{k+m+1-j} \binom{k+m+1-j}{l} B_{k+m+1-j-l} \\
&\quad - 2 \sum_{j=1}^{\max\{k,m\}} \left[\binom{k}{j} + (-1)^{j+1}\binom{m}{j}\right] \frac{1}{k+m+1-j} \\
&\quad \times \sum_{l=0}^{k+m+1-j} \binom{k+m+1-j}{l} B_{k+m+1-j-l} \delta_{0,l} \\
&= \sum_{j=1}^{\max\{k,m\}} \sum_{l=0}^{k+m+1-j} \frac{1}{k+m+1-j} \left[\binom{k}{j} + (-1)^{j+1}\binom{m}{j}\right] \\
&\quad \times \binom{k+m+1-j}{l} B_{k+m+1-j-l} E_l \\
&\quad + 2 \sum_{j=1}^{\max\{k,m\}} \frac{1}{k+m+1-j} \left[\binom{k}{j} + (-1)^{j+1}\binom{m}{j}\right] \delta_{1,(k+m+1-j)} \\
&= \sum_{j=1}^{\max\{k,m\}} \sum_{l=0}^{k+m+1-j} \frac{1}{k+m+1-j} \left[\binom{k}{j} + (-1)^{j+1}\binom{m}{j}\right] \\
&\quad \times \binom{k+m+1-j}{l} B_{k+m+1-j-l} E_l + 2\left[\binom{k}{k+m} + (-1)^{k+m+1}\binom{m}{k+m}\right].
\end{aligned} \tag{2.14}
$$

On the other hand, we get

$$
I_4 = (-1)^{k+m} \sum_{l=0}^{k} (-1)^l \binom{k}{l} E_{k+m-l} + \frac{(-1)^{m+1}}{(k+m+1)\binom{k+m}{k}}. \tag{2.15}
$$

By (2.14) and (2.15), we obtain the following theorem.

**Theorem 2.4.** *For $k, m \in \mathbb{Z}_+$, one has*

$$\sum_{j=1}^{\max\{k,m\}} \sum_{l=0}^{k+m+1-j} \frac{1}{k+m+1-j}\left[\binom{k}{j} + (-1)^{j+1}\binom{m}{j}\right]\binom{k+m+1-j}{l} \times B_{k+m+1-j-l}E_l + 2\left[\binom{k}{k+m} + (-1)^{k+m+1}\binom{m}{k+m}\right] - \frac{(-1)^{m+1}}{(k+m+1)\binom{k+m}{k}} = (-1)^{k+m}\sum_{l=0}^{k}(-1)^l\binom{k}{l}E_{k+m-l}. \quad (2.16)$$


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