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Research Article

Solution of Fuzzy Matrix Equation System

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The main is to develop a method to solve an arbitrary fuzzy matrix equation system by using the embedding approach. Considering the existing solution to $n \times n$ fuzzy matrix equation system is done. To illustrate the proposed model a numerical example is given, and obtained results are discussed.

1. Introduction

The concept of fuzzy numbers and fuzzy arithmetic operations was first introduced by Zadeh [1], Dubois, and Prade [2]. We refer the reader to [3] for more information on fuzzy numbers and fuzzy arithmetic. Fuzzy systems are used to study a variety of problems including fuzzy metric spaces [4], fuzzy differential equations [5], fuzzy linear systems [6–8], and particle physics [9, 10].

One of the major applications of fuzzy number arithmetic is treating fuzzy linear systems [11–20], several problems in various areas such as economics, engineering, and physics boil down to the solution of a linear system of equations. Friedman et al. [21] introduced a general model for solving a fuzzy $n\times n$ linear system whose coefficient matrix is crisp, and the right-hand side column is an arbitrary fuzzy number vector. They used the parametric form of fuzzy numbers and replaced the original fuzzy $n \times n$ linear system by a crisp $2n \times 2n$ linear system and studied duality in fuzzy linear systems $Ax = Bx + y$ where A and B are real $n \times n$ matrix, the unknown vector x is vector consisting of n fuzzy numbers, and the constant y is vector consisting of n fuzzy numbers, in [22]. In [6–8, 23, 24] the authors presented conjugate gradient, LU decomposition method for solving general fuzzy linear systems, or symmetric fuzzy linear systems. Also, Abbasbandy et al. [25] investigated the existence of a minimal solution of general dual fuzzy linear equation system of the form $Ax + f = Bx + c$, where A and B are real $m \times n$ matrices, the unknown vector x is vector consisting of n fuzzy numbers, and the constants f and c are vectors consisting of m fuzzy numbers.

In this paper, we give a new method for solving a $n \times n$ fuzzy matrix equation system whose coefficients matrix is crisp, and the right-hand side matrix is an arbitrary fuzzy number matrix by using the embedding method given in Cong-Xin and Min [26] and replace the original $n \times n$ fuzzy linear system by two $n \times n$ crisp linear systems. It is clear that, in large systems, solving $n \times n$ linear system is better than solving $2n \times 2n$ linear system. Since perturbation analysis is very important in numerical methods. Recently, Ezzati [27] presented the perturbation analysis for $n \times n$ fuzzy linear systems. Now, according to the presented method in this paper, we can investigate perturbation analysis in two crisp matrix equation systems instead of $2n \times 2n$ linear system as the authors of Ezzati [27] and Wang et al. [28].

2. Preliminaries

Parametric form of an arbitrary fuzzy number is given in [29] as follows. A fuzzy number u in parametric form is a pair $(\underline{u}, \overline{u})$ of functions $\underline{u}(r), \overline{u}(r),\ 0 \le r \le 1$, which satisfy the following requirements:

(1) $\underline{u}(r)$ is a bounded left continuous nondecreasing function over $[0,1]$,

(2) $\overline{u}(r)$ is a bounded left continuous nonincreasing function over $[0,1]$, and

(3) $\underline{u}(r) \le \overline{u}(r),\ 0 \le r \le 1$.

The set of all these fuzzy numbers is denoted by E which is a complete metric space with Hausdorff distance. A crisp number α is simply represented by $\underline{u}(r) = \overline{u}(r) = \alpha,\ 0 \le r \le 1$.

For arbitrary fuzzy numbers $x = (\underline{x}(r), \overline{x}(r)),\ y = (\underline{y}(r), \overline{y}(r))$, and real number k, we may define the addition and the scalar multiplication of fuzzy numbers by using the extension principle as [29]

(a) $x = y$ if and only if $\underline{x}(r) = \underline{y}(r)$ and $\overline{x}(r) = \overline{y}(r)$,

(b) $x + y = (\underline{x}(r) + \underline{y}(r), \overline{x}(r) + \overline{y}(r))$, and

(c) $kx = \begin{cases} (k\underline{x}, k\overline{x}), & k \ge 0, \\ (k\overline{x}, k\underline{x}), & k < 0. \end{cases}$

Definition 2.1. The $n \times n$ linear system is as follows:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= y_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= y_2, \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= y_n, \end{aligned} \tag{2.1}$$

where the given matrix of coefficients $A = (a_{ij})$, $1 \le i, j \le n$ is a real $n \times n$ matrix, the given $y_i \in E$, $1 \le i \le n$, with the unknowns $x_j \in E$, $1 \le j \le n$ is called a fuzzy linear system (FLS). The operations in (2.1) is described in next section.

Here, a numerical method for finding solution [21] of a fuzzy $n \times n$ linear system is given.

Definition 2.2 (see [21]). A fuzzy number vector $(x_1, x_2, \ldots, x_n)^t$ given by

$$x_j = \left(\underline{x}_j(r), \overline{x}_j(r)\right); \quad 1 \leq j \leq n, \qquad 0 \leq r \leq 1 \tag{2.2}$$

is called a solution of the fuzzy linear system (2.1) if

$$\begin{aligned} \underline{\sum_{j=1}^{n} a_{ij} x_j} &= \sum_{j=1}^{n} \underline{a_{ij} x_j} = \underline{y}_i, \\ \overline{\sum_{j=1}^{n} a_{ij} x_j} &= \sum_{j=1}^{n} \overline{a_{ij} x_j} = \overline{y}_i. \end{aligned} \tag{2.3}$$

If, for a particular i, $a_{ij} > 0$, for all j, we simply get

$$\sum_{j=1}^{n} a_{ij} \underline{x}_j = \underline{y}_i, \quad \sum_{j=1}^{n} a_{ij} \overline{x}_j = \overline{y}_i. \tag{2.4}$$

Finally, we conclude this section by a reviewing on the proposed method for solving fuzzy linear system [21].

The authors [21] wrote the linear system of (2.1) as follows:

$$SX = Y, \tag{2.5}$$

where s_{ij} are determined as follows:

$$\begin{aligned} a_{ij} \geq 0 &\Longrightarrow s_{ij} = a_{ij}, \qquad s_{i+n,j+n} = a_{ij}, \\ a_{ij} < 0 &\Longrightarrow s_{i,j+n} = -a_{ij}, s_{i+n,j} = -a_{ij}, \end{aligned} \tag{2.6}$$

and any s_{ij} which is not determined by (2.1) is zero and

$$X = \begin{bmatrix} \underline{x}_1 \\ \vdots \\ \underline{x}_n \\ -\overline{x}_1 \\ \vdots \\ -\overline{x}_n \end{bmatrix}, \qquad Y = \begin{bmatrix} \underline{y}_1 \\ \vdots \\ \underline{y}_n \\ -\overline{y}_1 \\ \vdots \\ -\overline{y}_n \end{bmatrix}. \tag{2.7}$$

The structure of S implies that $s_{ij} \geq 0$, $1 \leq i, j \leq 2n$ and that

$$S = \begin{pmatrix} B & C \\ C & B \end{pmatrix}, \tag{2.8}$$

where B contains the positive entries of A, and C contains the absolute values of the negative entries of A, that is, $A = B - C$.

Theorem 2.3 (see [21]). *The inverse of nonnegative matrix*

$$S = \begin{pmatrix} B & C \\ C & B \end{pmatrix} \tag{2.9}$$

is

$$S^{-1} = \begin{pmatrix} D & E \\ E & D \end{pmatrix}, \tag{2.10}$$

where

$$D = \frac{1}{2}\left[(B+C)^{-1} + (B-C)^{-1}\right], \qquad E = \frac{1}{2}\left[(B+C)^{-1} - (B-C)^{-1}\right]. \tag{2.11}$$

Corollary 2.4 (see [30]). *The solution of* (2.5) *is obtained by*

$$X = S^{-1}Y. \tag{2.12}$$

3. Fuzzy Matrix Equation System

A matrix system such as

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{pmatrix} = \begin{pmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ y_{21} & y_{22} & \cdots & y_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ y_{n1} & y_{n2} & \cdots & y_{nn} \end{pmatrix}, \tag{3.1}$$

where a_{ij}, $1 \le i, j \le n$, are real numbers, the elements y_{ij} in the right-hand matrix are fuzzy numbers, and the unknown elements x_{ij} are ones, is called a fuzzy matrix equation system (FMES).

Using matrix notation, we have

$$AX = Y. \tag{3.2}$$

A fuzzy number matrix

$$X = (x_1, \ldots, x_j, \ldots, x_n) \tag{3.3}$$

is called a solution of the fuzzy matrix system (2.1) if

$$Ax_j = y_j, \quad 1 \le j \le n. \tag{3.4}$$

In this section, we propose a new method for solving FMES.

Theorem 3.1. *Suppose that the inverse of matrix A exists and $x_j = (x_{j1}, x_{j2}, \dots, x_{jn})^T$ is a solution of this equation. Then $\underline{x_j} + \overline{x_j} = (\underline{x_{j_1}} + \overline{x}_{j1}, \underline{x_{j_2}} + \overline{x}_{j2}, \dots, \underline{x_{j_n}} + \overline{x}_{jn})^T$ is the solution of the following systems:*

$$A\left(\underline{x_j} + \overline{x_j}\right) = \underline{y_j} + \overline{y_j}, \quad j = 1, 2, \dots, n, \tag{3.5}$$

where $\underline{y_j} + \overline{y_j} = (\underline{y_{j_1}} + \overline{y}_{j1}, \underline{y_{j_2}} + \overline{y}_{j2}, \dots, \underline{y_{j_n}} + \overline{y}_{jn})^T$, $j = 1, 2, \dots, n$.

Proof. It is the same as the proof of Theorem 3 in [27].

For solving (3.2), we first solve the following system:

$$\begin{gathered}
a_{11}\left(\underline{x_{j_1}} + \overline{x}_{j1}\right) + \cdots + a_{1n}\left(\underline{x_{j_n}} + \overline{x}_{jn}\right) = \left(\underline{y_{j_1}} + \overline{y}_{j1}\right), \\
a_{21}\left(\underline{x_{j_1}} + \overline{x}_{j1}\right) + \cdots + a_{2n}\left(\underline{x_{j_n}} + \overline{x}_{jn}\right) = \left(\underline{y_{j_2}} + \overline{y}_{j2}\right), \\
\vdots \\
a_{n1}\left(\underline{x_{j_1}} + \overline{x}_{j1}\right) + \cdots + a_{nn}\left(\underline{x_{j_n}} + \overline{x}_{jn}\right) = \left(\underline{y_{j_n}} + \overline{y}_{jn}\right), \\
j = 1, 2, \dots, n.
\end{gathered} \tag{3.6}$$

Using matrix notation, we have

$$A\left(\underline{X} + \overline{X}\right) = \left(\underline{Y} + \overline{Y}\right). \tag{3.7}$$

Suppose that the solution of (3.7) is as

$$d_j = \begin{bmatrix} d_{j1} \\ d_{j2} \\ \vdots \\ d_{jn} \end{bmatrix} = \underline{x_j} + \overline{x_j} = \begin{bmatrix} \underline{x_{j_1}} + \overline{x}_{j1} \\ \underline{x_{j_2}} + \overline{x}_{j2} \\ \vdots \\ \underline{x_{j_n}} + \overline{x}_{jn} \end{bmatrix}, \quad j = 1, 2, \dots, n. \tag{3.8}$$

Let matrices B and C have defined as Section 2. Now using matrix notation for (3.7), we get in parametric form $(B - C)(\underline{X}(r) + \overline{X}(r)) = (\underline{Y}(r) + \overline{Y}(r))$. We can write this system as follows:

$$\begin{gathered}
B\underline{X}(r) - C\overline{X}(r) = \underline{Y}(r), \\
B\overline{X}(r) - C\underline{X}(r) = \overline{Y}(r).
\end{gathered} \tag{3.9}$$

By substituting $\overline{X}(r) = D - \underline{X}(r)$ and $\underline{X}(r) = D - \overline{X}(r)$ in the first and second equation of above system, respectively, we have

$$(B + C)\underline{X}(r) = \underline{Y}(r) + CD, \tag{3.10}$$

$$(B + C)\overline{X}(r) = \overline{Y}(r) + CD, \tag{3.11}$$

therefore, we have

$$\begin{aligned} \underline{X}(r) &= (B + C)^{-1}\left(\underline{Y}(r) + CD\right), \\ \overline{X}(r) &= (B + C)^{-1}\left(\overline{Y}(r) + CD\right). \end{aligned} \tag{3.12}$$

Therefore, we can solve fuzzy matrix equation system (3.2) by solving (3.7)–(3.10). □

Theorem 3.2. *Let in* (3.3) $j = 1$, *also* g *and* G *are the number of multiplication operations that are required to calculate*

$$X = \left(\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_n, -\overline{x}_1, -\overline{x}_2, \ldots, -\overline{x}_n\right)^T = S^{-1}Y, \tag{3.13}$$

(the proposed method in Friedman et al. [21]) and

$$x_j = \left(\underline{x_{j}}_1, \underline{x_{j}}_2, \ldots, \underline{x_{j}}_n, \overline{x}_{j1}, \overline{x}_{j2}, \ldots, \overline{x}_{jn}\right)^T, \tag{3.14}$$

from (3.7)–(3.10), *respectively. Then* $G \le g$ *and* $g - G = n^2$.

Proof. According to Section 2, we have

$$S^{-1} = \begin{pmatrix} D & E \\ E & D \end{pmatrix}, \tag{3.15}$$

where

$$D = \frac{1}{2}\left[(B + C)^{-1} + (B - C)^{-1}\right], \qquad E = \frac{1}{2}\left[(B + C)^{-1} - (B - C)^{-1}\right]. \tag{3.16}$$

Therefore, for determining S^{-1}, we need to compute $(B + C)^{-1}$ and $(B - C)^{-1}$. Now, assume that M is $n \times n$ matrix and denote by $h(M)$ the number of multiplication operations that are required to calculate M^{-1}. It is clear that

$$h(S) = h(B + C) + h(B - C) = 2h(A), \tag{3.17}$$

and hence

$$g = 2h(A) + 4n^2. \tag{3.18}$$

For computing $\underline{x_j}+\overline{x_j} = (\underline{x_{j_1}}+\overline{x_{j1}}, \underline{x_{j_2}}+\overline{x_{j2}}, \ldots, \underline{x_{j_n}}+\overline{x_{jn}})^T$ from (3.7) and $\underline{x_j} = (\underline{x_{j_1}}, \underline{x_{j_2}}, \ldots, \underline{x_{j_n}})^T$ from (3.10) the number of multiplication operations is $h(A) + n^2$ and $h(B + C) + 2n^2$, respectively. Clearly $h(B + C) = h(A)$, so

$$G = 2h(A) + 3n^2, \tag{3.19}$$

and hence $g - G = n^2$. This proves theorem. □

Remark 3.3. In (3.3) if $j = 1$, then this paper is similar to [27].

Example 3.4. Consider the 2×2 fuzzy matrix equation system as follows:

$$\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \begin{pmatrix} (3r-3, 3-3r) & (4r-4, 6-6r) \\ (2r+1, 5-2r) & (3r, 7-4r) \end{pmatrix}. \tag{3.20}$$

By using (3.7) and (3.10), we have

$$\begin{pmatrix} \underline{x_{11}}(r) + \overline{x_{11}}(r) & \underline{x_{12}}(r) + \overline{x_{12}}(r) \\ \underline{x_{21}}(r) + \overline{x_{21}}(r) & \underline{x_{22}}(r) + \overline{x_{22}}(r) \end{pmatrix} = \begin{pmatrix} 2 & 3-r \\ 4 & 4 \end{pmatrix},$$
$$\begin{pmatrix} \underline{x_{11}}(r) & \underline{x_{12}}(r) \\ \underline{x_{21}}(r) & \underline{x_{22}}(r) \end{pmatrix} = \begin{pmatrix} r & r \\ 1+r & 2r \end{pmatrix}, \tag{3.21}$$

and hence

$$\begin{pmatrix} \overline{x_{11}}(r) & \overline{x_{12}}(r) \\ \overline{x_{21}}(r) & \overline{x_{22}}(r) \end{pmatrix} = \begin{pmatrix} 2-r & 3-2r \\ 3-r & 4-2r \end{pmatrix}. \tag{3.22}$$

Obviously, x_{11}, x_{12}, x_{21} and x_{22}, are fuzzy numbers.

4. Conclusions

In this paper, we propose a general model for solving fuzzy matrix equation system. The original system with matrix coefficient A is replaced by two $n \times n$ crisp matrix equation systems.

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