Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2012, Article ID 763952, 13 pages
doi:10.1155/2012/763952

Research Article

Coupled Fixed Point Theorems for Nonlinear Contractions in Partial Metric Spaces

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Received 28 March 2012; Revised 22 August 2012; Accepted 26 August 2012

Academic Editor: Hari Mohan Srivástava

We establish some results on the existence and uniqueness of coupled fixed point involving nonlinear contractive conditions in complete-ordered partial metric spaces.

1. Introduction

The concept of partial metric which is a generalized metric space was introduced by Matthews [1] in 1994, in which the distance between two identical elements needs not be zero. The existence of fixed point for contraction-type mappings on such spaces was considered by many authors [1–12]. A modified version of a Banach contraction mapping principle, more suitable to solve certain problems arising in computer science using the concept of partial metric space is given in [1].

Gnana Bhaskar and Lakshmikantham [13] introduced the concept of coupled fixed point of a mapping $F : X\times X \to X$ and proved some interesting coupled fixed point theorems for mapping satisfying the mixed monotone property. Later in [14], Lakshmikantham and Ćirić investigated some more coupled fixed point theorems in partially ordered sets. For more on coupled fixed point theory, we refer the reader to [2, 14–20].

First, we start by recalling some definitions and properties of partial metric spaces.

Definition 1.1 (see [9]). A partial metric on a nonempty set X is a function $p : X \times X \to R^+$ such that for all $x, y, z \in X$:

(p_1) $x = y \Leftrightarrow p(x, x) = p(x, y) = p(y, y)$,

(p_2) $p(x, x) \leq p(x, y)$,

(p_3) $p(x,y) = p(y,x)$,

(p_4) $p(x,y) \leq p(x,z) + p(z,y) - p(z,z)$.

A partial metric space is a pair (X,p) such that X is a non empty set and p is a partial metric on X. Each partial metric p on X generates a T_0 topology τ_p on X which has as a base the family of open p-balls $\{B_p(x,\epsilon), x \in X, \epsilon > 0\}$, where $B_p(x,\epsilon) = \{y \in X : p(x,y) < p(x,x)+\epsilon\}$ for all $x \in X$ and $\epsilon > 0$. Matthews observed in [1, page 187] that a sequence (x_n) in a partial metric space (X,p) converges to some $x \in X$ with respect to p if and only if $p(x,x) = \lim_{n\to\infty} p(x,x_n)$. It is clear that if $p(x,y) = 0$, then from $(p_1), (p_2)$, and (p_3), $x = y$. But if $x = y$, $p(x,y)$ may not be 0.

If p is a partial metric on X, then the function $p_s : X \times X \to R^+$ given by

$$p_s(x,y) = 2p(x,y) - p(x,x) - p(y,y), \tag{1.1}$$

is a metric on X.

Example 1.2 (see, e.g., [1, 7]). Consider $X = R^+$ with $p(x,y) = \max\{x,y\}$. Then, (R^+,p) is a partial metric space.

It is clear that p is not a (usual) metric. Note that in this case $p_s(x,y) = |x-y|$.

Definition 1.3 (see [1, Definition 5.2]). Let (X,p) be a partial metric space and let (x_n) be a sequence in X. Then, $\{x_n\}$ is called a Cauchy sequence if $\lim_{n,m\to\infty} p(x_m,x_n)$ exists (and is finite).

Definition 1.4 (see [1, Definition 5.3]). A partial metric space (X,p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p, to a point $x \in X$, such that $p(x,x) = \log_{n,m\to\infty} p(x_m,x_n)$.

Example 1.5 (see [12]). Let $X := [0,1] \cup [2,3]$ and define $p : X \times X \to [0,\infty)$ by

$$p(x,y) = \begin{cases} \max\{x,y\}, & \{x,y\} \cap [2,3] \neq \phi, \\ |x-y|, & \{x,y\} \subset [0,1], \end{cases} \tag{1.2}$$

Then, (X,p) is a complete partial metric space.

It is well known (see, e.g., [1, page 194]) that a sequence in a partial metric space (X,p) is a Cauchy sequence in (X,p) if and only if it is a Cauchy sequence in the metric space (X,p_s), and that a partial metric space (X,p) is complete if and only the metric space (X,p_s) is complete. Furthermore, $\lim_{n\to\infty} p_s(x,x_n) = 0$ if and only if

$$p(x,x) = \lim_{n\to\infty} p(x,x_n) = \lim_{n,m\to\infty} p(x_m,x_n). \tag{1.3}$$

Let (X,p) be a partial metric. We endow $X \times X$ with the partial metric q defined for $(x,y),(u,v) \in X \times X$ by

$$q((x,y),(u,v)) = p(x,u) + p(y,v). \tag{1.4}$$

A mapping $F : X \times X \to X$ is said to be continuous at $(x, y) \in X \times X$, if for every $\epsilon > 0$, there exists $\delta > 0$ such that $F(Bq((x, y), \delta)) \subseteq Bp(F(x, y), \epsilon)$.

In this paper, we establish some results on the existence and uniqueness of a coupled fixed point involving nonlinear contractive conditions in complete-ordered partial metric spaces analogous to some other results in [17, 18].

Before presenting our main results, we recall some basic concepts.

Definition 1.6 (see [8]). An element $x, y \in X \times X$ is said to be a coupled fixed point of the mapping $F : X \times X \to X$ if $F(x, y) = x$ and $F(y, x) = y$.

Definition 1.7 (see, Gnana Bhashkar and Lakshmikantham [13]). Let $(X, \leq)$ be a partially ordered set and $F : X \times X \to X$. The mapping F is said to has the mixed monotone property if

$$\begin{aligned} &x_1, x_2 \in X, \qquad x_1 \leq x_2 \Longrightarrow F(x_1, y) \leq F(x_2, y) \quad \text{for any } y \in X, \\ &y_1, y_2 \in X, \qquad y_1 \leq y_2 \Longrightarrow F(x, y_1) \geq F(x, y_2) \quad \text{for any } x \in X. \end{aligned} \tag{1.5}$$

Now, let us denote by Φ the set of all nondecreasing continuous functions $\varphi : [0, \infty) \to [0, \infty)$ that satisfy

(i) $\varphi(t) = 0$ if and only if $t = 0$,

(ii) $\varphi(t + s) \leq \varphi(t) + \varphi(s)$, for all $t, s \in [0, \infty)$.

Again, let Ψ denote all functions $\psi : [0, \infty) \to [0, \infty)$ which satisfy $\lim_{t \to r} \psi(t) > 0$ for all $r > 0$ and $\lim_{t \to 0^+} \psi(t) = 0$. It is an easy matter to see that $\Phi \subseteq \Psi$ and $(1/2)\varphi(t) \leq \varphi(t/2)$ for any $t \in [0, \infty)$.

2. Main Results

The aim of this work is to prove the following theorem.

Theorem 2.1. *Let $(X, \leq)$ be a partially ordered set and suppose that there is a partial metric p on X such that (X, p) is a complete partial metric space. Let $F : X \times X \to X$ be a mapping having the mixed monotone property on X and assume that there exist $\varphi \in \Phi$ and $\psi \in \Psi$ such that*

$$\varphi(d(F(x, y), F(u, v))) \leq \varphi(\alpha p(x, u) + \beta p(y, v)) - \psi(\alpha p(x, u) + \beta p(y, v)), \tag{2.1}$$

for all $x, y, u, v \in X$ with $x \geq u$, $y \leq v$, and $\alpha + \beta < 1$. Suppose either F is continuous or X has the following properties:

(i) *if a nondecreasing sequence $x_n \to x$, then $x_n \leq x$ for all n,*

(ii) *if a nonincreasing sequence $x_n \to x$, then $x \leq x_n$ for all n.*

If there exist $x_0, y_0 \in X$ such that $x_0 \leq F(x_0, y_0)$, and $F(y_0, x_0) \leq y_0$, then there exist $x, y \in X$ such that $x = F(x, y)$ and $y = F(y, x)$, that is, F has a coupled fixed point. Furthermore, $p(x, x) = p(y, y) = 0$.

Proof. Choose $x_0, y_0 \in X$ and set $x_1 = F(x_0, y_0)$ and $F(y_0, x_0) = y_1$. Since $x_0 \leq F(x_0, y_0)$ and $y_0 \geq F(y_0, x_0)$, letting $x_2 = F(x_1, y_1)$ and $y_2 = F(y_1, x_1)$, we denote

$$F^2(x_0, y_0) = F(F(x_0, y_0), F(y_0, x_0)) = F(x_1, y_1) = x_2,$$
$$F^2(y_0, x_0) = F(F(y_0, x_0), F(x_0, y_0)) = F(y_1, x_1) = y_2, \tag{2.2}$$

and due to the mixed monotone property of F, we have

$$x_2 = F(x_1, y_1) \geq F(x_0, y_0) = x_1, \qquad y_2 = F(y_1, x_1) \leq F(y_0, x_0) = y_1. \tag{2.3}$$

Further, for $n = 1, 2, \ldots$, we can easily verify that

$$x_0 \leq F(x_0, y_0) = x_1 \leq F^2(x_0, y_0) = x_2 \leq \cdots \leq F(F^n(x_0, y_0), F^n(y_0, x_0)) = x_{n+1},$$
$$y_0 \geq F(y_0, x_0) = y_1 \geq F^2(y_0, x_0) = y_2 \geq \cdots \geq F(F^n(y_0, x_0), F^n(x_0, y_0)) = y_{n+1}. \tag{2.4}$$

Since $x_n \geq x_{n-1}$ and $y_n \leq y_{n-1}$, from (2.1), we have

$$\begin{aligned} \varphi(p(x_n, x_{n+1})) &= \varphi(p(F(x_{n-1}, y_{n-1}), F(x_n, y_n))) \\ &\leq \varphi(\alpha p(x_{n-1}, x_n) + \beta p(y_{n-1}, y_n)) - \psi(\alpha p(x_{n-1}, x_n) + \beta p(y_{n-1}, y_n)) \\ &\leq \varphi(\alpha p(x_{n-1}, x_n) + \beta p(y_{n-1}, y_n)). \end{aligned} \tag{2.5}$$

Similarly, since $y_{n-1} \geq y_n$ and $x_{n-1} \leq x_n$, from (2.1), we also have

$$\begin{aligned} \varphi(p(y_n, y_{n+1})) &= \varphi(p(F(y_{n-1}, x_{n-1}), F(y_n, x_n))) \\ &\leq \varphi(\alpha p(y_{n-1}, y_n) + \beta p(x_{n-1}, x_n)) - \psi(\alpha p(y_{n-1}, y_n) + \beta p(x_{n-1}, x_n)) \\ &\leq \varphi(\alpha p(y_{n-1}, y_n) + \beta p(x_{n-1}, x_n)). \end{aligned} \tag{2.6}$$

Consequently, since φ is nondecreasing, using (2.5) and (2.6), we get

$$p(x_n, x_{n+1}) \leq \alpha p(x_{n-1}, x_n) + \beta p(y_{n-1}, y_n), \tag{2.7}$$

$$p(y_n, y_{n+1}) \leq \alpha p(y_{n-1}, y_n) + \beta p(x_{n-1}, x_n). \tag{2.8}$$

By adding (2.7) to (2.8), we have

$$p(x_n, x_{n+1}) + p(y_n, y_{n+1}) \leq (\alpha + \beta)(p(x_{n-1}, x_n) + p(y_{n-1}, y_n)). \tag{2.9}$$

Now, we will show that both x_n and y_n are Cauchy sequences. Note that

$$\begin{aligned} p(x_n, x_{n+1}) + p(y_n, y_{n+1}) &\le (\alpha+\beta)(p(x_{n-1}, x_n) + p(y_{n-1}, y_n)) \\ &\le (\alpha+\beta)^2(p(x_{n-2}, x_{n-1}) + p(y_{n-2}, y_{n-1})) \\ &\vdots \\ &\le (\alpha+\beta)^n(p(x_0, x_1) + p(y_0, y_1)). \end{aligned} \tag{2.10}$$

Consequently, if $p(x_0, x_1) + p(y_0, y_1) = 0$, then $x_1 = x_0 = F(x_0, y_0)$ and $y_1 = y_0 = F(y_0, x_0)$, that means (x_0, y_0) is a coupled fixed point of F. If $p(x_0, x_1) + p(y_0, y_1) > 0$, for each $m > n$, combining (2.7) and (2.8) using property p_4, we have

$$\begin{aligned} p(x_n, x_m) &\le p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2}) + \cdots + p(x_{m-1}, x_m) - \sum_{k=n+1}^{m-1} p(x_k, x_k) \\ &\le p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2}) + \cdots + p(x_{m-1}, x_m), \\ p(y_n, y_m) &\le p(y_n, y_{n+1}) + p(y_{n+1}, y_{n+2}) + \cdots + p(y_{m-1}, y_m) - \sum_{k=n+1}^{m-1} p(y_k, y_k) \\ &\le p(y_n, y_{n+1}) + p(y_{n+1}, y_{n+2}) + \cdots + p(y_{m-1}, y_m). \end{aligned} \tag{2.11}$$

Thus,

$$\begin{aligned} p(x_n, x_m) + p(y_n, y_m) &\le p(x_n, x_{n+1}) + p(y_n, y_{n+1}) + p(x_{n+1}, x_{n+2}) + p(y_{n+1}, y_{n+2}) \\ &\quad + \cdots + p(x_{m-1}, x_m) + p(y_{m-1}, y_m) \\ &\le \left((\alpha+\beta)^n + (\alpha+\beta)^{n+1} + \cdots + (\alpha+\beta)^{m-1}\right)(p(x_0, x_1) + p(y_0, y_1)) \\ &\le \frac{(\alpha+\beta)^n}{1-(\alpha+\beta)}(p(x_0, x_1) + p(y_0, y_1)). \end{aligned} \tag{2.12}$$

By definition of p_s, we have $p_s(x, y) \le 2p(x, y)$, so

$$\begin{aligned} p_s(x_n, x_m) + p_s(y_n, y_m) &\le 2p(x_n, x_m) + 2p(y_n, y_m) \\ &\le \frac{2(\alpha+\beta)^n}{1-(\alpha+\beta)}(p(x_0, x_1) + p(y_0, y_1)). \end{aligned} \tag{2.13}$$

Consequently,

$$p_s(x_n, x_m) \le p_s(x_n, x_m) + p_s(y_n, y_m) \le \frac{2(\alpha+\beta)^n}{1-(\alpha+\beta)}\left(p(x_0, x_1) + p(y_0, y_1)\right),$$
$$p_s(y_n, y_m) \le p_s(x_n, x_m) + p_s(y_n, y_m) \le \frac{2(\alpha+\beta)^n}{1-(\alpha+\beta)}\left(p(x_0, x_1) + p(y_0, y_1)\right), \tag{2.14}$$

which implies that both (x_n) and (y_n) are Cauchy sequences in the metric space (X, p_s). Since the metric space (X, p_s) is complete, it follows that there exist $x, y \in X$ such that

$$\lim_{n\to\infty} p_s(x_n, x) = \lim_{n\to\infty} p_s(y_n, y) = 0. \tag{2.15}$$

Therefore, using property p_2 and the fact that (X, p) is complete if and only if (X, p_s) is complete, using (1.3), we have

$$p(x, x) = \lim_{n\to\infty} p(x_n, x) = \lim_{n\to\infty} p(x_n, x_n),$$
$$p(y, y) = \lim_{n\to\infty} p(y_n, y) = \lim_{n\to\infty} p(y_n, y_n). \tag{2.16}$$

From p_2 and (2.10), we have

$$p(x_n, x_n) \le p(x_n, x_{n+1}) \le (\alpha+\beta)^n \left(p(x_0, x_1) + p(y_0, y_1)\right). \tag{2.17}$$

Since $\alpha+\beta < 1$, we get, $\lim_{n\to\infty} p(x_n, x_n) = 0$. Similarly, one can show that $\lim_{n\to\infty} p(y_n, y_n)$. Therefore,

$$p(x, x) = \lim_{n\to\infty} p(x_n, x) = \lim_{n\to\infty} p(x_n, x_n) = 0,$$
$$p(y, y) = \lim_{n\to\infty} p(y_n, y) = \lim_{n\to\infty} p(y_n, y_n) = 0. \tag{2.18}$$

Finally, we will show that $x = F(x, y)$ and $y = F(y, x)$.

(a) Assume that F is continuous on X. In particular, F is continuous at (x, y), hence for any $\epsilon > 0$, there exists $\delta > 0$ such that if $(u, v) \in X \times X$ verifying $q((x, y), (u, v)) < q((x, y), (x, y)) + \delta$, meaning that

$$p(x, u) + p(y, v) < p(x, x) + p(y, y) + \delta = \delta, \tag{2.19}$$

because $p(x, x) = p(y, y) = 0$, then we have

$$p(F(x, y), F(u, v)) < p(F(x, y), F(x, y)) + \frac{\epsilon}{2}. \tag{2.20}$$

Since $\lim_{n\to\infty} p(x_n, x) = \lim_{n\to\infty} p(y_n, y) = 0$, for $\gamma = \min\{\delta/2, \epsilon/2\} > 0$, there exist $n_0, m_0 \in N$ such that, for $n \geq n_0$, $m \geq m_0$, $p(x_n, x) < \gamma$ and $p(y_n, y) < \gamma$. Therefore, for $n \in N$, $n \geq \max(n_0, m_0)$, we have

$$p(x_n, x) + p(y_n, y) < 2\gamma < \delta, \tag{2.21}$$

so we get

$$p(F(x,y), F(x_n, y_n)) < p(F(x,y), F(x,y)) + \frac{\epsilon}{2}. \tag{2.22}$$

Now, for any $n \geq \max(n_0, m_0)$,

$$\begin{aligned} p(F(x,y), x) &\leq p(F(x,y), x_{n+1}) + p(x_{n+1}, x) \\ &= p(F(x,y), F(x_n, y_n)) + p(x_{n+1}, x) \\ &< p(F(x,y), F(x,y)) + \frac{\epsilon}{2} + \gamma \\ &< p(F(x,y), F(x,y)) + \epsilon. \end{aligned} \tag{2.23}$$

On the other hand, inserting $p(x,x) = p(y,y) = 0$ in (2.1), we get

$$\begin{aligned} \varphi(p(F(x,y), F(x,y))) &\leq \varphi(\alpha p(x,x) + \beta p(y,y)) - \psi(\alpha p(x,x) + \beta p(y,y)) \\ &= \varphi(0) - \psi(0) = -\psi(0) \leq 0, \end{aligned} \tag{2.24}$$

which implies $p(F(x,y), F(x,y)) = 0$, so for any $\epsilon > 0$, $p(F(x,y), x) < \epsilon$. This implies that $F(x,y) = x$. Similarly, we can show that $F(y,x) = y$.

(b) Assume that X satisfies the two conditions given by (i) and (ii). Since (x_n), (y_n) are a nondecreasing sequences and $x_n \to x$, $y_n \to y$, we have $x_n \leq x$ and $y_n \geq y$ for all n. By the condition (p_4), we have

$$p(x, F(x,y)) \leq p(x, x_{n+1}) + p(x_{n+1}, F(x,y)) = p(x, x_{n+1}) + p(F(x_n, y_n), F(x,y)). \tag{2.25}$$

Therefore,

$$\begin{aligned} \varphi(p(x, F(x,y))) &\leq \varphi(p(x, x_{n+1})) + \varphi(p(F(x_n, y_n), F(x,y))) \\ &\leq \varphi(p(x, x_{n+1})) + \varphi(\alpha p(x_n, x) + \beta p(y_n, y)) \\ &\quad - \psi(\alpha p(x_n, x) + \beta p(y_n, y)). \end{aligned} \tag{2.26}$$

Taking the limit as $n \to +\infty$ in the above inequality, using (2.18), and the properties of φ and ψ, we get $\varphi(p(x, F(x,y))) = 0$. Thus, $p(x, F(x,y)) = 0$. Hence, $x = F(x,y)$. Similarly, one can show that $y = F(y,x)$. Thus, we proved that F has a coupled fixed point. □

Corollary 2.2. *Let $(X, \leq)$ be a partially ordered set and suppose that there is a partial metric p on X such that (X, p) is a complete partial metric space. Let $F : X \times X \to X$ be a mapping having the mixed monotone property on X. Supposed that*

$$p(F(x,y), F(u,v)) \leq \alpha p(x,u) + \beta p(y,v) \tag{2.27}$$

for all $x, y, u, v \in X$ with $x \geq u, y \leq v$ and $\alpha + \beta < 1$. Suppose either F is continuous or X has the following properties:

(i) *if a nondecreasing sequence $x_n \to x$, then $x_n \leq x$ for all n,*

(ii) *if a nonincreasing sequence $x_n \to x$, then $x \leq x_n$ for all n. If there exist $x_0, y_0 \in X$ such that $x_0 \leq F(x_0, y_0)$ and $F(y_0, x_0) \leq y_0$ then, there exist $x, y \in X$ such that $x = F(x, y)$ and $y = F(y, x)$, that is, F has a coupled fixed point. Also, $p(x,x) = p(y,y) = 0$.*

Proof. For $\alpha + \beta < 1$, taking $\varphi(t) = t$ and $\psi(t) = 0$, we get the result. □

The following main theorem for Gnana Bhaskar and Lakshmikantham in [13] proved the next theorem.

Theorem 2.3 (see Gnana Bhaskar and Lakshmikantham [13]). *Let $(X, \leq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Let $F : X \times X \to X$ be a mapping having the mixed monotone property on X. Assume that there exists a $k \in [0, 1)$ with*

$$d(F(x,y), F(u,v)) \leq \frac{k}{2}(d(x,u) + d(y,v)), \tag{2.28}$$

for all $x, y, u, v \in X$ with $x \leq u$ and $v \leq y$. suppose either F is continuous or X has the following properties:

(i) *if a nondecreasing sequence $x_n \to x$, then $x_n \leq x$ for all n,*

(ii) *if a nonincreasing sequence $x_n \to x$, then $x \leq x_n$ for all n.*

If there exist $x_0, y_0 \in X$ such that $x_0 \leq F(x_0, y_0)$ and $F(y_0, x_0) \leq y_0$ then there exist $x, y \in X$ such that $x = F(x, y)$ and $y = F(y, x)$, that is, F has a coupled fixed point.

Note that for $k \in [0, 1)$, and $\alpha = \beta = k/2$ in Corollary 2.2, we get analogous to Theorem 2.3 in complete-ordered partial metric space.

Theorem 2.4. *Let $(X, \leq)$ be a partially ordered set having the property that for every $(x, y), (z, t) \in X \times X$, there exists a (u, v) in $X \times X$ such that $u \leq x, z$ and $v \leq y, t$ and suppose that there is a partial metric p on X such that (X, p) is a complete partial metric space. Let $F : X \times X \to X$ be a mapping having the mixed monotone property on X and assume that there exist $\varphi \in \Phi$ and $\psi \in \Psi$ such that*

$$\varphi(p(F(x,y), F(u,v))) \leq \varphi(\alpha p(x,u) + \beta p(y,v)) - \psi(\alpha p(x,u) + \beta p(y,v)) \tag{2.29}$$

for all $x, y, u, v \in X$ with $x \geq u$, $y \leq v$, and $\alpha + \beta < 1$. Suppose either F is continuous or X has the following properties:

(i) *if a nondecreasing sequence $x_n \to x$, then $x_n \leq x$ for all n,*

(ii) *if a nonincreasing sequence $x_n \to x$, then $x \leq x_n$ for all n.*

If there exist $x_0, y_0 \in X$ such that $x_0 \leq F(x_0, y_0)$ and $F(y_0, x_0) \leq y_0$ then F has a unique coupled fixed point.

Proof. From Theorem 2.1, the set of coupled fixed points of F is non-empty. Suppose (x, y) and (z, t) are coupled fixed points of F, that is, $x = F(x, y)$, $y = F(y, x)$, $z = F(z, t)$ and $t = F(t, z)$. We shall show that $x = z$ and $y = t$.

By assumption, there exists $(u, v) \in X \times X$ such that $u \leq x, z$ and $v \leq y, t$. We define sequences $(u_n), (v_n)$ as follows:

$$u_0 = u, \qquad v_0 = v, \qquad u_{n+1} = F(u_n, v_n), \qquad v_{n+1} = F(v_n, u_n) \quad \forall n. \tag{2.30}$$

Since we may assume that $(u_0, v_0) = (u, v)$. By using the mathematical induction, it is easy to prove that $u_n \leq x$ and $y \leq v_n$ for any $n \in N$. From (2.1), we have

$$\begin{aligned}
\varphi(p(x, u_{n+1})) &= \varphi(p(F(x, y), F(u_n, v_n))) \\
&\leq \varphi(\alpha p(x, u_n) + \beta p(y, v_n)) - \psi(\alpha p(x, u_n) + \beta p(y, v_n)) \\
&\leq \varphi(\alpha p(x, u_n) + \beta p(y, v_n)), \\
\varphi(p(v_{n+1}, y)) &= \varphi(p(F(v_n, u_n), F(y, x))) \\
&\leq \varphi(\alpha p(v_n, y) + \beta p(u_n, x)) - \psi(\alpha p(v_n, y) + \beta p(u_n, x)) \\
&\leq \varphi(\alpha p(v_n, y) + \beta p(u_n, x)).
\end{aligned} \tag{2.31}$$

Since φ is nondecreasing, from the above inequalities, we have

$$p(x, u_{n+1}) \leq \alpha p(x, u_n) + \beta p(y, v_n), \tag{2.32}$$

$$p(v_{n+1}, y) \leq \alpha p(v_n, y) + \beta p(u_n, x). \tag{2.33}$$

Adding (2.32) to (2.33), we get

$$p(x, u_{n+1}) + p(y, v_{n+1}) \leq (\alpha + \beta)(p(y, v_n) + p(x, u_n)). \tag{2.34}$$

Therefore,

$$\begin{aligned} p(x,u_{n+1}),p(y,v_{n+1}) &\leq p(x,u_{n+1})+p(y,v_{n+1}) \\ &\leq (\alpha+\beta)(p(x,u_n)+p(y,v_n)) \\ &\leq (\alpha+\beta)^2(p(x,u_{n-1})+p(y,v_{n-1})) \\ &\vdots \\ &\leq (\alpha+\beta)^n(p(x,u_1)+p(y,v_1)), \end{aligned} \tag{2.35}$$

that is, since $\alpha+\beta<1$, the sequences $\{p(y,v_n)$ and $p(x,u_n)\}$ are convergent and

$$\lim_{n\to\infty} p(x,u_n) = \lim_{n\to\infty} p(y,v_n) = 0. \tag{2.36}$$

Similarly, one can show that $\lim_{n\to\infty} p(z,u_n) = \lim_{n\to\infty} p(t,v_n) = 0$. Since

$$\begin{aligned} p(x,z) &\leq p(x,u_n)+p(u_n,z), \\ p(y,t) &\leq p(y,v_n)+p(v_n,t), \end{aligned} \tag{2.37}$$

letting $n \to \infty$, we obtain $p(x,z)=p(y,t)=0$, so $x=z$ and $y=t$. □

Theorem 2.5. *Let $(X,\leq)$ be a partially ordered set such that for every $(x,y),(z,t)\in X\times X$, there exists a (u,v) in $X\times X$ such that $u\leq x,z$ and $v\leq y,t$ and suppose there is a partial metric p on X such that (X,p) is a complete partial metric space. Let $F: X\times X \to X$ be a mapping having the mixed monotone property on X and assume that there exist $\varphi\in\Phi$ and $\psi\in\Psi$ such that*

$$\varphi(p(F(x,y),F(u,v))) \leq \varphi(\alpha p(x,u)+\beta p(y,v)) - \psi(\alpha p(x,u)+\beta p(y,v)), \tag{2.38}$$

for all $x,y,u,v\in X$ with $x\geq u$, $y\leq v$ and $\alpha+\beta<1$. Suppose either F is continuous or X has the following properties:

(i) *if a nondecreasing sequence $x_n \to x$, then $x_n\leq x$ for all n,*

(ii) *if a nonincreasing sequence $x_n \to x$, then $x\leq x_n$ for all n.*

If there exist $x_0,y_0\in X$ such that $x_0\leq F(x_0,y_0)$ and $F(y_0,x_0)\leq y_0$, then F has a unique coupled fixed point. In addition, if $x_0\leq y_0$ or $y_0\leq x_0$, then $x=F(x,y)=F(y,x)=y$ where (x,y) is a coupled fixed point of F.

Proof. Following the proof of Theorem 2.4, F has a unique coupled fixed point (x,y). We only have to show that $x=y$. Assume $y_0\leq x_0$. Using the mathematical induction, one can show that $x_n\geq y_n$ for any $n\in N$. Note that, by (p_4),

$$\begin{aligned} p(x,y) &\leq p(x,x_{n+1})+p(x_{n+1},y_{n+1})+p(y_{n+1},y) \\ &= p(x,x_{n+1})+p(y_{n+1},y)+p(F(x_n,y_n),F(y_n,x_n)). \end{aligned} \tag{2.39}$$

Therefore, using the condition (p_3), (2.1), and a property of φ,

$$\begin{aligned}\varphi(p(x,y)) &\le \varphi(p(x,x_{n+1})) + \varphi(p(y_{n+1},y)) + \varphi(p(F(x_n,y_n),F(y_n,x_n))) \\ &\le \varphi(p(x,x_{n+1}) + \varphi p(y_{n+1},y)) + \varphi((\alpha+\beta)p(x_n,y_n)) - \psi((\alpha+\beta)p(x_n,y_n)).\end{aligned} \tag{2.40}$$

From $\lim_{n\to\infty} p(x_n,x) = \lim_{n\to\infty} p(y_n,y) = 0$, we have $\lim_{n\to\infty} p(x_n,y_n) = p(x,y)$. Assume that $p(x,y) \neq 0$. Letting $n \to +\infty$ in (2.40) we get

$$\begin{aligned}\varphi(p(x,y)) &\le 2\varphi(0) + \varphi((\alpha+\beta)p(x,y)) - \lim_{n\to\infty}\psi((\alpha+\beta)p(x_n,y_n)) \\ &= \varphi((\alpha+\beta)p(x,y)) - \lim_{p(x_n,y_n)\to p(x,y)}\psi((\alpha+\beta)p(x_n,y_n)).\end{aligned} \tag{2.41}$$

Since $\alpha+\beta<1$, and φ is nondecreasing it follows that

$$\varphi(p(x,y)) \le \varphi(p(x,y)) - \lim_{p(x_n,y_n)\to p(x,y)}\psi((\alpha+\beta)p(x_n,y_n)), \tag{2.42}$$

that is, $\lim_{p(x_n,y_n)\to p(x,y)}\psi((\alpha+\beta)p(x_n,y_n)) \le 0$, which is a contradiction. Thus, $p(x,y)=0$, so $x=y$. □

Example 2.6 (see [17]). Let $X=[0,1]$ with usual order. Define $p:[0,1]\times[0,1]\to R^+$ by $p(x,y)=\max\{x,y\}$ and $F:[0,1]\times[0,1]\to[0,1]$ by $F(x,y)=(1/8)x$. Then,

(i) $(X,\le,p)$ is a complete partially ordered partial metric space,

(ii) F has the mixed monotone property,

(iii) for $x,y,u,v\in X$ with $x\ge u$ and $y\le v$, we have

$$p(F(x,y),F(u,v)) \le \frac{1}{8}(p(x,u)+p(y,v)), \tag{2.43}$$

(iv) F is continuous.

Proof. The proofs of (i), (ii), and (iii) are clear. To prove (iv), letting $(x,y)\in X\times X$ and $\epsilon>0$, we claim $F(B_q((x,y),8\epsilon)) \subseteq B_q(F(x,y),\epsilon)$. To prove our claim, let $(s,t)\in B_q((x,y),8\epsilon)$, then

$$q((s,t),(x,y)) < q((x,y),(x,y)) + 8\epsilon. \tag{2.44}$$

So,

$$p(s,x)+p(t,y) < p(x,x)+p(y,y)+8\epsilon. \tag{2.45}$$

Since $y\le p(t,y), p(x,x)=x$ and $p(y,y)=y$, we have $p(s,x)<x+8\epsilon$. Therefore, and hence $p(F(s,t),F(x,y)) \le p(F(x,y),p(x,y))$. So, $F(s,t)\in Bp(F(x,y),\epsilon)$. We deduce that all the

hypotheses of Theorem 2.1 are satisfied with $\varphi(t) = t$, $\psi(t) = 0$ and $\alpha = \beta = 1/8$. Therefore, F has a coupled fixed point. Here, $(0,0)$ is the coupled fixed point of F. □

3. Application

In this part, from previous obtained results, we will deduce some coupled fixed point results for mappings satisfying a contraction of integral type in a complete partial metric space.

Let Γ be the set of all functions $\alpha : [0,+\infty) \to [0,+\infty)$ satisfying the following conditions:

(i) α is a Lebesgue integrable mapping on each compact subset of $[0,+\infty)$,

(ii) for all $\epsilon > 0$, we have $\int_0^{\epsilon} \alpha(s)ds > 0$,

(iii) α is subadditive on each $[a,b] \subset [0,+\infty)$, that is,

$$\int_0^{a+b} \alpha(s)ds \leq \int_0^{a} \alpha(s)ds + \int_0^{b} \alpha(s)ds. \tag{3.1}$$

Let $N \in N^*$ be fixed. Let $\{\alpha_i\}1 \leq i \leq N$ be a family of N functions that belong to Γ. For all $t \geq 0$, we denote (I_i), $i = 1,\ldots,N$ as follows:

$$I_1(t) = \int_0^{t} \alpha_1(s)ds, \qquad I_2(t) = \int_0^{I_1(t)} \alpha_2(s)ds,\ldots, \qquad I_N(t) = \int_0^{I_{N-1}(t)} \alpha_N(s)ds. \tag{3.2}$$

We have the following result.

Theorem 3.1. *Let $(X,\leq)$ be a partially ordered set and suppose there is a partial metric p on Xsuch that (X,p) is a complete partial metric space. Let $F : X \times X \to X$ be a mapping having the mixed monotone property on X. Assume that there exist $\varphi \in \Phi$ and $\psi \in \Psi$ such that*

$$I_N(\varphi(p(F(x,y),F(u,v)))) \leq I_N(\varphi((\alpha p(x,u) + \beta p(y,v)))) - I_N(\psi(\alpha p(x,u) + \beta p(y,v))), \tag{3.3}$$

for all $x,y,u,v \in X$ with $x \geq u$, $y \leq v$, and $\alpha + \beta < 1$. Suppose either F is continuous, or X has the following properties:

(i) *if a nondecreasing sequence $x_n \to x$, then $x_n \leq x$ for all n,*

(ii) *if a nonincreasing sequence $x_n \to x$, then $x \leq x_n$ for all n.*

If there exist $x_0, y_0 \in X$ such that $x_0 \leq F(x_0,y_0)$ and $F(y_0,x_0) \leq y_0$, then there exist $x,y \in X$ such that $x = F(x,y)$ and $y = F(y,x)$, that is, F has a coupled fixed point.

Proof. Take $\tilde{\varphi} = I_N \circ \varphi$ and $\tilde{\psi} = I_N \circ \psi$. Note that the (α_i), $i = 1,\ldots,N$ are taken to be subadditive on each $[a,b] \subset [0,\infty)$ in order to get $\tilde{\varphi}(a+b) \leq \tilde{\varphi}(a) + \tilde{\varphi}(b)$. Moreover, it is easy to show that $\widetilde{\varphi}$ is continuous, nondecreasing and verifies $\widetilde{\varphi}(t) = 0 \Leftrightarrow t = 0$. We get that $\widetilde{\varphi} \in \Phi$. Also, we can find that $\widetilde{\psi} \in \Psi$. From (3.3), we have

$$\widetilde{\varphi}(p(F(x,y),F(u,v))) \leq \widetilde{\varphi}(\alpha p(x,u) + \beta p(y,v)) - \widetilde{\psi}(\alpha p(x,u) + \beta p(y,v)). \tag{3.4}$$

Now, applying Theorem 2.1, we obtain the desired result. □

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