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Research Article

Common Fixed Points of Generalized Meir-Keeler Type Condition and Nonexpansive Mappings

R. K. Bisht

Department of Mathematics, Kumaun University, D. S. B. Campus, Nainital 263002, India

Correspondence should be addressed to R. K. Bisht, ravindra.bisht@yahoo.com

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The aim of the present paper is to obtain common fixed point theorems by employing the recently introduced notion of weak reciprocal continuity. The new notion is a proper generalization of reciprocal continuity and is applicable to compatible mappings as well as noncompatible mappings. We demonstrate that weak reciprocal continuity ensures the existence of common fixed points under contractive conditions, which otherwise do not ensure the existence of fixed points. Our results generalize and extend Banach contraction principle and Meir-Keeler-type fixed point theorem.

1. Introduction

In his earlier works, Pant [1, 2] introduced the notion of reciprocal continuity and obtained the first results that established a situation in which a collection of mappings has a fixed point, which is a point of discontinuity for all the mappings. These papers are the genesis of a large number of papers (e.g., [3–16]) that employ or deal with reciprocal continuity to study fixed points of discontinuous mappings in various settings. Imdad and Ali [4] used this concept in the setting of non-self-mappings. Singh et al. [9, 10] have obtained applications of reciprocal continuity for hybrid pair of mappings. Balasubramaniam et al. [14] (see also [15]) extended the study of reciprocal continuity to fuzzy metric spaces. Kumar and Pant [16] studied this concept in the setting of probabilistic metric space. Muralisankar and Kalpana [11] established a common fixed point theorem in an intuitionistic fuzzy metric space using contractive condition of integral type.

In 1986, Jungck [17] generalized the notion of weakly commuting maps by introducing the concept of compatible maps.

Definition 1.1. Two self-maps f and g of a metric space (X, d) are called compatible [17] if $\lim_n d(fgx_n, gfx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_n fx_n = \lim_n gx_n = t$ for some t in X.

The definition of compatibility implies that the mappings f and g will be noncompatible if there exists a sequence $\{x_n\}$ in X such that $\lim_n fx_n = \lim_n gx_n = t$ for some t in X but $\lim_n d(fgx_n, gfx_n)$ is either nonzero or nonexistent.

Definition 1.2. Two self-maps f and g are called pointwise R-weakly commuting [1] (see also [18, 19]) on X if given x in X there exists $R > 0$ such that $d(fgx, gfx) \leq Rd(fx, gx)$.

Definition 1.3. Two self-maps f and g are called pointwise R-weakly commuting of type (A_f) [20] (see also [21]) on X if given x in X there exists $R > 0$ such that $d(fgx, ggx) \leq Rd(fx, gx)$.

Definition 1.4. Two self-maps f and g are called pointwise R-weakly commuting of type (A_g) [20] on X if given x in X there exists $R > 0$ such that $d(ffx, gfx) \leq Rd(fx, gx)$.

Definition 1.5. A pair (f, g) of self-mappings defined on a nonempty set X is said to be weakly compatible [22] if the pair commutes on the set of coincidence points, that is, $fx = gx$ $(x \in X)$ implies $fgx = gfx$.

It is well known now that pointwise R-weak commutativity and analogous notions of pointwise R-weak commutativity of type (A_f) or pointwise R-weak commutativity of type (A_g) are equivalent to commutativity at coincidence points and in the setting of metric spaces these notions are equivalent to weak compatibility. On the other hand, pointwise R-weak commutativity and analogous notions of pointwise R-weak commutativity of type (A_f) or (A_g) are more useful in establishing common fixed point theorems since they not only imply commutativity at coincidence points but may also help in the determination of coincidence points [19, 21].

In a recent work, Al-Thagafi and Shahzad [23] generalized the notion of nontrivial weakly compatible maps by introducing the notion of occasionally weakly compatible mappings.

Definition 1.6. A pair (f, g) of self-mappings defined on a nonempty set X is said to be occasionally weakly compatible [23] (in short owc) if there exists a point x in X, which is a coincidence point of f and g at which f and g commute.

Definition 1.7. Two self-mappings f and g of a metric space (X, d) are called conditionally commuting [24] if they commute on a nonempty subset of the set of coincidence points whenever the set of their coincidences is nonempty.

From the definition itself, it is clear that if two maps are weakly compatible or owc then they are necessarily conditionally commuting; however, the conditionally commuting mappings are not necessarily weakly compatible or owc [24].

Definition 1.8. Let f and $g(f \neq g)$ be two self-maps of a metric space (X, d), then f is called g-absorbing [25] if there exists some positive real number R such that $d(gx, gfx) \leq Rd(fx, gx)$ for all x in X. Similarly, g will be called f-absorbing if there exists some positive real number R such that $d(fx, fgx) \leq Rd(fx, gx)$ for all x in X.

It is well known that the absorbing maps are neither a subclass of compatible maps nor a subclass of noncompatible maps [25].

Definition 1.9. Two self-mappings f and g of a metric space (X, d) are called reciprocally continuous [1, 2] if and only if $fgx_n \to ft$ and $gfx_n \to gt$ whenever $\{x_n\}$ is a sequence such that $\lim_n fx_n = \lim_n gx_n = t$ for some t in X.

If f and g are both continuous, then they are obviously reciprocally continuous but the converse is not true [1, 2]. The notion of reciprocal continuity is mainly applicable to compatible mapping satisfying contractive conditions [7]. To widen the scope of the study of fixed points from the class of compatible mappings satisfying contractive conditions to a wider class including compatible as well as noncompatible mappings satisfying contractive, nonexpansive, or Lipschitz-type condition Pant et al. [7] generalized the notion of reciprocal continuity by introducing the new concept of weak reciprocal continuity as follows.

Definition 1.10. Two self-mappings f and g of a metric space (X, d) are called weakly reciprocally continuous [7] iff $fgx_n \to ft$ or $gfx_n \to gt$, whenever $\{x_n\}$ is a sequence in X such that $\lim_n fx_n = \lim_n gx_n = t$ for some t in X.

We now give examples of compatible and weakly reciprocally continuous mappings with or without common fixed points.

Example 1.11. Let X = [0, 1] and d be the usual metric on X. Define $f, g : X \to X$ by

$$fx = x, \quad \forall x, \qquad gx = \frac{x}{2} \quad \text{if } x > 0, \; g(0) = 1. \tag{1.1}$$

Then it can be verified that f and g are compatible as well as weakly reciprocally continuous mappings but do not have a common fixed point.

Example 1.12. Let X = [0, 1] and d be the usual metric on X. Define $f, g : X \to X$ by

$$fx = (1 - x), \quad \forall x, \qquad gx = \text{fractional part of } (1 - x). \tag{1.2}$$

It may be noted that f and g are compatible as well as weakly reciprocally continuous mappings and have infinitely many common fixed points. Examples of noncompatible weakly reciprocally continuous mappings are given on the following pages.

If f and g are reciprocally continuous, then they are obviously weakly reciprocally continuous but, as shown in Example 2.2 below, the converse is not true. As an application of weak reciprocal continuity we prove common fixed point theorems under contractive conditions that extend the scope of the study of common fixed point theorems from the class of compatible continuous mappings to a wider class of mappings, which also includes noncompatible and discontinuous mappings. Our results also demonstrate the usefulness of the notion of the absorbing maps in fixed point considerations.

2. Main Results

Theorem 2.1. *Let f and g be weakly reciprocally continuous pointwise R-weakly commuting of type (A_f) self-mappings of a complete metric space (X, d) such that*

(i) $fX \subseteq gX$,

(ii) $d(fx, fy) \leq kd(gx, gy), k \in [0, 1)$.

If g is f-absorbing or f is g-absorbing, then f and g have a unique common fixed point.

Proof. Let x_0 be any point in X. Define sequences $\{x_n\}$ and $\{y_n\}$ in X by

$$y_n = fx_n = gx_{n+1}. \tag{2.1}$$

We claim that $\{y_n\}$ is a Cauchy sequence. Using (ii), we obtain

$$\begin{aligned} d(y_n, y_{n+1}) &= d(fx_n, fx_{n+1}) \leq kd(gx_n, gx_{n+1}) \\ &= kd(y_{n-1}, y_n) \leq \cdots \leq k^n d(y_0, y_1). \end{aligned} \tag{2.2}$$

Moreover, for every integer $p > 0$, we get

$$\begin{aligned} d(y_n, y_{n+p}) &\leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + \cdots + d(y_{n+p-1}, y_{n+p}) \\ &\leq d(y_n, y_{n+1}) + kd(y_n, y_{n+1}) + \cdots + k^{p-1}d(y_n, y_{n+1}) \\ &= \left(1 + k + k^2 + \cdots + k^{p-1}\right)d(y_n, y_{n+1}) \\ &\leq \left(\frac{1}{1-k}\right)d(y_n, y_{n+1}) \leq \left(\frac{k^n}{1-k}\right)d(y_0, y_1). \end{aligned} \tag{2.3}$$

This means that $d(y_n, y_{n+p}) \to 0$ as $n \to \infty$. Therefore, $\{y_n\}$ is a Cauchy sequence. Since X is complete, there exists a point t in X such that $y_n \to t$. Moreover, $y_n = fx_n = gx_{n+1} \to t$.

Suppose that g is f-absorbing. Now, weak reciprocal continuity of f and g implies that $fgx_n \to ft$ or $gfx_n \to gt$. Let $gfx_n \to gt$. By virtue of (2.1), this also yields $ggx_{n+1} = gfx_n \to gt$. Since g is f-absorbing, $d(fx_n, fgx_n) \leq Rd(fx_n, gx_n)$. On letting $n \to \infty$, we obtain $fgx_n \to t$. Using (ii), we get $d(ft, fgx_n) \leq kd(gt, ggx_n)$. On making $n \to \infty$ we get $fgx_n \to ft$. Hence $t = ft$. Since $fX \subseteq gX$, there exists u in X such that $t = ft = gu$. Now using (ii), we obtain $d(fx_n, fu) \leq kd(gx_n, gu)$. On letting $n \to \infty$, we get $fu = gu$. Since f and g are pointwise R-weak commutative of type (A_f), we have $d(fgu, ggu) \leq R_1 d(fu, gu) = 0$ for some $R_1 > 0$, that is, $fgu = ggu$. Thus $fgu = gfu = ggu = ffu$. Finally using (ii), we obtain $d(fu, ffu) \leq kd(gu, gfu) = kd(fu, ffu)$, that is, $(1-k)d(fu, ffu) = 0$. Hence $fu = ffu = gfu$ and fu is a common fixed point of f and g.

Next suppose that $fgx_n \to ft$. Since g is f-absorbing, $d(fx_n, fgx_n) \leq Rd(fx_n, gx_n)$. On letting $n \to \infty$, we get $t = ft$. Since $fX \subseteq gX$, there exists u in X such that $t = ft = gu$. Now using (ii), we obtain $d(fx_n, fu) \leq kd(gx_n, gu)$. On letting $n \to \infty$, we get $fu = t$. Thus $fu = gu$. Since f and g are pointwise R-weak commutative of type (A_f), we have $d(fgu, ggu) \leq R_1 d(fu, gu) = 0$ for some $R_1 > 0$, that is, $fgu = ggu$. Thus $fgu = gfu =$

$ggu = ffu$. Finally using (ii), we obtain $d(fu, ffu) \leq kd(gu, gfu) = kd(fu, ffu)$, that is, $(1-k)d(fu, ffu) = 0$. Hence $fu = ffu = gfu$ and fu is a common fixed point of f and g.

Finally suppose that f is g-absorbing. Now, weak reciprocal continuity of f and g implies that $fgx_n \to ft$ or $gfx_n \to gt$. Let us first assume that $gfx_n \to gt$. Since f is g-absorbing, $d(gx_n, gfx_n) \leq Rd(fx_n, gx_n)$. On making $n \to \infty$, we get $t = gt$. Using (ii) we get $d(fx_n, ft) \leq kd(gx_n, gt)$. On letting $n \to \infty$, we get $fx_n \to ft$. Hence $t = ft = gt$ and t is a common fixed point of f and g.

Next suppose that $fgx_n \to ft$. Then $fX \subseteq gX$ implies that $ft = gu$ for some $u \in X$ and $fgx_n \to gu$. By virtue of (2.1), this also yields $ffx_{n-1} \to gu$. Since f is g-absorbing, $d(gx_n, gfx_n) \leq Rd(fx_n, gx_n)$. On letting $n \to \infty$, we get $gfx_n \to t$. Now, using (ii), we get $d(fx_n, ffx_n) \leq kd(gx_n, gfx_n)$. On making $n \to \infty$, we obtain $t = gu$. Again, by virtue of (ii), $d(fx_n, fu) \leq kd(gx_n, gu)$. Making $n \to \infty$ we get $fu = t$. Hence $t = fu = gu$. Since f and g are pointwise R-weak commutative of type (A_f), we have $d(fgu, ggu) \leq R_1 d(fu, gu) = 0$ for some $R_1 > 0$, that is, $fgu = ggu$. Thus $fgu = gfu = ggu = ffu$. Finally using (ii), we obtain $d(fu, ffu) \leq kd(gu, gfu) = kd(fu, ffu)$, that is, $(1-k)d(fu, ffu) = 0$. Hence $fu = ffu = gfu$ and fu is a common fixed point of f and g. □

Uniqueness of the common fixed point theorem follows easily in each of the two cases. We now give an example to illustrate the above theorem.

Example 2.2. Let X $= [2, 20]$ and d be the usual metric on X. Define $f, g : \mathrm{X} \to \mathrm{X}$ as follows:

$$\begin{gathered} fx = 2 \quad \text{if } x = 2 \text{ or } x > 5, \qquad fx = 6 \quad \text{if } 2 < x \leq 5, \\ g2 = 2, \quad gx = 12 \quad \text{if } 2 < x \leq 5, \qquad gx = \frac{(x+1)}{3} \quad \text{if } x > 5. \end{gathered} \tag{2.4}$$

Then f and g satisfy all the conditions of Theorem 2.1 and have a unique common fixed point at $x = 2$. It can be verified in this example that f and g satisfy the contraction condition (ii) for $k = 4/5$. The mappings f and g are pointwise R-weakly commuting of type (A_f) maps as they commute at their only coincidence point $x = 2$. Furthermore, f is g-absorbing with $R = 29/18$. It can also be noted that f and g are weakly reciprocally continuous. To see this, let $\{x_n\}$ be a sequence in X such that $fx_n \to t$, $gx_n \to t$ for some t. Then $t = 2$ and either $x_n = 2$ for each n from some place onwards or $x_n = 5 + \varepsilon_n$, where $\varepsilon_n \to 0$ as $n \to \infty$. If $x_n = 2$ for each n from some place onwards, $fgx_n \to 2 = f2$ and $gfx_n \to 2 = g2$. If $x_n = 5 + \varepsilon_n$, then $fx_n \to 2$, $gx_n = (2 + \epsilon_n/3) \to 2$, $fgx_n = f(2 + \epsilon_n/3) \to 6 \neq f2$, and $gfx_n \to g2 = 2$. Thus $\lim_{n\to\infty} gfx_n = g2$ but $\lim_{n\to\infty} fgx_n \neq f2$. Hence f and g are weakly reciprocally continuous. It is also obvious that f and g are not reciprocally continuous mappings.

Remark 2.3. Putting g equal to identity map, we get the famous Banach fixed point theorem as a particular case of the above theorem.

We now establish a common fixed point theorem for a pair of mappings satisfying an $(\in, \delta)$ type contractive condition. It is now well known (e.g., Example 2.4 below) that an $(\in, \delta)$ contractive condition does not ensure the existence of a fixed point.

Example 2.4 (see [26]). Let $X = [0,2]$ and d be the Euclidean metric on X. Define $f : X \to X$ by

$$fx = \frac{(1+x)}{2} \quad \text{if } x < 1, \qquad fx = 0 \quad \text{if } x \geq 1. \tag{2.5}$$

Then f satisfies the contractive condition

$$\varepsilon \leq \max\{d(x,y), d(x,fx), d(y,fy)\} < \varepsilon + \delta \Longrightarrow d(fx,fy) < \varepsilon \tag{2.6}$$

with $\delta(\varepsilon) = 1$ for $\varepsilon \geq 1$ and $\delta(\varepsilon) = 1 - \varepsilon$ for $\varepsilon < 1$ but f does not have a fixed point.

In view of the above example, the next theorem demonstrates the usefulness of weak reciprocal continuity and shows that the new notion ensures the existence of a common fixed point under an (ε, δ) contractive condition.

Theorem 2.5. *Let f and g be weakly reciprocally continuous pointwise R-weakly commuting of type (A_f) self-mappings of a complete metric space (X,d) such that*

(i) $fX \subseteq gX$;

(ii) $d(fx,fy) < d(gx,gy)$ *whenever* $gx \neq gy$;

(iii) *given* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that*

$$\varepsilon < d(gx,gy) < \varepsilon + \delta \Longrightarrow d(fx,fy) \leq \varepsilon. \tag{2.7}$$

If g is f-absorbing or f is g-absorbing, then f and g have a unique common fixed point.

Proof. Let x_0 be any point in X. Define sequences $\{x_n\}$ and $\{y_n\}$ in X by

$$y_n = fx_n = gx_{n+1}. \tag{2.8}$$

We claim that $\{y_n\}$ is a Cauchy sequence. Using (ii), we obtain

$$d(y_n, y_{n+1}) = d(fx_n, fx_{n+1}) < d(gx_n, gx_{n+1}) = d(y_{n-1}, y_n). \tag{2.9}$$

Thus $\{d(y_n, y_{n+1})\}$ is a strictly decreasing sequence of positive real numbers and, therefore, tends to a limit $r \geq 0$, that is, $\lim_{n\to\infty} d(y_n, y_{n+1}) = r$, $r \geq 0$. We assert that $r = 0$. For, if not, suppose that $r > 0$. Then given $\delta > 0$, no matter small δ may be, there exists a positive integer N such that for each $n \geq N$, we have

$$r < d(y_n, y_{n+1}) = d(fx_n, fx_{n+1}) < r + \delta, \tag{2.10}$$

that is,

$$r < d(gx_{n+1}, gx_{n+2}) < r + \delta. \tag{2.11}$$

Selecting δ in (2.11) in accordance with (iii), for each $n \geq N$, we get $d(fx_{n+1}, fx_{n+2}) \leq r$, that is, $d(y_{n+1}, y_{n+2}) \leq r$, a contradiction to (2.11). Therefore, $\lim_{n\to\infty} d(y_n, y_{n+1}) = 0$. We now show that $\{y_n\}$ is a Cauchy sequence. Suppose it is not. Then there exist an $\varepsilon > 0$ and a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that $d(y_{n_i}, y_{n_{i+1}}) \geq 2\varepsilon$. Select δ in (iii) so that $0 < \delta \leq \varepsilon$. Since $\lim_{n\to\infty} d(y_n, y_{n+1}) = 0$, there exists an integer N such that $d(y_n, y_{n+1}) < \delta/6$ whenever $n \geq N$.

Let $n_i \geq N$. Then, there exist integers m_i satisfying $n_i < m_i < n_{i+1}$ such that $d(y_{n_i}, y_{m_i}) \geq \varepsilon + (\delta/3)$. If not, then

$$\begin{aligned} d(y_{n_i}, y_{n_{i+1}}) &\leq d(y_{n_i}, y_{n_{i+1}-1}) + d(y_{n_{i+1}-1}, y_{n_{i+1}}) \\ &< \varepsilon + \left(\frac{\delta}{3}\right) + \left(\frac{\delta}{6}\right) < 2\varepsilon, \end{aligned} \tag{2.12}$$

a contradiction. Let m_i be the smallest integer such that $d(y_{n_i}, y_{m_i}) \geq \varepsilon + (\delta/3)$. Then $d(y_{n_i}, y_{m_i-2}) < \varepsilon + (\delta/3)$ and

$$\begin{aligned} \varepsilon + \left(\frac{\delta}{3}\right) \leq d(y_{n_i}, y_{m_i}) &\leq d(y_{n_i}, y_{m_i-2}) + d(y_{m_i-2}, y_{m_i-1}) + d(y_{m_i-1}, y_{m_i}) \\ &< \varepsilon + \left(\frac{\delta}{3}\right) + \left(\frac{\delta}{6}\right) + \left(\frac{\delta}{6}\right) < \varepsilon + \left(\frac{2\delta}{3}\right), \end{aligned} \tag{2.13}$$

that is, $\varepsilon < \varepsilon + (\delta/3) \leq d(gx_{n_i+1}, gx_{m_i+1}) < \varepsilon + (2/3)\delta$. In view of (iii), this yields $d(y_{n_i+1}, y_{m_i+1}) \leq \varepsilon$. But then

$$\begin{aligned} d(y_{n_i}, y_{m_i}) &\leq d(y_{n_i}, y_{n_i+1}) + d(y_{n_i+1}, y_{m_i+1}) + d(y_{m_i+1}, y_{m_i}) \\ &< \left(\frac{\delta}{6}\right) + \varepsilon + \left(\frac{\delta}{6}\right) = \varepsilon + \left(\frac{\delta}{3}\right), \end{aligned} \tag{2.14}$$

which contradicts (2.13). Hence $\{y_n\}$ is a Cauchy sequence. Since X is complete, there exists a point t in X such that $y_n \to t$. Moreover, $y_n = fx_n = gx_{n+1} \to t$.

Suppose that g is f-absorbing. Now, weak reciprocal continuity of f and g implies that $fgx_n \to ft$ or $gfx_n \to gt$. Let $gfx_n \to gt$. By virtue of (2.8), this also yields $ggx_{n+1} = gfx_n \to gt$. Since g is f-absorbing, $d(fx_n, fgx_n) \leq Rd(fx_n, gx_n)$. On letting $n \to \infty$, we get $fgx_n \to t$. Using (ii), we get $d(ft, fgx_n) < d(gt, ggx_n)$. On making $n \to \infty$, we get $fgx_n \to ft$. Hence $t = ft$. Since $fX \subseteq gX$, there exists u in X such that $t = ft = gu$. Now using (ii), we obtain $d(fx_n, fu) < d(gx_n, gu)$. On letting $n \to \infty$, we get $fu = t$. Thus $fu = gu$. Since f and g are pointwise R-weak commutative of type (A_f), we have $d(fgu, ggu) \leq R_1 d(fu, gu) = 0$ for some $R_1 > 0$, that is, $fgu = ggu$. Thus $fgu = gfu = ggu = ffu$. If $fu \neq ffu$, then using (ii) we get $d(fu, ffu) < d(gu, gfu) = d(fu, ffu)$, a contradiction. Hence $fu = ffu = gfu$ and fu is a common fixed point of f and g.

Next suppose that $fgx_n \to ft$. Since g is f-absorbing, $d(fx_n, fgx_n) \leq Rd(fx_n, gx_n)$. On letting $n \to \infty$, we get $t = ft$. Since $fX \subseteq gX$, there exists u in X such that $t = ft = gu$. Now using (ii), we obtain $d(fx_n, fu) < d(gx_n, gu)$. On letting $n \to \infty$, we get $fu = t$. Thus $fu = gu$. Since f and g are pointwise R-weak commutative of type (A_f), we have $d(fgu, ggu) \leq R_1 d(fu, gu) = 0$ for some $R_1 > 0$, that is, $fgu = ggu$. Thus $fgu = gfu = ggu = ffu$. If $fu \neq ffu$ then using (ii) we get $d(fu, ffu) < d(gu, gfu) = d(fu, ffu)$, a contradiction. Hence $fu = ffu = gfu$ and fu is a common fixed point of f and g.

When f is assumed g-absorbing, the proof follows on similar lines as in the corresponding part of Theorem 2.1. □

We now give an example to illustrate Theorem 2.5.

Example 2.6. Let $X = [2, 20]$ and d be the usual metric on X. Define $f, g : X \to X$ as follows:

$$fx = 2 \quad \text{if } x = 2 \text{ or } x > 5, \qquad fx = 6 \quad \text{if } 2 < x \leq 5,$$
$$g2 = 2, \quad gx = \frac{(x+31)}{3} \quad \text{if } 2 < x \leq 5, \qquad gx = \frac{(x+1)}{3} \quad \text{if } x > 5. \tag{2.15}$$

Then f and g satisfy all the conditions of Theorem 2.5 and have a unique common fixed point at $x = 2$. It can be seen in this example that f and g satisfy the condition (ii) and the condition

$$\varepsilon < d(gx, gy) < \varepsilon + \delta \Longrightarrow d(fx, fy) \leq \varepsilon \tag{2.16}$$

with $\delta(\varepsilon) = 1$ for $\varepsilon \geq 4$ and $\delta(\varepsilon) = 4 - \varepsilon$ for $\varepsilon < 4$. Furthermore, f is g-absorbing with $R = 2$. It can also be noted that f and g are weakly reciprocally continuous. To see this, let $\{x_n\}$ be a sequence in X such that $fx_n \to t, gx_n \to t$ for some t. Then $t = 2$ and either $x_n = 2$ for each n from some place onwards or $x_n = 5 + \varepsilon_n$ where $\varepsilon_n \to 0$ as $n \to \infty$. If $x_n = 2$ for each n from some place onwards, $fgx_n \to 2 = f2$ and $gfx_n \to 2 = g2$. If $x_n = 5 + \varepsilon_n$, then $fx_n \to 2, gx_n = (2 + \epsilon_n/3) \to 2, fgx_n = f(2 + \epsilon_n/3) \to 6 \neq f2$, and $gfx_n \to g2 = 2$. Thus $\lim_{n\to\infty} gfx_n = g2$ but $\lim_{n\to\infty} fgx_n \neq f2$. Hence f and g are weakly reciprocally continuous. It is also obvious that f and g are not reciprocally continuous mappings. Further, f and g are pointwise R-weakly commuting of type (A_f) maps as they commute at their only coincidence point $x = 2$.

Remark 2.7. Theorem 2.5 generalizes the well-known fixed point theorem of Meir and Keeler [27].

It may be observed that the mappings f and g in Examples 2.2 and 2.6 are noncompatible mappings. However, in the case of noncompatible mappings there is an alternative method of proving the existence of fixed points [6, 7, 11, 19, 24, 26, 28–32]. This alternative method was introduced by Pant [19, 26, 28–30] and is also applicable under strictly contractive [19, 26, 31–33], nonexpansive [7], and Lipschitz-type conditions [6, 24, 30]. The existence of such a method is important since there is no general method for studying the fixed points of nonexpansive or Lipschitz-type mapping pairs in ordinary metric spaces.

In the area of fixed point theory, Lipschitz type mappings constitute a very important class of mappings and include contraction mappings, contractive mappings and, nonexpansive mappings as subclasses. The next theorem provides a good illustration of the applicability of recently introduced notions of conditional commutativity and weak reciprocal continuity to establish a situation in which a pair of mappings may possess common fixed points as well as coincidence points, which may not be common fixed points.

Theorem 2.8. *Let f and g be weakly reciprocally continuous noncompatible self-mappings of a metric space (X, d) satisfying*

(i) $fX \subseteq gX$,

(ii) $d(fx, fy) \leq kd(gx, gy), k \geq 0$.

If f and g are conditionally commuting and g is f-absorbing or f is g-absorbing, then f and g have a common fixed point.

Proof. Since f and g are noncompatible maps, there exists a sequence $\{x_n\}$ in X such that $fx_n \to t$ and $gx_n \to t$ for some t in X but either $\lim_n d(fgx_n, gfx_n) \neq 0$ or the limit does not exist. Since $fX \subseteq gX$, for each x_n there exists y_n in X such that $fx_n = gy_n$. Thus $fx_n \to t$, $gx_n \to t$ and $gy_n \to t$ as $n \to \infty$. By virtue of this and using (ii), we obtain $fy_n \to t$. Therefore, we have

$$fx_n = gy_n \longrightarrow t, \qquad gx_n \longrightarrow t, \qquad fy_n \longrightarrow t. \tag{2.17}$$

Suppose that g is f-absorbing. Then $d(fx_n, fgx_n) \leq Rd(fx_n, gx_n)$ and $d(fy_n, fgy_n) \leq Rd(fy_n, gy_n)$. On letting $n \to \infty$, these inequalities yield

$$fgx_n \longrightarrow t, \qquad fgy_n (= ffx_n) \longrightarrow t. \tag{2.18}$$

Weak reciprocal continuity of f and g implies that $fgx_n \to ft$ or $gfx_n \to gt$. Let $gfx_n \to gt$. By virtue of (ii), we get $d(ffx_n, ft) \leq kd(gfx_n, gt)$. On letting $n \to \infty$, we get $ffx_n \to ft$. In view of (2.18), this yields $t = ft$. Since $fX \subseteq gX$, there exists u in X such that $t = ft = gu$. Now using (ii), we obtain $d(fx_n, fu) \leq kd(gx_n, gu)$. On letting $n \to \infty$, we get $fu = t$. Thus $fu = gu$. Conditional commutativity of f and g implies that f and g commute at u, or there exists a coincidence point v of f and g at which f and g commute. Suppose f and g commute at the coincidence point v. Then $fv = gv$ and $fgv = gfv$. Also $ffv = fgv = gfv = ggv$. Since g is f-absorbing $d(fv, fgv) \leq Rd(fv, gv)$. This yields $fv = fgv$. Hence $fv = ffv = gfv$ and fv is a common fixed point of f and g.

Next suppose that $fgx_n \to ft$. In view of (2.18), we get $t = ft$. Since $fX \subseteq gX$, there exists u in X such that $t = ft = gu$. Now using (ii), we obtain $d(fx_n, fu) \leq kd(gx_n, gu)$. On letting $n \to \infty$, we get $fu = t$. Thus $fu = gu$. This, in view of conditional commutativity and f-absorbing property of g, implies that f and g have a common fixed point.

Now suppose that f is g-absorbing. Then $d(gx_n, gfx_n) \leq Rd(fx_n, gx_n)$ and $d(gy_n, gfy_n) \leq Rd(fy_n, gy_n)$. On letting $n \to \infty$, these inequalities yield

$$gfx_n (= ggy_n) \longrightarrow t, \qquad gfy_n \longrightarrow t. \tag{2.19}$$

Weak reciprocal continuity of f and g implies that $fgy_n \to ft$ or $gfy_n \to gt$. Let us first assume that $gfy_n \to gt$. In view of (2.19), this yields $t = gt$. Using (ii) we get $d(fx_n, ft) \leq kd(gx_n, gt)$. On letting $n \to \infty$, we obtain $t = ft$. Hence $t = ft = gt$ and t is a common fixed point of f and g.

Next suppose that $fgy_n \to ft$. Then $fX \subseteq gX$ implies that $ft = gu$ for some $u \in X$. Therefore, $fgy_n \to ft = gu$. Using (ii) and in view of (2.19), we get $d(fy_n, fgy_n) \leq kd(gy_n, ggy_n)$. On letting $n \to \infty$, we get $t = gu$. Again, by virtue of (ii), we obtain $d(fy_n, fu) \leq kd(gy_n, gu)$. Making $n \to \infty$, we get $t = fu$. Hence $fu = gu$. Conditional commutativity of f and g implies that f and g commute at u or there exists a coincidence point v of f and g at which f and g commute. Suppose f and g commute at the coincidence point v. Then $fv = gv$ and $fgv = gfv$. Also $ffv = fgv = gfv = ggv$. Since f is g-absorbing, $d(gv, gfv) \leq Rd(fv, gv)$. This yields $gv = gfv$. Hence $fv = ffv = gfv$ and fv is a common fixed point of f and g. This completes the proof of the theorem. □

We now give examples to illustrate Theorem 2.8.

Example 2.9. Let $X = [0,1]$ and d be the usual metric on X. Define $f, g : X \to X$ by $fx = (1/2) - |x - (1/2)|$,

$$gx = \frac{2}{3}(1 - x). \tag{2.20}$$

Then f and g satisfy all the conditions of the above theorem and have two coincidence points $x = 1, 2/5$ and a common fixed point $x = 2/5$. It may be verified in this example that $f(X) = [0, 1/2]$, $g(X) = [0, 2/3]$ and $fX \subseteq gX$. Also that f and g are noncompatible but conditionally commuting maps. Furthermore, f and g are conditionally commuting since they commute at their coincidence point $2/5$. To see that f and g are noncompatible, let us consider the sequence $\{x_n\}$ given by $x_n = 1 - (1/n)$. Then $fx_n \to 0$, $gx_n \to 0$, $fgx_n \to 0$, and $gfx_n \to 2/3$. Hence f and g are noncompatible. It may also be verified that f and g are not pointwise R-weakly commuting of type (A_f) as they do not commute at the coincidence point $x = 1$, since $f(g(1)) = 0$ and $g(f(1)) = 2/3$. It is also easy to verify that f and g satisfy the Lipschitz-type condition $d(fx, fy) \le (3/2)d(gx, gy)$ together with f-absorbing condition $d(fx, fgx) \le d(fx, gx)$ for all x. It can also be noted that f and g are weakly reciprocally continuous since both f and g are continuous.

In Example 2.9, f and g are not pointwise R-weakly commuting of type (A_f) as they do not commute at the coincidence point $x = 1$. We now give an example of pointwise R-weakly commuting of type (A_f) maps satisfying Theorem 2.8.

Example 2.10. Let $X = [0,1]$ and d be the usual metric on X. Define $f, g : X \to X$ as follows:

$$\begin{aligned} fx &= \frac{1}{2} - \left| x - \frac{1}{2} \right|, \\ gx &= \left(\frac{2}{3}\right) \text{ fractional part of } (1 - x). \end{aligned} \tag{2.21}$$

Then f and g satisfy all the conditions of the above theorem and have three coincidence points $x = 0, 2/5, 1$ and two common fixed point $x = 0, 2/5$. It may be verified in this example that $f(X) = [0, 1/2]$, $g(X) = [0, 2/3)$ and $fX \subseteq gX$. Also, f and g are pointwise R-weakly commuting of type (A_f) maps, hence also conditionally commuting, since they commute at each of their coincidence points, namely, $x = 0, 2/5, 1$. To see that f and g are noncompatible, let us consider the sequence $\{x_n\}$ given by $x_n = 1 - 1/n$. Then $fx_n \to 0$, $gx_n \to 0$, $fgx_n \to 0$, and $gfx_n \to 2/3$. Hence f and g are noncompatible. It is also easy to verify that f and g satisfy the Lipschitz-type condition $d(fx, fy) \le (3/2)d(gx, gy)$. The mapping g is f-absorbing since $d(fx, fgx) \le d(fx, gx)$ for all x. It can also be noted that f and g are weakly reciprocally continuous. To see this, let $\{x_n\}$ be a sequence in X such that $fx_n \to t$, $gx_n \to t$ for some t. Then $t = 0$ and either $x_n = 0$ for each n or $x_n \to 1$. If $x_n = 0$ for each n, then $fx_n \to 0$, $gx_n \to 0$, $fgx_n \to 0 = f(0)$, and $gfx_n \to 0 = g(0)$. If $x_n \to 1$, then $fx_n \to 0$, $gx_n \to 0$, $fgx_n \to 0 = f(0)$, and $gfx_n \to 2/3 \ne g(0)$. Thus $\lim_{n\to\infty} fgx_n = f(0)$ but $\lim_{n\to\infty} gfx_n \ne g(0)$. Hence f and g are weakly reciprocally continuous.

Putting $k = 1$ in Theorem 2.8, we get a common fixed point theorem for a non-expansive-type mapping pair.

Corollary 2.11. *Let f and g be weakly reciprocally continuous noncompatible self-mappings of a metric space (X, d) satisfying*

(i) $fX \subseteq gX$,

(ii) $d(fx, fy) \leq d(gx, gy)$.

If f and g are conditionally commuting and g is f-absorbing or f is g-absorbing, then f and g have a common fixed point.

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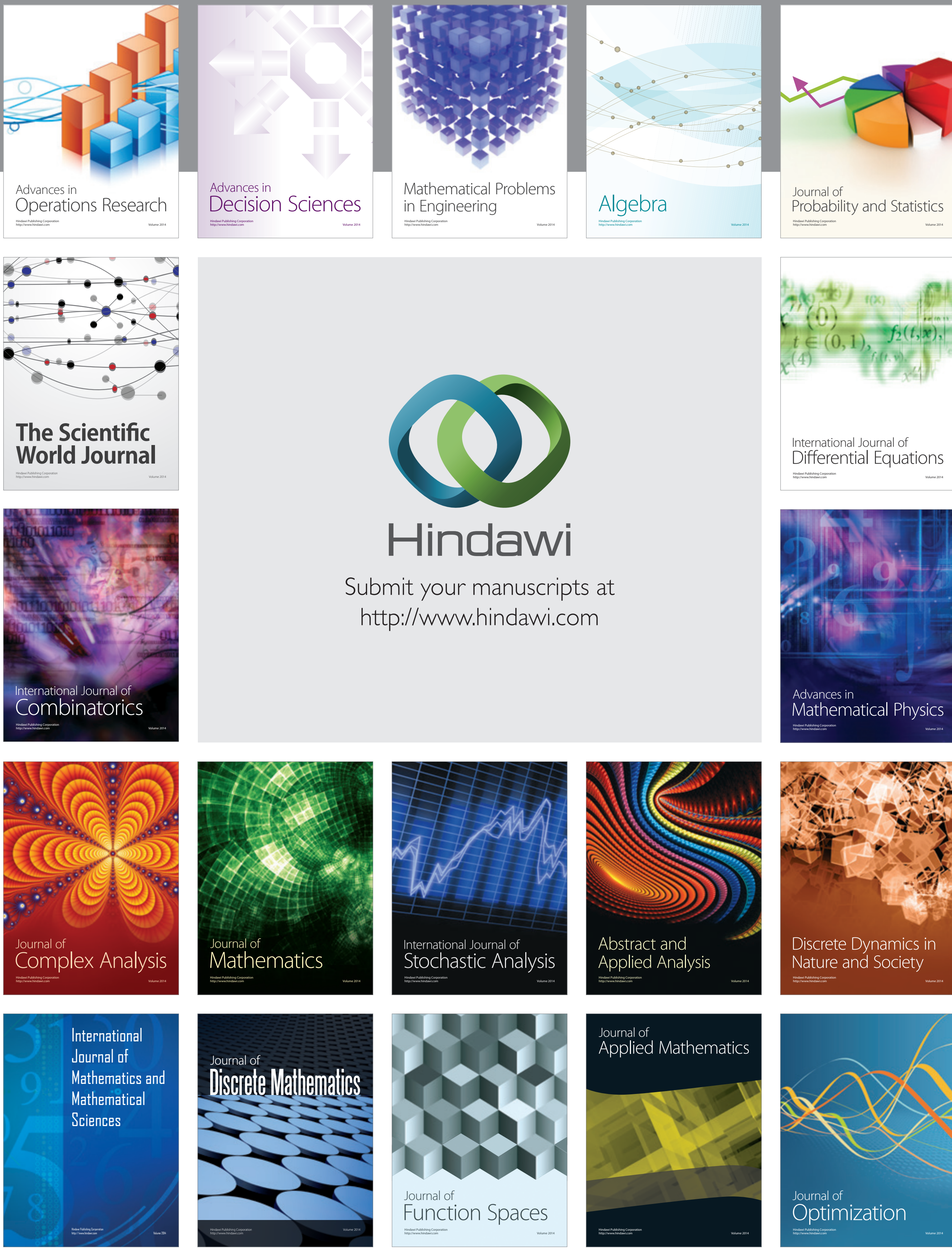
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