

Research Article

Generalized (q, w) -Euler Numbers and Polynomials Associated with p -Adic q -Integral on \mathbb{Z}_p

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We generalize the Euler numbers and polynomials by the generalized (q, w) -Euler numbers $E_{n,q,w}(a)$ and polynomials $E_{n,q,w}(x : a)$. We observe an interesting phenomenon of “scattering” of the zeros of the generalized (q, w) -Euler polynomials $E_{n,q,w}(x : a)$ in complex plane.

1. Introduction

Recently, many mathematicians have studied in the area of the Euler numbers and polynomials (see [1–15]). The Euler numbers and polynomials possess many interesting properties and arising in many areas of mathematics and physics. In [14], we introduced that Euler equation $E_n(x) = 0$ has symmetrical roots for $x = 1/2$ (see [14]). It is the aim of this paper to observe an interesting phenomenon of “scattering” of the zeros of the generalized (q, w) -Euler polynomials $E_{n,q,w}(x : a)$ in complex plane. Throughout this paper, we use the following notations. By \mathbb{Z}_p , we denote the ring of p -adic rational integers, \mathbb{Q}_p denotes the field of p -adic rational numbers, \mathbb{C}_p denotes the completion of algebraic closure of \mathbb{Q}_p , \mathbb{N} denotes the set of natural numbers, \mathbb{Z} denotes the ring of rational integers, \mathbb{Q} denotes the field of rational numbers, \mathbb{C} denotes the set of complex numbers, and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. Let v_p be the normalized exponential valuation of \mathbb{C}_p with $|p|_p = p^{-v_p(p)} = p^{-1}$. When one talks of q -extension, q is considered in many ways such as an indeterminate, a complex number $q \in \mathbb{C}$, or p -adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$ one normally assume that $|q| < 1$. If $q \in \mathbb{C}_p$, we normally assume that $|q - 1|_p < p^{-1/(p-1)}$ so that $q^x = \exp(x \log q)$ for $|x|_p \leq 1$

$$[x]_q = \frac{1 - q^x}{1 - q}, \quad [x]_{-q} = \frac{1 - (-q)^x}{1 + q}. \quad (1.1)$$

Compared with [1, 4, 5]. Hence, $\lim_{q \rightarrow 1} [x] = x$ for any x with $|x|_p \leq 1$ in the present p -adic case. Let d be a fixed integer, and let p be a fixed prime number. For any positive integer N , we set

$$\begin{aligned} X &= \lim_{\leftarrow N} \left(\frac{\mathbb{Z}}{dp^N \mathbb{Z}} \right), \\ X^* &= \bigcup_{\substack{0 < a < dp \\ (a,p)=1}} (a + dp\mathbb{Z}_p), \\ a + dp^N \mathbb{Z}_p &= \left\{ x \in X \mid x \equiv a \pmod{dp^N} \right\}, \end{aligned} \quad (1.2)$$

where $a \in \mathbb{Z}$ lies in $0 \leq a < dp^N$. For any positive integer N ,

$$\mu_q(a + dp^N \mathbb{Z}_p) = \frac{q^a}{[dp^N]_q} \quad (1.3)$$

is known to be a distribution on X , compared with [1–10, 14]. For

$$g \in UD(\mathbb{Z}_p) = \{g \mid g : \mathbb{Z}_p \rightarrow \mathbb{C}_p \text{ is uniformly differentiable function}\}. \quad (1.4)$$

Kim defined the fermionic p -adic q -integral on \mathbb{Z}_p

$$I_{-q}(g) = \int_{\mathbb{Z}_p} g(x) d\mu_{-q}(x) = \lim_{N \rightarrow \infty} \frac{1}{[p^N]_{-q}} \sum_{0 \leq x < p^N} g(x) (-q)^x. \quad (1.5)$$

From (1.5), we also obtain

$$qI_{-q}(g_1) + I_{-q}(g) = [2]_q g(0), \quad (1.6)$$

where $g_1(x) = g(x+1)$ (see [1–3]).

From (1.6), we obtain

$$q^n I_{-q}(g_n) + (-1)^{n-1} I_{-q}(g) = [2]_q \sum_{l=0}^{n-1} (-1)^{n-1-l} q^l g(l), \quad (1.7)$$

where $g_n(x) = g(x+n)$.

As well-known definition, the Euler polynomials are defined by

$$\begin{aligned} F(t) &= \frac{2}{e^t + 1} = e^{Et} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}, \\ F(t, x) &= \frac{2}{e^t + 1} e^{xt} = e^{E(x)t} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}, \end{aligned} \quad (1.8)$$

with the usual convention of replacing $E^n(x)$ by $E_n(x)$. In the special case, $x = 0$, $E_n(0) = E_n$ are called the n -th Euler numbers (cf. [1–15]).

Our aim in this paper is to define the generalized (q, w) -Euler numbers $E_{n,q,w}(a)$ and polynomials $E_{n,q,w}(x : a)$. We investigate some properties which are related to the generalized (q, w) -Euler numbers $E_{n,q,w}(a)$ and polynomials $E_{n,q,w}(x : a)$. Especially, distribution of roots for $E_{n,q,w}(x : a) = 0$ is different from $E_n(x) = 0$'s. We also derive the existence of a specific interpolation function which interpolate the generalized (q, w) -Euler numbers $E_{n,q,w}(a)$ and polynomials $E_{n,q,w}(x : a)$.

2. The Generalized (q, w) -Euler Numbers and Polynomials

Our primary goal of this section is to define the generalized (q, w) -Euler numbers $E_{n,q,w}(a)$ and polynomials $E_{n,q,w}(x : a)$. We also find generating functions of the generalized (q, w) -Euler numbers $E_{n,q,w}(a)$ and polynomials $E_{n,q,w}(x : a)$. Let a be strictly positive real number.

The generalized (q, w) -Euler numbers and polynomials $E_{n,q,w}(a)$, $E_{n,q,w}(x : a)$ are defined by

$$\sum_{n=0}^{\infty} E_{n,q,w}(a) \frac{t^n}{n!} = \int_{\mathbb{Z}_p} w^{ax} e^{axt} d\mu_{-q}(x), \tag{2.1}$$

$$\sum_{n=0}^{\infty} E_{n,q,w}(x : a) \frac{t^n}{n!} = \int_{\mathbb{Z}_p} w^{ay} e^{(ay+x)t} d\mu_{-q}(y), \quad \text{for } t, w \in \mathbb{C}, \tag{2.2}$$

respectively.

From above definition, we obtain

$$\begin{aligned} E_{n,q,w}(a) &= \int_{\mathbb{Z}_p} w^{ax} (ax)^n d\mu_{-q}(x), \\ E_{n,q,w}(x : a) &= \int_{\mathbb{Z}_p} w^{ay} (x + ay)^n d\mu_{-q}(y). \end{aligned} \tag{2.3}$$

Let $g(x) = w^{ax} e^{axt}$. By (1.6) and using p -adic q -integral on \mathbb{Z}_p , we have

$$\begin{aligned} qI_{-q}(g_1) + I_{-q}(g) &= \int_{\mathbb{Z}_p} w^{a(x+1)} e^{a(x+1)t} d\mu_{-q}(x) + \int_{\mathbb{Z}_p} w^{ax} e^{axt} d\mu_{-q}(x) \\ &= (qw^a e^{at} + 1) \int_{\mathbb{Z}_p} w^{ax} e^{axt} d\mu_{-q}(x) \\ &= [2]_q. \end{aligned} \tag{2.4}$$

Hence, by (2.1), we obtain

$$\sum_{n=0}^{\infty} E_{n,q,w}(a) \frac{t^n}{n!} = \frac{[2]_q}{qw^a e^{at} + 1}. \tag{2.5}$$

By (1.6), (2.2) and $g(y) = w^{ay} e^{(ay+x)t}$, we have

$$\sum_{n=0}^{\infty} E_{n,q,w}(x : a) \frac{t^n}{n!} = \frac{[2]_q}{qw^a e^{at} + 1} e^{xt}. \quad (2.6)$$

After some elementary calculations, we obtain

$$\sum_{n=0}^{\infty} E_{n,q,w}(x : a) \frac{t^n}{n!} = [2]_q \sum_{n=0}^{\infty} (-1)^n q^n w^{an} e^{ant} e^{xt}. \quad (2.7)$$

From (2.6), we have

$$\begin{aligned} E_{n,q,w}(x : a) &= \sum_{k=0}^n \binom{n}{k} x^{n-k} E_{k,q,w}(a) \\ &= (x + E_{q,w}(a))^n, \end{aligned} \quad (2.8)$$

with the usual convention of replacing $(E_{q,w}(a))^n$ by $E_{n,q,w}(a)$.

3. Basic Properties for the Generalized (q, w) -Euler Numbers and Polynomials

By (2.5), we have

$$\begin{aligned} \frac{\partial}{\partial x} \sum_{n=0}^{\infty} E_{n,q,w}(x : a) \frac{t^n}{n!} &= \frac{\partial}{\partial x} \left(\frac{[2]_q}{qw^a e^{at} + 1} e^{xt} \right) \\ &= t \sum_{n=0}^{\infty} E_{n,q,w}(x : a) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} n E_{n-1,q,w}(x : a) \frac{t^n}{n!}. \end{aligned} \quad (3.1)$$

By (3.1), we have the following differential relation.

Theorem 3.1. For positive integers n , one has

$$\frac{\partial}{\partial x} E_{n,q,w}(x : a) = n E_{n-1,q,w}(x : a). \quad (3.2)$$

By Theorem 3.1, we easily obtain the following corollary.

Corollary 3.2 (integral formula). Consider that

$$\int_p^q E_{n-1,q,w}(x : a) dx = \frac{1}{n} (E_{n,q,w}(q : a) - E_{n,q,w}(p : a)). \quad (3.3)$$

By (2.5), one obtains

$$\begin{aligned} \sum_{n=0}^{\infty} E_{n,q,w}(x+y : a) \frac{t^n}{n!} &= \frac{[2]_q}{q\omega^a e^{at} + 1} e^{(x+y)t} \\ &= \sum_{n=0}^{\infty} E_{n,q,w}(x : a) \frac{t^n}{n!} \sum_{k=0}^{\infty} y^k \frac{t^k}{k!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^n \binom{n}{k} E_{k,q,w}(x : a) y^{n-k} \right) \frac{t^n}{n!}. \end{aligned} \tag{3.4}$$

By comparing coefficients of $t^n/n!$ in the above equation, we arrive at the following addition theorem.

Theorem 3.3 (addition theorem). For $n \in \mathbb{Z}_+$,

$$E_{n,q,w}(x+y : a) = \sum_{k=0}^n \binom{n}{k} E_{k,q,w}(x : a) y^{n-k}. \tag{3.5}$$

By (2.5), for $m \equiv 1 \pmod{2}$, one has

$$\begin{aligned} \sum_{n=0}^{\infty} \left(m^n \frac{[2]_q}{[2]_{q^m}} \sum_{k=0}^{m-1} (-1)^k q^k \omega^{ak} E_{n,q^m,\omega^m} \left(\frac{x+ak}{m} : a \right) \right) \frac{t^n}{n!} \\ &= \sum_{k=0}^{m-1} (-1)^k q^k \omega^{ak} \left(\sum_{n=0}^{\infty} E_{n,q^m,\omega^m} \left(\frac{x+ak}{m} : a \right) \right) \frac{(mt)^n}{n!} \\ &= \sum_{k=0}^{m-1} \left((-1)^k q^k \omega^{ak} \frac{[2]_q}{q^m \omega^{ma} e^{mat} + 1} e^{(x+ak)t} \right) \\ &= \frac{[2]_q}{1 + q\omega^a e^{at}} e^{xt} \\ &= \sum_{n=0}^{\infty} E_{n,q,w}(x : a) \frac{t^n}{n!}. \end{aligned} \tag{3.6}$$

By comparing coefficients of $t^n/n!$ in the above equation, we arrive at the following multiplication theorem.

Theorem 3.4 (multiplication theorem). For $m, n \in \mathbb{N}$

$$E_{n,q,w}(x : a) = m^n \frac{[2]_q}{[2]_{q^m}} \sum_{k=0}^{m-1} (-1)^k q^k \omega^{ak} E_{n,q^m,\omega^m} \left(\frac{x+ak}{m} : a \right). \tag{3.7}$$

From (1.6), one notes that

$$\begin{aligned}
 [2]_q &= \int_{\mathbb{Z}_p} qw^{ax+a} e^{(ax+a)t} d\mu_{-q}(x) + \int_{\mathbb{Z}_p} w^{ax} e^{axt} d\mu_{-q}(x) \\
 &= \sum_{n=0}^{\infty} \left(qw^a \int_{\mathbb{Z}_p} w^{ax} (ax+a)^n d\mu_{-q}(x) + \int_{\mathbb{Z}_p} w^{ax} (ax)^n d\mu_{-q}(x) \right) \frac{t^n}{n!} \\
 &= \sum_{n=0}^{\infty} (qw^a E_{n,q,w}(a : a) + E_{n,q,w}(a)) \frac{t^n}{n!}.
 \end{aligned} \tag{3.8}$$

From the above, we obtain the following theorem.

Theorem 3.5. For $n \in \mathbb{Z}_+$, we have

$$qw^a E_{n,q,w}(a : a) + E_{n,q,w}(a) = \begin{cases} [2]_{q'} & \text{if } n = 0, \\ 0, & \text{if } n > 0. \end{cases} \tag{3.9}$$

By (2.8) in the above, we arrive at the following corollary.

Corollary 3.6. For $n \in \mathbb{Z}_+$, one has

$$qw^a (a + E_{q,w}(a))^n + E_{n,q,w}(a) = \begin{cases} [2]_{q'} & \text{if } n = 0, \\ 0, & \text{if } n > 0, \end{cases} \tag{3.10}$$

with the usual convention of replacing $(E_{q,w}(a))^n$ by $E_{n,q,w}(a)$.

From (1.7), one notes that

$$\begin{aligned}
 &\sum_{m=0}^{\infty} \left([2]_q \sum_{l=0}^{n-1} (-1)^{n-1-l} q^l w^{al} (al)^m \right) \frac{t^m}{m!} \\
 &= q^n \int_{\mathbb{Z}_p} w^{ax+an} e^{(ax+an)t} d\mu_{-q}(x) + (-1)^{n-1} \int_{\mathbb{Z}_p} w^{ax} e^{axt} d\mu_{-q}(x) \\
 &= \sum_{m=0}^{\infty} \left(q^n w^{an} \int_{\mathbb{Z}_p} w^{ax} (ax+an)^m d\mu_{-q}(x) + (-1)^{n-1} \int_{\mathbb{Z}_p} w^{ax} (ax)^m d\mu_{-q}(x) \right) \frac{t^m}{m!} \\
 &= \sum_{m=0}^{\infty} \left(q^n w^{an} E_{m,w}(an : a) + (-1)^{n-1} E_{m,w}(a) \right) \frac{t^m}{m!}.
 \end{aligned} \tag{3.11}$$

By comparing coefficients of $t^n/n!$ in the above equation, we arrive at the following theorem.

Theorem 3.7. For $n \in \mathbb{Z}_+$, one has

$$q^n \omega^{an} E_{m,\omega}(na : a) + (-1)^{n-1} E_{m,\omega}(a) = [2]_q \sum_{l=0}^{n-1} (-1)^{n-1-l} \omega^{al} q^l (al)^m. \tag{3.12}$$

4. The Analogue of the q -Euler Zeta Function

By using the generalized (q, ω) -Euler numbers and polynomials, the generalized (q, ω) -Euler zeta function and the generalized Hurwitz (q, ω) -Euler zeta functions are defined. These functions interpolate the generalized (q, ω) -Euler numbers and (q, ω) -Euler polynomials, respectively. Let

$$F_{q,\omega}(x : a)(t) = [2]_q \sum_{n=0}^{\infty} (-1)^n q^n \omega^{an} e^{ant} e^{xt} = \sum_{n=0}^{\infty} E_{n,q,\omega}(x : a) \frac{t^n}{n!}. \tag{4.1}$$

By applying derivative operator, $d^k / dt^k|_{t=0}$ to the above equation, we have

$$\left. \frac{d^k}{dt^k} F_{q,\omega}(x : a)(t) \right|_{t=0} = [2]_q \sum_{n=0}^{\infty} (-1)^n q^n \omega^{an} (an + x)^k, \quad (k \in \mathbb{N}), \tag{4.2}$$

$$E_{k,q,\omega}(x : a) = [2]_q \sum_{n=0}^{\infty} (-1)^n q^n \omega^{an} (an + x)^k. \tag{4.3}$$

By using the above equation, we are now ready to define the generalized (q, ω) -Euler zeta functions.

Definition 4.1. For $s \in \mathbb{C}$, one defines

$$\zeta_{q,\omega}^{(a)}(x : s) = 2 \sum_{n=1}^{\infty} \frac{(-1)^n q^n \omega^{an}}{(an + x)^s}. \tag{4.4}$$

Note that $\zeta_w^{(a)}(x, s)$ is a meromorphic function on \mathbb{C} . Note that, if $\omega \rightarrow 1, q \rightarrow 1$, and $a = 1$, then $\zeta_{q,\omega}^{(a)}(x : s) = \zeta(x : s)$ which is the Hurwitz Euler zeta functions. Relation between $\zeta_w^{(a)}(x : s)$ and $E_{k,\omega}(x : a)$ is given by the following theorem.

Theorem 4.2. For $k \in \mathbb{N}$, one has

$$\zeta_{q,\omega}^{(a)}(x : -k) = E_{k,\omega}(x : a). \tag{4.5}$$

By using (4.2), one notes that

$$\left. \frac{d^k}{dt^k} F_{q,\omega}(0 : a)(t) \right|_{t=0} = [2]_q \sum_{n=0}^{\infty} (-1)^n q^n \omega^{an} (an)^k, \quad (k \in \mathbb{N}). \tag{4.6}$$

Hence, one obtains

$$E_{k,q,w}(a) = [2]_q \sum_{n=0}^{\infty} (-1)^n q^n w^{an} (an)^k. \quad (4.7)$$

By using the above equation, one is now ready to define the generalized Hurwitz (q, w) -Euler zeta functions.

Definition 4.3. Let $s \in \mathbb{C}$. One defines

$$\zeta_{q,w}^{(a)}(s) = 2 \sum_{n=1}^{\infty} \frac{(-1)^n q^n w^{an}}{(an)^s}. \quad (4.8)$$

Note that $\zeta_{q,w}^{(a)}(s)$ is a meromorphic function on \mathbb{C} . Obverse that, if $w \rightarrow 1$, $q \rightarrow 1$, and $a = 1$, then $\zeta_w^{(a)}(s) = \zeta(s)$ which is the Euler zeta functions. Relation between $\zeta_w^{(a)}(s)$ and $E_{k,w}(s)$ is given by the following theorem.

Theorem 4.4. For $k \in \mathbb{N}$, one has

$$\zeta_{q,w}^{(a)}(-k) = E_{k,q,w}(a). \quad (4.9)$$

5. Zeros of the Generalized (q, w) -Euler Polynomials $E_{n,q,w}(x : a)$

In this section, we investigate the reflection symmetry of the zeros of the generalized (q, w) -Euler polynomials $E_{n,q,w}(x : a)$.

In the special case, $w = 1$ and $q \rightarrow 1$, $E_{n,q,w}(x : a)$ are called generalized Euler polynomials $E_n(x : a)$. Since

$$\begin{aligned} \sum_{n=0}^{\infty} E_n(a-x : a) \frac{(-t)^n}{n!} \\ &= \frac{2}{e^{-at} + 1} e^{(a-x)(-t)} \\ &= \frac{2}{e^{at} + 1} e^{xt} = \sum_{n=0}^{\infty} E_n(x : a) \frac{t^n}{n!}, \end{aligned} \quad (5.1)$$

we have

$$E_n(x : a) = (-1)^n E_n(a-x : a) \quad \text{for } n \in \mathbb{N}. \quad (5.2)$$

We observe that $E_n(x : a)$, $x \in \mathbb{C}$ has $\text{Re}(x) = a/2$ reflection symmetry in addition to the usual $\text{Im}(x) = 0$ reflection symmetry analytic complex functions.

Let

$$F_{q,w}(x : t) = \frac{[2]_q}{q\omega^a e^{at} + 1} e^{xt} = \sum_{n=0}^{\infty} E_{n,q,w}(x : a) \frac{t^n}{n!}. \quad (5.3)$$

Then, we have

$$\begin{aligned} F_{q^{-1},w^{-1}}(a-x : -t) &= \frac{[2]_{q^{-1}}}{q^{-1}\omega^{-a} e^{-at} + 1} e^{(a-x)(-t)} \\ &= \omega^a \frac{[2]_q}{q\omega^a e^{at} + 1} e^{xt} \\ &= \omega^a \sum_{n=0}^{\infty} E_{n,q,w}(x : a) \frac{t^n}{n!}. \end{aligned} \quad (5.4)$$

Hence, we arrive at the following complement theorem.

Theorem 5.1 (complement theorem). For $n \in \mathbb{N}$,

$$E_{n,q^{-1},w^{-1}}(a-x : a) = (-1)^n \omega^a E_{n,q,w}(x : a). \quad (5.5)$$

Throughout the numerical experiments, we can finally conclude that $E_{n,q,w}(x : a)$, $x \in \mathbb{C}$ has not $\text{Re}(x) = a/2$ reflection symmetry analytic complex functions. However, we observe that $E_{n,q,w}(x : a)$, $x \in \mathbb{C}$ has $\text{Im}(x) = 0$ reflection symmetry (see Figures 1, 2, and 3). The obvious corollary is that the zeros of $E_{n,q,w}(x : a)$ will also inherit these symmetries.

$$\text{If } E_{n,q,w}(x_0 : a) = 0, \text{ then } E_{n,q,w}(x_0^* : a) = 0, \quad (5.6)$$

where $*$ denotes complex conjugation (see Figures 1, 2, and 3).

We investigate the beautiful zeros of the generalized (q, w) -Euler polynomials $E_{n,q,w}(x : a)$ by using a computer. We plot the zeros of the generalized Euler polynomials $E_{n,q,w}(x : a)$ for $n = 30$, $a = 1, 2, 3, 4$, and $x \in \mathbb{C}$ (Figure 1). In Figure 1 (top-left), we choose $n = 30$, $q = 1/2$, $w = 1$, and $a = 1$. In Figure 1 (top-right), we choose $n = 30$, $q = 1/2$, $w = 2$, and $a = 2$. In Figure 1 (bottom-left), we choose $n = 30$, $q = 1/2$, $w = 3$, and $a = 3$. In Figure 1 (bottom-right), we choose $n = 30$, $q = 1/2$, $w = 4$, and $a = 4$.

We plot the zeros of the generalized Euler polynomials $E_{n,q,w}(x : a)$ for $n = 30$, $a = 2$, $w = 2$, and $x \in \mathbb{C}$ (Figure 2).

In Figure 2 (top-left), we choose $n = 30$, $q = 1/10$, $w = 2$, and $a = 2$. In Figure 2 (top-right), we choose $n = 30$, $q = 3/10$, $w = 2$, and $a = 2$. In Figure 2 (bottom-left), we choose $n = 30$, $q = 7/10$, $w = 2$, and $a = 2$. In Figure 2 (bottom-right), we choose $n = 30$, $q = 9/10$, $w = 2$ and $a = 2$.

Plots of real zeros of $E_{n,q,w}(x : a)$ for $1 \leq n \leq 25$ structure are presented (Figure 3).

In Figure 3 (top-left), we choose $q = 1/2$, $w = 1$, and $a = 2$. In Figure 3 (top-right), we choose $q = 1/2$, $w = 2$, and $a = 2$. In Figure 3 (bottom-left), we choose $q = 1/2$, $w = 3$, and $a = 2$. In Figure 3 (bottom-right), we choose $q = 1/2$, $w = 4$, and $a = 2$.

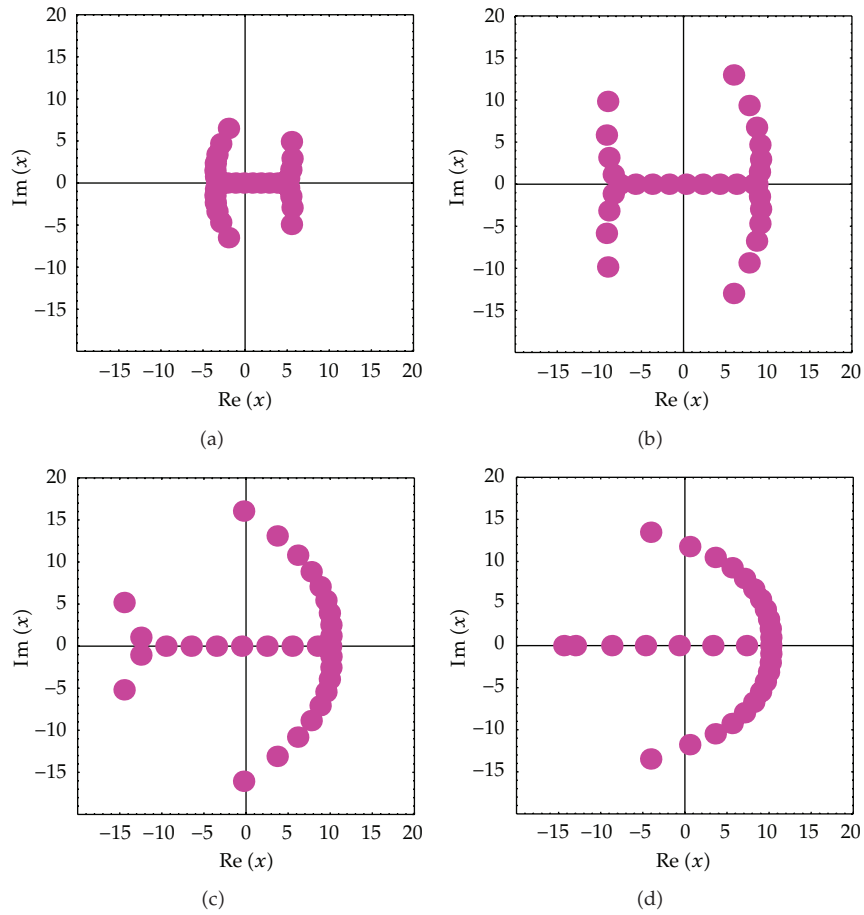


Figure 1: Zeros of $E_{n,q,w}(x : a)$ for $a = 1, 2, 3, 4$.

Stacks of zeros of $E_{n,q,w}(x : a)$ for $1 \leq n \leq 30, q = 1/2, w = 4$, and $a = 4$ from a 3-D structure are presented (Figure 4).

Our numerical results for approximate solutions of real zeros of the generalized $E_{n,q,w}(x : a)$ are displayed (Tables 1 and 2).

We observe a remarkably regular structure of the complex roots of the generalized (q, w) -Euler polynomials $E_{n,q,w}(x : a)$. We hope to verify a remarkably regular structure of the complex roots of the generalized (q, w) -Euler polynomials $E_{n,q,w}(x : a)$ (Table 1).

Next, we calculated an approximate solution satisfying $E_{n,q,w}(x : a), q = 1/2, w = 2, a = 2, x \in \mathbb{R}$. The results are given in Table 2.

Figure 5 shows the generalized (q, w) -Euler polynomials $E_{n,q,w}(x : a)$ for real $-9/10 \leq q \leq 9/10$ and $-5 \leq x \leq 5$, with the zero contour indicated in black (Figure 5). In Figure 5 (top-left), we choose $n = 1, w = 2$, and $a = 2$. In Figure 5 (top-right), we choose $n = 2, w = 2$, and $a = 2$. In Figure 5 (bottom-left), we choose $n = 3, w = 2$, and $a = 2$. In Figure 5 (bottom-right), we choose $n = 4, w = 2$, and $a = 2$.

Finally, we will consider the more general problems. How many roots does $E_{n,q,w}(x : a)$ have? This is an open problem. Prove or disprove: $E_{n,q,w}(x : a) = 0$ has n distinct solutions. Find the numbers of complex zeros $C_{E_{n,q,w}(x:a)}$ of $E_{n,q,w}(x : a), \text{Im}(x : a) \neq 0$. Since n is

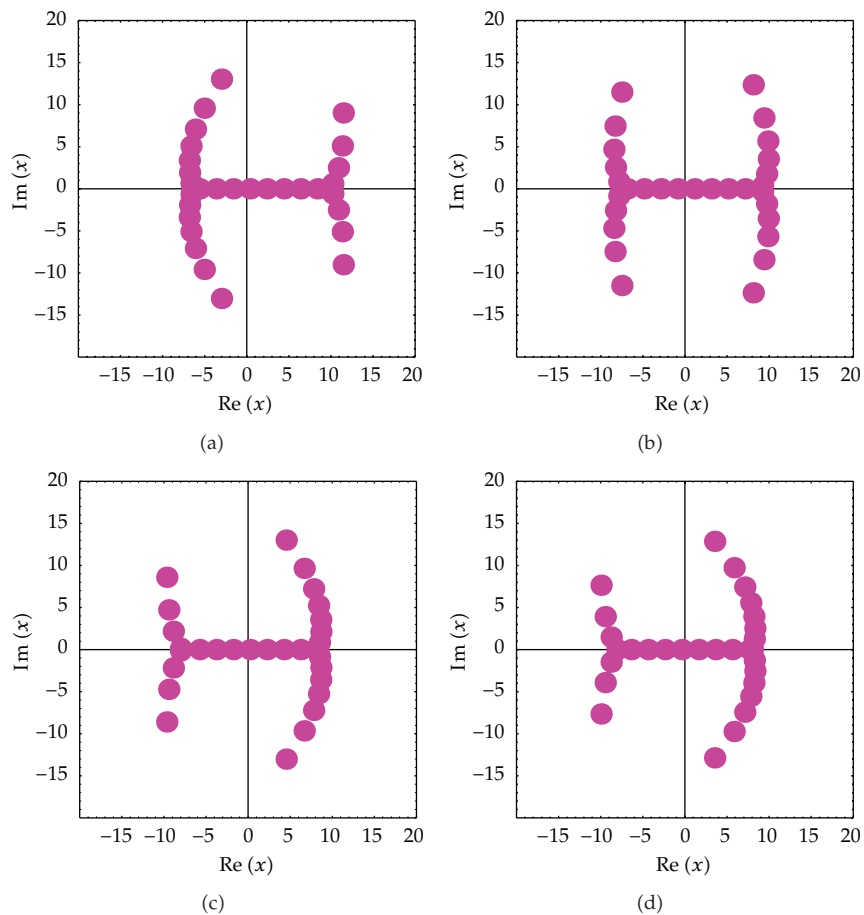


Figure 2: Zeros of $E_{n,q,w}(x : a)$ for $q = 1/10, 3/10, 7/10, 9/10$.

Table 1: Numbers of real and complex zeros of $E_{n,q,w}(x : a)$.

n	$q = 1/2, w = 2, a = 2$		$q = 1/2, w = 4, a = 4$	
	Real zeros	Complex zeros	Real zeros	Complex zeros
1	1	0	1	0
2	2	0	2	0
3	3	0	1	2
4	2	2	2	2
5	3	2	1	4
6	4	2	2	4
7	3	4	3	4
8	4	4	2	6
9	3	6	3	6
10	4	6	2	8
11	5	6	3	8
12	6	6	4	8
13	5	8	3	10

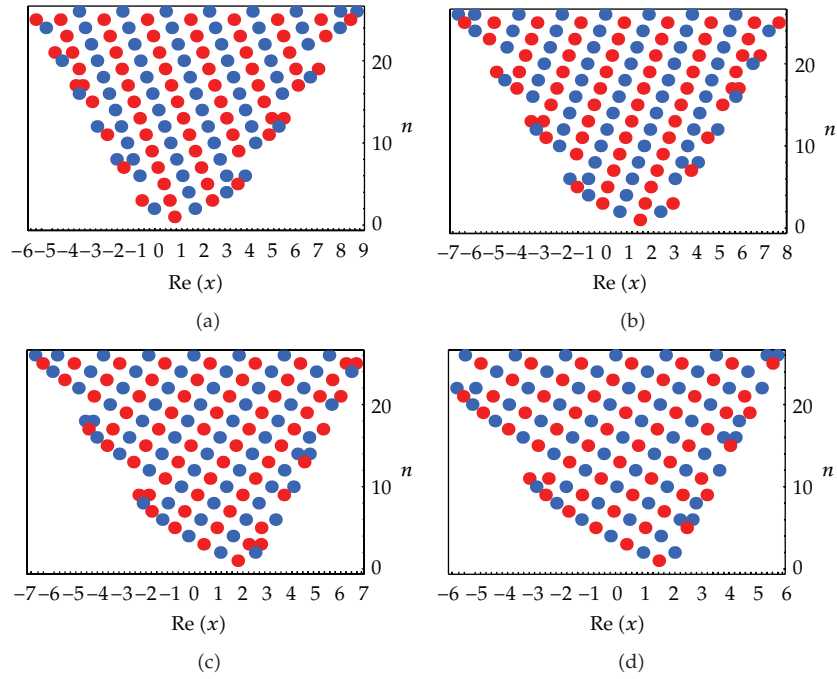


Figure 3: Real zeros of $E_{n,q,w}(x : a)$ for $1 \leq n \leq 25$.

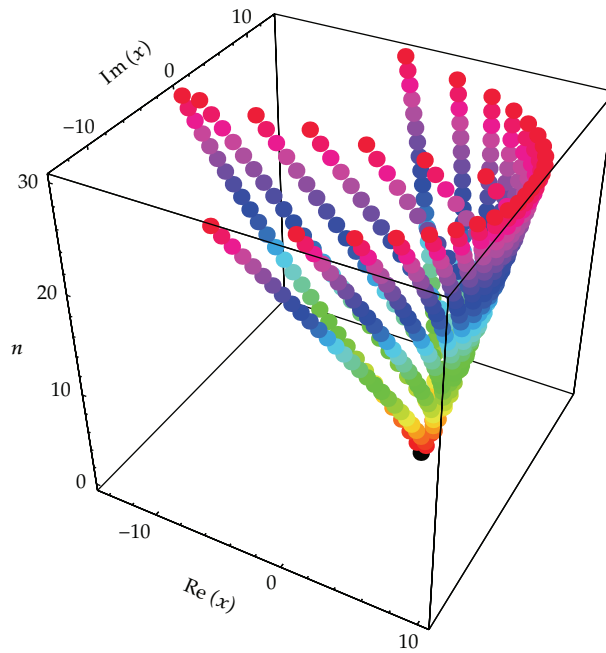


Figure 4: Stacks of zeros of $E_{n,q,w}(x : a)$ for $1 \leq n \leq 30$.

Table 2: Approximate solutions of $E_{n,q,w}(x : a) = 0, x \in \mathbb{R}$.

n	x
1	1.3333
2	0.3905, 2.2761
3	-0.4011, 1.560, 2.841
4	-1.0546, 0.6907
5	-1.5732, -0.17085, 1.829
6	-1.9151, -1.0557, 0.9680, 2.94
7	0.10585, 2.106, 3.68
8	-0.7557, 1.2442, 3.26, 4.00
9	-1.6091, 0.3825, 2.382
10	-2.392, -0.4793, 1.521, 3.52
11	-3.013, -1.3411, 0.6590, 2.66, 4.4

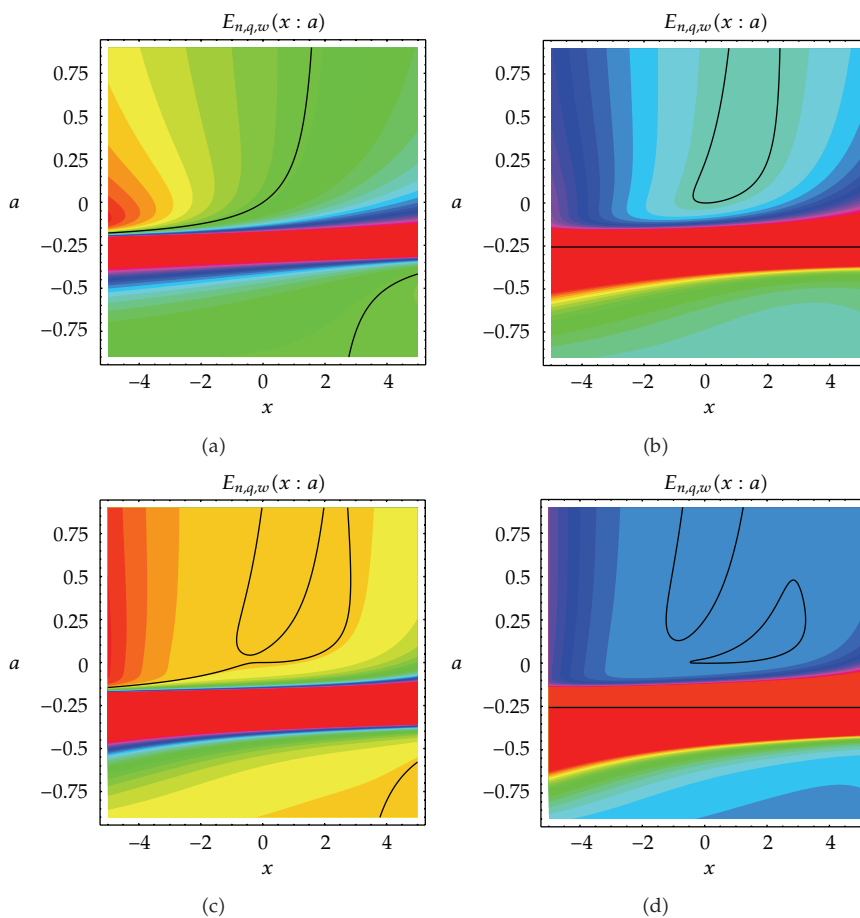


Figure 5: Zero contour of $E_{n,q,w}(x : a)$.

the degree of the polynomial $E_{n,q,w}(x : a)$, the number of real zeros $R_{E_{n,q,w}(x:a)}$ lying on the real plane $\text{Im}(x : a) = 0$ is then $R_{E_{n,q,w}(x:a)} = n - C_{E_{n,q,w}(x:a)}$, where $C_{E_{n,q,w}(x:a)}$ denotes complex zeros. See Table 1 for tabulated values of $R_{E_{n,q,w}(x:a)}$ and $C_{E_{n,q,w}(x:a)}$. We plot the zeros of $E_{n,q,w}(x : a)$, respectively (Figures 1–5). These figures give mathematicians an unbounded capacity to create visual mathematical investigations of the behavior of the roots of the $E_{n,q,w}(x : a)$. Moreover, it is possible to create a new mathematical ideas and analyze them in ways that generally are not possible by hand. The authors have no doubt that investigation along this line will lead to a new approach employing numerical method in the field of research of (q, w) -Euler polynomials $E_{n,q,w}(x : a)$ to appear in mathematics and physics.

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