Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2012, Article ID 908104, 10 pages
doi:10.1155/2012/908104

Research Article

On the Rational Approximation of Analytic Functions Having Generalized Types of Rate of Growth

Devendra Kumar

Department of Mathematics, Research and Post Graduate Studies, M.M.H. College, Model Town, Ghaziabad 201001, UP, India

Correspondence should be addressed to Devendra Kumar, d_kumar001@rediffmail.com

Received 21 March 2012; Accepted 2 July 2012

Academic Editor: Narendra Govil

The present paper is concerned with the rational approximation of functions holomorphic on a domain $G \subset C$, having generalized types of rates of growth. Moreover, we obtain the characterization of the rate of decay of product of the best approximation errors for functions f having fast and slow rates of growth of the maximum modulus.

1. Introduction

Let K be a compact subset of the extended complex plane C and let E_n be the error in the best uniform approximation of a function f (holomorphic on K) on K in the class R_n of all rational functions of order n:

$$E_n = E_n(f, K) = \inf_{r \in R_n} \|f - r\|_K \tag{1.1}$$

for each nonnegative integer n, where $\|\cdot\|_K$ is the supremum norm on K.

In view of Walsh's inequality [1], if f is holomorphic on $C \setminus M$, where M is a compact set in C and $M \cap K = \phi$, then

$$\limsup_{n \to \infty} E_n^{1/n} \leq \frac{1}{d}, \tag{1.2}$$

where $d = \exp(1/C(K,M))$ and $C(K,M)$ is the capacity of the condenser (K,M), (see [2–4], for the definition and properties of the capacity).

The theory of Hankel operators permits one [5–7] to estimate the order of decrease of the product $E_1 E_2 \cdots E_n$:

$$\limsup_{n\to\infty} (E_1 E_2 \cdots E_n)^{1/n^2} \le \frac{1}{d}. \tag{1.3}$$

The last relation implies Walsh's inequality (1.2) and the following upper estimate for $\liminf_{n\to\infty} E_n^{1/n}$:

$$\liminf_{n\to\infty} E_n^{1/n} \le \frac{1}{d^2}. \tag{1.4}$$

The present paper is concerned to results that make the inequalities (1.2), (1.3) and (1.4) more precise for analytic functions having generalized types of the rate of growth of the maximum modulus in the domain of analyticity of f.

The generalized order $\rho(\alpha,\beta,f)$ of the rate of growth of entire functions f was introduced by Šeremeta [8], who obtained a characterization of $\rho(\alpha,\beta,f)$ in terms of the coefficients of the power series of f. In [8], the relationship between the generalized order of entire functions f and the degree of polynomial approximation of f was studied. The coefficient characterization of a generalized order of the rate of growth of functions analytic in a disk has been discussed in several papers [9–12]. The degree of rational approximation of entire functions of a finite generalized order is investigated in [6].

Now let us consider the Dirichlet problem in the domain $C \setminus (K \cup M)$ with boundary function equal to 1 on ∂M and to 0 on ∂K. Here, K and M be disjoint compact sets with connected complements in the extended complex plane C such that their boundaries consist of finitely many closed analytic Jordan curves. Since the domain $C \setminus (K \cup M)$ is regular with respect to the Dirichlet problem, this problem is solvable. Let $w(z)$ be the solution which is extended by continuity to $C : w(z) = 1$ for $z \in M$ and $w(z) = 0$ for $z \in K$. For $0 < \varepsilon < 1$, let $\gamma(\varepsilon) = \{z : w(z) = \varepsilon\}$.

Let α and β be continuous positive functions on $[a,\infty)$ satisfying the following properties:

(i) $\lim_{x\to\infty}\alpha(x) = +\infty$, and $\lim_{x\to\infty}\beta(x) = +\infty$;

(ii) $\lim_{x\to\infty}(\beta(x + o(x))/\beta(x)) = 1$;

(iii) $\alpha^{-1}(\log(1/\mho\beta(x)))/\alpha^{-1}(\log(1/\mho'\beta(x))) = o(x)$ as $x \to 0$ for all $\mho' > \mho > 0$.

Let f be holomorphic on $G = C \setminus M$. We define the generalized order $\rho(\alpha,\beta,f)$ and generalized type $T(\alpha,\beta,f)$ of f in the domain G by the formulae:

(a) $\rho(\alpha,\beta,f) = \limsup_{\varepsilon\to 1}(\alpha(\log\|f\|_{\gamma(\varepsilon)})/\beta(\log(1/(1-\varepsilon))))$,

(b) $T(\alpha,\beta,f) = \limsup_{\varepsilon\to 1}(\alpha(\|f\|_{\gamma(\varepsilon)})/[(1/(1-\varepsilon))]^{\rho(\alpha,\beta,f)})$,

where $\|f\|_{\gamma(\varepsilon)} = \max_{z\in\gamma(\varepsilon)}|f(z)|$.

It is easy to see that for the functions $\alpha(x) = \log_p x$, $p \ge 2$, and $\beta(x) = x$ properties (i)–(iii) will hold. The following theorem gives the characterization of the rate of decay of product $E_0 E_1 \cdots E_n$ for functions f having fast rates of growth of the maximum modulus. So to avoid some trivial cases, we will assume that $\lim_{\varepsilon\to 1}\|f\|_{\gamma(\varepsilon)} = \infty$.

Theorem 1.1. *Suppose that f is holomorphic on G, α and β satisfy conditions (i)–(iii), and f has generalized order $\rho(\alpha,\beta,f) > 0$ and generalized type $T(\alpha,\beta,f)$ in the domain G. Then,*

$$\limsup_{n\to\infty} \exp\alpha(n)\left[\beta\left(\log^{+}\left(\left((E_0E_1\cdots E_n)^{1/n(n+1)}\ d\right)\right)\right)\right]^{\rho} \leq T(\alpha,\beta,f), \tag{1.5}$$

where $\log^{+} x = \max(0,\log x)$ *for* $x \geq 0$.

Proof. Let us assume that $T(\alpha,\beta,f) < \infty$. Fix arbitrary numbers $T'' > T' > T(\alpha,\beta,f)$. For $n = 1,2,\ldots$, we set

$$\delta_n = \min\left(\frac{1}{4},\beta^{-1}\left[T''\exp(-\alpha(n))\right]^{1/\rho}\right). \tag{1.6}$$

We have $\delta_n \to 0$ as $n \to \infty$. Using (1.5) for all sufficiently large values of n, $n \geq n_1$, we set

$$\begin{aligned}\log\|f\|_{\gamma_{2,n}} &\leq \alpha^{-1}\left\{\log\left(T'\left[\beta(\delta_n)\right]^{-\rho}\right)\right\}\\ &= \alpha^{-1}\left\{\rho\log\left(\frac{1}{(T'^{1/\rho}\beta(\delta_n))}\right)\right\}.\end{aligned} \tag{1.7}$$

From (1.6), we have

$$n = \alpha^{-1}\left\{\log\left(T''\left[\beta(\delta_n)\right]^{-\rho}\right)\right\}. \tag{1.8}$$

In (1.7), $\gamma_{2,n}$ defined as subsets of the extended complex plane C:

$$\begin{aligned}\gamma_{k,n} &= \{z : w(z) = \varepsilon_{k,n}\},\\ D_n &= \{z : w(z) > \varepsilon_{0,n}\},\end{aligned} \tag{1.9}$$

where $\varepsilon_{0,n} = k/2n$, $\varepsilon_{1,n} = k/n$, $\varepsilon_{2,n} = 1-\delta_n$, $k = 0,1,2$, and $n = 1,2,\ldots$. It is given [13] that $\gamma_{0,n}$, $\gamma_{1,n}$, and $\gamma_{2,n}$, $n = 1,2,\ldots$, consist of finitely many closed analytic curves whose lengths are bounded from above by a positive quantity not depending on n. It is assumed that $\gamma_{0,n}$ and $\gamma_{2,n}$ are positively oriented with respect to D_n and $\{z : w(z) > \varepsilon_{2,n}\}$, respectively.

In view of (1.8) for $n \geq \max(n_0,n_1)$, we may use the inequality 3.1 of [13] in the form:

$$\begin{aligned}(E_0E_1\cdots E_n)^{1/n(n+1)}d &\leq \left(C^n m(n+1)!n^{8n}\right)^{1/n(n+1)}\\ &\quad\times\exp\left(\frac{\alpha^{-1}\left(\rho\log\left(1/T'^{1/\rho}\beta(\delta_n)\right)\right)}{\alpha^{-1}(\rho\log(1/T''^{1/\rho}\beta(\delta_n)))}+\frac{\delta_n}{C(K,M)}\right).\end{aligned} \tag{1.10}$$

Now, using property (iii), we get

$$\log^+\Big((E_0E_1\cdots E_n)^{1/n(n+1)}d\Big) \le \frac{\delta_n}{C(K,M)} + o(\delta_n). \tag{1.11}$$

It gives

$$\limsup_{n\to\infty} \exp\alpha(n)\Big[\beta\Big(\log^+(E_0E_1\cdots E_n)^{1/n(n+1)}d\Big)\Big]^\rho \le T''. \tag{1.12}$$

On letting $T'' \to T(\alpha,\beta,f)$, the proof is complete. □

In the consequence of Theorem 1.1, we have the following.

Corollary 1.2. *With the assumption of Theorem 1.1, the following inequalities are valid:*

$$\limsup_{n\to\infty} \exp(\alpha(n))\Big[\beta\Big(\log^+E_n^{1/n}d\Big)\Big]^\rho \le T(\alpha,\beta,f), \tag{1.13}$$

$$\limsup_{n\to\infty} \exp(\alpha(n))\Big[\beta\Big(\log^+E_n^{1/n}d^2\Big)\Big]^\rho \le T(\alpha,\beta,f). \tag{1.14}$$

Proof. Using the fact $E_n \le E_{n-1} \le \cdots \le E_0$, we obtain (1.13) immediately from (1.5). To prove (1.14), let us suppose that

$$\liminf_{n\to\infty} \exp\alpha(n)\Big[\beta\Big(\log^+\Big(E_n^{1/n}d^2\Big)\Big)\Big]^\rho > T' > T(\alpha,\beta,f). \tag{1.15}$$

Then, for sufficiently large values of n, we get

$$\beta\Big[\log^+\Big(E_n^{1/n}d^2\Big)\Big] > \left[\frac{T'}{\exp\alpha(n)}\right]^{1/\rho} \tag{1.16}$$

or

$$\log^+E_nd^{2n} \ge n\beta^{-1}\left[\frac{T'}{\exp(\alpha(n))}\right]^{1/\rho}. \tag{1.17}$$

Since the functions α and β are increasing, (1.17) gives

$$\begin{aligned}\log^+\Big((E_0E_1\cdots E_n)^{1/n(n+1)}\ d\Big) &\ge \frac{\Big(\sum_{k=0}^n k\beta^{-1}\Big\{\ [T'/\exp\alpha(n)\ ]^{1/\rho}+c\Big\}\Big)}{n(n+1)} \\ &\ge \beta^{-1}\left\{\left[\frac{T'}{\exp\alpha(n)}\right]^{1/\rho}+\frac{c}{n(n+1)}\right\},\end{aligned} \tag{1.18}$$

where c is a constant. Using (ii), we get

$$\left[\beta\left(\log^{+}(E_0E_1\cdots E_n)^{1/n(n+1)}d\right)\right]^{\rho} \geq \left[\frac{T'}{\exp\alpha(n)}\right] \tag{1.19}$$

or

$$\liminf_{n\to\infty} \exp\alpha(n)\left[\beta\left(\log^{+}(E_0E_1\cdots E_n)^{1/n(n+1)}\right)\right]^{\rho} \geq T' > T(\alpha,\beta,f) \tag{1.20}$$

which contradicts (1.5). Thus, (1.14) is valid. □

2. Rational Approximation of Analytic Functions Having Slow Rates of Growth

For a function f analytic in a domain G, the type of f in G can be defined by (b) for $\alpha(x) = \log x$ and $\beta(x) = x$:

$$T = \limsup_{\varepsilon\to 1} \frac{\log\|f\|_{\Upsilon(\varepsilon)}}{(1/1-\varepsilon)^{\rho}}. \tag{2.1}$$

For $\alpha(x) = \log x$ and $\beta(x) = x$, the property (iii) fails to hold. However, we have the following:

$$\frac{\alpha^{-1}(c\log(1/\beta(x)))}{\alpha^{-1}((c+1)\log(1/\beta(x)))} = x, \tag{2.2}$$

and we may repeat the arguments involving (1.10), we get

$$\begin{aligned}(E_0E_1\cdots E_n)^{1/n(n+1)}d &\leq \left(c^n(n+1)!n^{8n}\right)^{1/n(n+1)} \\ &\quad\times \exp\left(\frac{\alpha^{-1}\left(\log\left[1/T'^{1/\rho}\beta(\delta_n)\right]^{\rho}\right)}{\alpha^{-1}(\log\left[1/T''^{1/\rho}\beta(\delta_n)\right]^{\rho})} + \frac{\delta_n}{C(K,M)}\right).\end{aligned} \tag{2.3}$$

Taking $T'' = T' + 1$, and $x = T''^{1/\rho}\delta_n$ in (2.2), for sufficiently large values of n we have

$$n\left((T'+1)^{1/\rho}\log^{+}(E_0E_1\cdots E_n)^{1/n(n+1)}d\right)\rho \leq T' \tag{2.4}$$

or

$$\limsup_{n\to\infty} n\left[\log^{+}(E_0E_1\cdots E_n)^{1/n(n+1)}d\right]^{\rho} \leq \frac{T}{T+1}. \tag{2.5}$$

We summarize the above facts in the following.

Theorem 2.1. *Let f have an order $\rho > 0$ and generalized type T in the domain G. Then,*

$$\limsup_{n\to\infty} n\left[\log^{+}(E_0E_1\cdots E_n)^{1/n(n+1)}d\right]^{\rho} \le \frac{T}{T+1}. \tag{2.6}$$

By the inequality $E_n \le E_{n-1} \le \cdots \le E_0$, one gets the following.

Corollary 2.2. *With the assumption of Theorem 2.1*

$$\limsup_{n\to\infty} n\left[\log^{+}\left(E_n^{1/n}d\right)\right]^{\rho} \le \frac{T}{T+1}. \tag{2.7}$$

Theorem 2.1 also gives us the following corollary.

Corollary 2.3. *With the assumption of Theorem 2.1,*

$$\liminf_{n\to\infty} n\left[\log^{+}\left(E_n^{1/n}d^2\right)\right]^{\rho} \le \frac{T}{T+1}. \tag{2.8}$$

Proof. Let

$$\liminf_{n\to\infty} n\left[\log^{+}\left(E_n^{1/n}d^2\right)\right]^{\rho} > T_1 > \frac{T}{T+1}. \tag{2.9}$$

Then, from the relation

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{n} k^{1-1/T_1}}{n^{2-1/T_1}} = \frac{1}{2-1/T_1}, \tag{2.10}$$

we obtain

$$\liminf_{n\to\infty} n\left[\log^{+}\left((E_1E_2\cdots E_n)^{1/n(n+1)}d\right)\right]^{\rho} \ge T_1 > \frac{T}{T+1}, \tag{2.11}$$

which contradicts the inequality (2.6). □

Now, we define α-type of f to classify functions having slow rates of growth.

A continuous positive function h on $[a, +\infty)$ belongs to the class Λ, if this function satisfies the following.

h is strictly increasing on $[a, +\infty)$,

$$\lim_{x\to\infty} h(x) = +\infty, \tag{2.12}$$

$$\lim_{x\to+\infty} \frac{h(cx)}{h(x)} = 1, \tag{2.13}$$

for any $c > 0$.

Let $\alpha \in \Lambda$. We define α-order and α-type of f in G by the formulae:

$$\rho(\alpha, f) = \limsup_{\varepsilon \to 1} \frac{\alpha\left(\log \|f\|_{\gamma(\varepsilon)}\right)}{\alpha(\log(1/(1-\varepsilon)))}, \tag{2.14}$$

$$T(\alpha, f) = \limsup_{\varepsilon \to 1} \frac{\alpha\left(\log \|f\|_{\gamma(\varepsilon)}\right)}{[\alpha(1/(1-\varepsilon))]^{\rho}}. \tag{2.15}$$

The following results are concerned with the degree of rational approximation of functions having α-type $T(\alpha, f)$. The functions $\alpha(x) = \log_p x$, $p \geq 1$, and $\alpha(x) = \exp(\log x)^{\delta}$, $0 < \delta < 1$, satisfy the condition $\alpha \epsilon \Lambda$. For $\alpha(x) = \log x$, the parameter $T(\alpha, f)$ is called the logarithmic type of f in G [14].

Theorem 2.4. *Let f, analytic in G, be of α-order $\rho(\alpha, f) \geq 1$, and α-type $T(\alpha, f)$, $\alpha \in \Lambda$. Then,*

$$\limsup_{n \to \infty} \frac{\alpha\left[\left((E_0 E_1 \cdots E_n)^{1/n(n+1)} d^n\right)\right]}{[\alpha(n)]^{\rho(\alpha, f)}} \leq T(\alpha, f). \tag{2.16}$$

Proof. The inequality (2.16) holds for $T(\alpha, f) = \infty$ obviously. Now, let $T(\alpha, f) < \infty$ and $\|f\|_{\gamma(\varepsilon)} \to \infty$ as $\varepsilon \to 1$. Fix $T' > T(\alpha, f)$. Then, for ε sufficiently close to 1, from (2.15), we have

$$\|f\|_{\gamma(\varepsilon)} \leq \alpha^{-1}\left[T'\left[\alpha\left(\frac{1}{1} - \varepsilon\right)\right]^{\rho}\right], \qquad \rho(\alpha, f) \equiv \rho. \tag{2.17}$$

Define $\delta_n = \min(1/4, 1/n)$, $n = 1, 2, \ldots$. Using [13, Equation (3.1)] with (2.17), for all sufficiently large values of n, $n \geq n_0$, we have

$$E_0 E_1 \cdots E_n d^{n(n+1)} \leq (n+1)! c^n n^{8n} \exp(n+1)\left(\log\left(\alpha^{-1}\left(T'[\alpha(n)]^{\rho}\right)\right) + \frac{1}{C(K, M)}\right). \tag{2.18}$$

Since α is strictly increasing, for $n \geq n_0$, we get

$$(E_0 E_1 \cdots E_n)^{1/n+1} d^n \leq c_1 \alpha^{-1}\left[T'[\alpha(n)]^{\rho}\right]. \tag{2.19}$$

In view of (2.13), (2.19) gives

$$\limsup_{n \to \infty} \frac{\alpha\left[(E_0 E_1 \cdots E_n)^{1/n+1} d^n\right]}{[\alpha(n)]^{\rho}} \leq T'. \tag{2.20}$$

In order to complete the proof, it remains to let T' tend to $T(\alpha, f)$. □

Now, we have the following corollaries.

Corollary 2.5. *With assumption of Theorem 2.4,*

$$\limsup_{n\to\infty}\frac{\alpha(E_n d^n)}{[\alpha(n)]^{\rho(\alpha,f)}} \leq T(\alpha,f). \tag{2.21}$$

The proof is immediate in view of $E_n \leq E_{n-1} \leq \cdots E_0$.
For $c > 0$, let

$$F[x,c,\rho] = \log\left(\alpha^{-1}\left(c[\alpha(x)]^{\rho}\right)\right). \tag{2.22}$$

Corollary 2.6. *Let a function f, analytic in G, be of α-order $\rho(\alpha,f) \geq 1$, and α-type $T(\alpha,f)$ where $\alpha \in \Lambda$ is continuously differentiable on $[a,+\infty)$ and for all $1 < c < \infty$ the function $x(F(x,c,\rho))' = O(1)$ as $x \to \infty$ or is increasing and*

$$\lim_{x\to\infty}\frac{x(F(x,c,\rho))'}{F(x,c,\rho)} = 0. \tag{2.23}$$

Then,

$$\liminf_{n\to\infty}\frac{\alpha(E_n d^{2n})}{[\alpha(n)]^{\rho(\alpha,f)}} \leq T(\alpha,f). \tag{2.24}$$

Proof. We may assume that $T(\alpha,f) < \infty$. Let

$$\liminf_{n\to\infty}\frac{\alpha(E_n d^{2n})}{[\alpha(n)]^{\rho}} > T' > T(\alpha,f). \tag{2.25}$$

For sufficiently large values of n,

$$\frac{\alpha\left((E_0E_1\cdots E_n)^{1/n+1}d^n\right)}{[\alpha(n)]^{\rho}} \geq \frac{\alpha\left(\exp\left[(1/(n+1))\left(\sum_{k=1}^{n} F[k,T',\rho] + c\right)\right]\right)}{[\alpha(n)]^{\rho}}. \tag{2.26}$$

Since $F[x,T',\rho]$ is increasing, we get

$$\begin{gathered}\sum_{k=1}^{n-1}F[k,T',\rho] \leq \int_1^n F[x,T',\rho]\,dx \leq \sum_{k=2}^{n}F[k,T',\rho],\\ \int_1^n F[x,T',\rho]\,dx = nF[n,T',\rho] - F[1,T',\rho] - \int_1^n x(F[x,T',\rho])'\,dx.\end{gathered} \tag{2.27}$$

We see that

$$\frac{1}{nF[n,T',\rho]}\int_1^n x(F[x,T',\rho])'\,dx \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{2.28}$$

Thus,

$$\frac{(1/(n+1))\sum_{k=1}^{n} F[k,T',\rho]}{F[n,T',\rho]} \longrightarrow 1 \quad \text{as } n \longrightarrow \infty. \tag{2.29}$$

From this and (2.26), we get

$$\liminf_{n\to\infty} \frac{\alpha\left(\left(E_0E_1\cdots E_n\right)^{1/n+1}d^n\right)}{[\alpha(n)]^\rho} \geq \frac{\alpha(\exp F[n,T',\rho])}{[\alpha(n)]^\rho} \geq T' > T(\alpha,F) \tag{2.30}$$

which contradicts (2.16). Hence the proof is complete. □

Remark 2.7. The function $\alpha(x) = \log_p x,\ p \geq 1$, and $\alpha(x) = \exp(\log x)^\rho,\ 0 < \delta < 1$, satisfy the assumptions of Corollary 2.6.

Acknowledgment

The author is extremely thankful to the reviewers for giving fruitful comments to improve the paper.

References

[1] J. L. Walsh, *Interpolation and Approximation by Rational Functions in the Complex Domain*, American Mathematical Society, Providence, RI, USA, 5th edition, 1965.
[2] N. S. Landkof, *Foundations of Modern Potential Theory*, Nauka, Morscow, Russia, 1966, English Translation in Springer, Berlin, Germany,1972.
[3] M. Tsuji, *Potential Theory in Modern Function Theory*, Maruzen, Tokyo, Japan, 1959.
[4] E. B. Saff and V. Totik, *Logarithmic Potentials with External Fields*, vol. 316, Springer, Berlin, Germany, 1997.
[5] O. G. Parfënov, "Estimates for singular numbers of the Carleson embedding operator," *Matematicheskiĭ Sbornik*, vol. 131, no. 173, pp. 501–518, 1986, English Translation in *Mathematics of the USSR-Sbornik*, vol. 59, 1988.
[6] V. A. Prokhorov, "Rational approximation of analytic functions," *Matematicheskiĭ Sbornik*, vol. 184, no. 2, pp. 3–32, 1993, English Translation in *Russian Academy of Sciences. Sbornik Mathematics*, vol. 78, 1994.
[7] V. A. Prokhorov and E. B. Saff, "Rates of best uniform rational approximation of analytic functions by ray sequences of rational functions," *Constructive Approximation*, vol. 15, no. 2, pp. 155–173, 1999.
[8] M. N. Šeremeta, "Connection between the growth of the maximum of the modulus of an entire function and the moduli of the coefficients of its power series expansion," *Izvestija Vysših Učebnyh Zavedeniĭ Matematika*, vol. 2, no. 57, pp. 100–108, 1967, English Translation in *American Mathematical Society Translations*, vol. 2, no. 88, 1970.
[9] B. J. Aborn and H. Shankar, "Generalized growth for functions analytic in a finite disc," *Pure and Applied Mathematika Sciences*, vol. 12, no. 1-2, pp. 83–94, 1980.
[10] G. P. Kapoor and A. Nautiyal, "On the coefficients of a function analytic in the unit disc having slow rate of growth," *Annali di Matematica Pura ed Applicata*, vol. 131, pp. 281–290, 1982.
[11] A. Nautiyal, "A remark on the generalized order of an analytic function," *Pure and Applied Mathematika Sciences*, vol. 17, no. 1-2, pp. 15–17, 1983.
[12] M. N. Šeremeta, "The connection between the growth of functions of order zero which are entire or analytic in a disc and their power series coefficients," *Izvestija Vysših Učebnyh Zavedeniĭ Matematika*, vol. 6, no. 73, pp. 115–121, 1968 (Russian).
[13] V. A. Prokhorov, "Rational approximation of analytic functions having generalized orders of rate of growth," *Journal of Computational Analysis and Applications*, vol. 5, no. 1, pp. 129–146, 2003.

[14] G. P. Kapoor and K. Gopal, "Decomposition theorems for analytic functions having slow rates of growth in a finite disc," *Journal of Mathematical Analysis and Applications*, vol. 74, no. 2, pp. 446–455, 1980.