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*Research Article*

# Discrete Mixed Petrov-Galerkin Finite Element Method for a Fourth-Order Two-Point Boundary Value Problem

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A quadrature-based mixed Petrov-Galerkin finite element method is applied to a fourth-order linear ordinary differential equation. After employing a splitting technique, a cubic spline trial space and a piecewise linear test space are considered in the method. The integrals are then replaced by the Gauss quadrature rule in the formulation itself. Optimal order *a priori* error estimates are obtained without any restriction on the mesh.

## 1. Introduction

In this paper, we develop a quadrature-based Petrov-Galerkin mixed finite element method for the following fourth-order boundary value problem:

$$\frac{d^2}{dx^2}\left[a(x)\frac{d^2u}{dx^2}\right] + b(x)u = f(x), \quad x \in I = (0,1), \tag{1.1}$$

subject to the boundary conditions

$$u(0) = 0, \qquad u(1) = 0; \qquad u''(0) = 0, \qquad u''(1) = 0, \tag{1.2}$$

where $a(x) \neq 0$, $x \in I$. Let $\alpha(x) = 1/a(x)$. We, hereafter, suppress the dependency of the independent variable $x$ on the functions $\alpha(x)$, $b(x)$, and $f(x)$. Therefore, we write $\alpha$, $b$, and $f$ instead of these functions.

Let us define the splitting of the above fourth-order equation as follows.

Set

$$u'' = \alpha v, \quad x \in I. \tag{1.3}$$

Then the differential equation (1.1) with the boundary conditions (1.2) can be written as a coupled system of equations as follows:

$$u'' = \alpha v, \quad x \in I, \text{ with } u(0) = u(1) = 0, \tag{1.4}$$

$$v'' + bu = f, \quad x \in I, \text{ with } v(0) = v(1) = 0. \tag{1.5}$$

In this paper, the error analysis will take place in the usual Sobolev space $W_p^m(I)$ defined on the domain $I = (0,1)$ with $H^m(I)$ denoting $W_2^m(I)$. The Sobolev norms are given below. For an open interval $E$ and a non negative integer $m$,

$$\begin{aligned} \|v\|_{W_p^m(E)} &= \left(\sum_{i=0}^{m} \left\|v^{(i)}\right\|_{L_p(E)}^p\right)^{1/p}, \quad \text{if } 1 \le p < \infty, \\ &= \max_{1\le i\le n} \left\|v^{(i)}\right\|_{L_\infty(E)}, \quad \text{if } p = \infty. \end{aligned} \tag{1.6}$$

We suppress the dependence of the norms on $I$ when $E = I$. Further, $H_0^m(I)$ denotes the function space

$$\{\phi \in H^m(I) : \phi(0) = \phi(1) = 0\}. \tag{1.7}$$

## 2. Continuous and Discrete $H^1$-Galerkin Formulation

Given $n > 1$, let

$$\Pi_n : 0 = x_0 < x_1 < \cdots < x_n = 1 \tag{2.1}$$

be an arbitrary partition of $[0,1]$ with the property that $h \to 0$ as $n \to \infty$, where $h = \max_{1\le k\le n} h_k$ and $h_k = x_k - x_{k-1},\ k = 1,\ldots,n$. Let $(u,v)$ represent the $L_2$ inner product, and let $\langle u,v\rangle_h$ represent the discrete inner product of any two functions $u, v \in L_2(I)$ and be defined as follows:

$$(u,v) = \int uv\,dx, \qquad \langle u,v\rangle_h = Q_h(uv), \tag{2.2}$$

where $Q_h$ is the fourth-order Gaussian quadrature rule:

$$Q_h(g) := \frac{1}{2}\sum_{i=1}^{n} h_k\left[g(x_{k,1}) + g(x_{k,2})\right]. \tag{2.3}$$

Here, $x_{k,i} = x_{k-1} + \xi_i h_k,\ i = 1,2,$ are the two Gaussian points in the subinterval $[x_{k-1}, x_k]$ with $\xi_1 = (1/2)(1 - 1/\sqrt{3}),\ \xi_2 = 1 - \xi_1$.

Let us now consider the following cubic spline space as trial space:

$$S_{h,3} = \left\{\varphi \in C^2(I) : \varphi|_{I_k} \in P_3(I_k),\ k = 1,2,\ldots,n\right\}, \tag{2.4}$$

where $P_r(I_k)$ is the space of polynomials of degree $r$ defined over the $k$th subinterval $I_k = [x_{k-1}, x_k]$.

The corresponding space with zero Dirichlet boundary condition is denoted by

$$\overset{0}{S}_{h,3} = \{\varphi \in S_{h,3} : \varphi(0) = \varphi(1) = 0\}. \tag{2.5}$$

Further, let us consider the following piecewise linear space

$$S_{h,1} = \{\varphi \in C(I) : \varphi|_{I_k} \in P_1(I_k),\ k = 1,2,\ldots,n\} \tag{2.6}$$

as the test space.

### 2.1. Weak Formulation

The weak formulation corresponding to the split equations (1.4) and (1.5) is defined, respectively, as follows.

Find $\{u, v\} \in H_0^2(I)$ such that

$$\left(u'', \phi\right) = (\alpha v, \phi), \quad \phi \in H^2(0,1), \tag{2.7}$$

$$\left(v'' + bu, \phi\right) = (f, \phi), \quad \phi \in H^2(0,1). \tag{2.8}$$

### 2.2. The Petrov-Galerkin Formulation

The Petrov-Galerkin formulation corresponding to the above weak formulation (2.7) and (2.8) is defined, respectively, as follows.

Find $\{u_h, v_h\} \in \overset{0}{S}_{h,3}$ such that

$$\begin{aligned} \left(u_h'', \phi_h\right) &= (\alpha v_h, \phi_h), \quad \phi_h \in S_{h,1}, \\ \left(v_h'' + bu_h, \phi_h\right) &= (f, \phi_h), \quad \phi_h \in S_{h,1}. \end{aligned} \tag{2.9}$$

The integrals in the above Petrov-Galerkin formulation are not evaluated exactly at the implementation level. We, therefore, define the following discrete Petrov-Galerkin procedure in which the integrals are replaced by the Gaussian quadrature in the scheme as follows.

### *2.3. Discrete Petrov-Galerkin Formulation*

The discrete Petrov-Galerkin formulation corresponding to (2.7) and (2.8) is defined, respectively, as follows.

Find $\{u_h, v_h\} \in \overset{0}{S}_{h,3}$ such that

$$\left\langle u_h'', \phi_h \right\rangle_h = \langle \alpha v_h, \phi_h \rangle_h, \quad \phi_h \in S_{h,1}, \tag{2.10}$$

$$\left\langle v_h'' + b u_h, \phi_h \right\rangle_h = \langle f, \phi_h \rangle_h, \quad \phi_h \in S_{h,1}. \tag{2.11}$$

The approximate solutions $u_h$ and $v_h$ without any conditions on boundary points are expressed as a linear combination of the B-splines as follows:

$$u_h(x) = \sum_{j=-1}^{n+1} \gamma_j B_j(x), \qquad v_h(x) = \sum_{j=-1}^{n+1} \delta_j B_j(x), \tag{2.12}$$

where the $j$th basis $B_j(x)$ of the cubic B-splines space $S_{h,3}$ for $j = -1, 0, 1, 2, \ldots, n, n+1$ is given below:

$$B_j(x) = \begin{cases} 0, & \text{if } x \le x_{j-2}, \\ \dfrac{1}{6h^3}(x - x_{j-2})^3, & \text{if } x_{j-2} \le x \le x_{j-1}, \\ \dfrac{1}{6h^3}\left(h^3 + 3h^2(x - x_{j-1}) + 3h(x - x_{j-1})^2 - 3(x - x_{j-1})^3\right), & \text{if } x_{j-1} \le x \le x_j, \\ \dfrac{1}{6h^3}\left(h^3 + 3h^2(x_{j+1} - x) + 3h(x_{j+1} - x)^2 - 3(x_{j+1} - x)^3\right), & \text{if } x_j \le x \le x_{j+1}, \\ \dfrac{1}{6h^3}(x_{j+2} - x)^3, & \text{if } x_{j+1} \le x \le x_{j+2}, \\ 0, & \text{if } x \ge x_{j+2}. \end{cases} \tag{2.13}$$

For $j = -1, 0$ and $j = n, n+1$, the basis functions are defined as in the above form, after extending the partition by introducing fictitious nodal points $x_{-3}, x_{-2}, x_{-1}$ on the left-hand side and $x_{n+1}, x_{n+2}, x_{n+3}$ on the right-hand side, respectively. Further, the $i$th basis $\phi_i(x)$ of the piecewise linear "hat" splines space $S_{h,1}$ for $i = 0, 1, 2, \ldots, n$ is given below:

$$\phi_i(x) = \begin{cases} 0, & \text{if } x \le x_{i-1}, \\ \dfrac{1}{h}(x - x_{i-1}), & \text{if } x_{i-1} \le x \le x_i, \\ \dfrac{1}{h}(x_{i+1} - x), & \text{if } x_i \le x \le x_{i+1}, \\ 0, & \text{if} x \ge x_{i+1}. \end{cases} \tag{2.14}$$

In a similar manner, for $i = 0$ and $i = n$, the basis functions are defined as in the above form, after extending the partition by introducing fictitious nodal point $x_{-1}$ on the left-hand side and $x_{n+1}$ on the right-hand side, respectively. The mixed discrete Petrov-Galerkin method for (2.10) and (2.11) without assuming boundary conditions in the trial space is given as follows:

$$\sum_{j=-1}^{n+1}\gamma_j\left\langle B_j'', \phi_i\right\rangle_h - \sum_{j=-1}^{n+1}\delta_j\langle \alpha B_j, \phi_i\rangle_h = 0, \quad i = 0,1,2,\ldots,n,$$
$$\sum_{j=-1}^{n+1}\gamma_j\langle bB_j, \phi_i\rangle_h + \sum_{j=-1}^{n+1}\delta_j\left\langle B_j'', \phi_i\right\rangle_h = \langle f, \phi_i\rangle_h, \quad i = 0,1,2,\ldots,n, \tag{2.15}$$

with the corresponding equations:

$$\sum_{j=-1}^{n+1}\gamma_j B_j(0) = 0, \qquad \sum_{j=-1}^{n+1}\gamma_j B_j(1) = 0,$$
$$\sum_{j=-1}^{n+1}\delta_j B_j(0) = 0, \qquad \sum_{j=-1}^{n+1}\delta_j B_j(0) = 0, \tag{2.16}$$

referring to the zero-boundary conditions:

$$u_h(0) = 0, \quad u_h(1) = 0, \quad v_h(0) = 0, \quad v_h(1) = 0. \tag{2.17}$$

The above set of equations (2.15)–(2.16) can be written as a set of $2n + 6$ equations in $2n + 6$ unknowns. Here, we study the effect of quadrature rule in the error analysis. Since we compute the approximations for the solution $u(x)$ as well as for its second derivative $v(x)$ with integrals replaced by Gaussian quadrature rule in the formulation, this work may be considered as a quadrature-based mixed Petrov-Galerkin method.

## 3. Overview of Discrete Petrov-Galerkin Method

Here, the integrals are replaced by composite two-point Gauss rule. Therefore, the resulting method may be described as a "qualocation" approximation, that is, a quadrature-based modification of the collocation method. Further, it may be considered as a Petrov-Galerkin method with a quadrature rule because the test space and trial space are different. Hence, it may be referred to as discrete Petrov-Galerkin method. One practical advantage of this procedure over the orthogonal spline collocation method described in Douglas Jr. and Dupont [1, 2] is that for a given partition there are only half the number of unknowns, and therefore it reduces the size of the matrix.

The qualocation method was first introduced and analysed by Sloan [3] for boundary integral equation on smooth curves. Later on Sloan et al. [4] extended this method to a class of linear second-order two-point boundary value problems and derived optimal error estimates without quasi-uniformity assumption on the finite element mesh. Then, Jones Doss and Pani [5] discussed the qualocation method for a second-order semilinear two-point boundary value problem. Further, Pani [6] expanded its scope by adapting the analysis to a semilinear parabolic initial and boundary value problem in a single space variable. Jones Doss and Pani [7] extended this method to the free boundary problem, that is, one-dimensional single-phase Stefan problem for which part of the boundary has to be found out along with the solution process. A quadrature-based Petrov-Galerkin method applied to higher dimensional boundary value problems is studied in Bialecki et al. [8, 9] and Ganesh and Mustapha [10].

The main idea of this paper is that a quadrature based approximation for a fourth order problem is analyzed in mixed Galerkin setting. The organization of this paper is as follows. In previous Sections 1 and 2, the problem is introduced; the weak and the Galerkin formulations are defined. Overview of discrete Petrov-Galerkin method is discussed in Section 3. Preliminaries required for our analysis are mentioned in Section 4. Error analysis is carried over in Section 5. Throughout this paper $C$ is a generic positive constant, whose dependence on the smoothness of the exact solution can be easily determined from the proofs.

## 4. Preliminaries

We assume that $\alpha$ and $b$ are such that

$$\alpha, b \in C^4\left(\overline{I}\right), \tag{4.1}$$

where $\overline{I} = [0,1]$. We assume that the problem consisting of the coupled equations (1.4) and (1.5) is uniquely solvable for a given sufficiently smooth function $f(x)$. It can be proved that the quadrature rule in (2.3) has an error bound of the form

$$E_h(g) = \left| Q_h(g) - \int g \right| \le C \sum_{i=1}^{n} h_k^4 \left\| g^{(4)} \right\|_{L_1(I_k)}. \tag{4.2}$$

This follows from Peano's kernel theorem (see [11]).

The following inequality is frequently used in our analysis. If $v \in W_p^m(E)$ with $p \in [1,\infty]$, then there exists a positive constant $C$ depending only on $m$ such that, for any $\delta$ satisfying $0 < \delta \le |E| \le 1$,

$$\|v\|_{W_p^i(E)} \le C\left[\delta^{m-i}\|v\|_{W_p^m(E)} + \delta^{-i}\|v\|_{L_p(E)}\right], \qquad 0 \le i \le m-1, \tag{4.3}$$

where $|E|$ denotes the length of $E$. For a detailed proof, one may refer to appendix of Sloan et al. [4] or Chapter 4 of Adams [12]. Let us use the following notation:

$$Lv := v''. \tag{4.4}$$

The adjoint operator $L^*$ with corresponding adjoint boundary condition is defined as follows:

$$\begin{gathered} L^*\phi = \phi'', \\ \phi(0) = \phi(1) = 0. \end{gathered} \tag{4.5}$$

Since $L$ is a self-adjoint operator, we mention below the regularity of $L^*$ (equal to $L$) in the $q$ norm. We make a stronger assumption as in Sloan et al. [4] that for arbitrary $q \in [1, \infty]$, there exists a positive constant $C$ such that

$$\|L^* u\|_{L_q(I)} \geq C \|u\|_{W_q^2(I)}. \tag{4.6}$$

We have the following inequality due to the Sobolev embedding theorem; the proof of which can be found in page 97, Adams [12],

$$\|\phi\|_{L_\infty(I_k)} \leq \|\phi\|_{W_p^1(I_k)}; \quad 1 \leq p \leq \infty, \ \phi \in W_p^1(I_k). \tag{4.7}$$

## 5. Convergence Analysis

Hereafter throughout this section, for $p$ and $q$ with $1 \leq p, \ q \leq \infty$,s and $p^{-1} + q^{-1} = 1$, we use the following notations:

$$\|v\|_{0,p} = \|v\|_{L_p}, \qquad \|v\|_{s,p} = \|v\|_{W_p^s}, \qquad \|v\|_{s,p,k} = \|v\|_{W_p^s(I_k)}. \tag{5.1}$$

Let us denote the error between $u$ and $u_h$ by $\varepsilon_h$ and the error between $v$ and $v_h$ by $e_h$, respectively, that is, $\varepsilon_h = u - u_h$ and $e_h = v - v_h$. Using (2.11) and (1.5), we obtain the following error equations:

$$\left\langle e_h'', \phi_h \right\rangle_h = \left\langle v'' - v_h'', \phi_h \right\rangle_h = \left\langle v'', \phi_h \right\rangle_h - \langle f - b u_h, \phi_h \rangle_h = -\langle b(u - u_h), \phi_h \rangle_h = -\langle b\varepsilon_h, \phi_h \rangle_h, \tag{5.2}$$

and therefore we get

$$\left\langle e_h'', \phi_h \right\rangle_h = -\langle b\varepsilon_h, \phi_h \rangle_h, \quad \phi_h \in S_{h,1}. \tag{5.3}$$

Further, using (2.10) and (1.4),

$$\left\langle \varepsilon_h'', \phi_h \right\rangle_h = \left\langle u'' - u_h'', \phi_h \right\rangle_h = \langle \alpha(v - v_h), \phi_h \rangle_h = \langle \alpha e_h, \phi_h \rangle_h, \tag{5.4}$$

and therefore we have

$$\left\langle \varepsilon_h'', \phi_h \right\rangle_h = \langle \alpha e_h, \phi_h \rangle_h, \quad \phi_h \in S_{h,1}. \tag{5.5}$$

The following lemma gives estimates for the error in the quadrature rule for the term $(e_h'' \chi_h)$ and $(\varepsilon_h'' \chi_h)$ for $\chi_h \in S_{h,1}$. These estimates are required for our error analysis later. The proof of the lemma is similar to the proof of Lemma 4.2 of Sloan et al. [4].

**Lemma 5.1.** *For all $\chi_h \in S_{h,1}$ and $h$ sufficiently small,*

(a) $E_h(e_h^{''}\chi_h) \leq Ch^4\|v\|_{6,p}\|\chi_h\|_{1,q'}$

(b) $E_h(e_h^{''}\chi_h) \leq Ch^3\|v\|_{6,p}\|\chi_h\|_{0,q'}$

(c) $E_h(\varepsilon_h^{''}\chi_h) \leq Ch^4\|u\|_{6,p}\|\chi_h\|_{1,q'}$

(d) $E_h(\varepsilon_h^{''}\chi_h) \leq Ch^3\|u\|_{6,p}\|\chi_h\|_{0,q'}$.

The following result gives estimate for $\varepsilon_h(\overline{x})$, where $\overline{x}$ is any arbitrary point in $I$. This estimate is crucial for our error analysis.

**Lemma 5.2.** *Let $u$ be the weak solution of* (1.4) *defined through* (2.7)*. Further, let $u_h$ be the corresponding discrete Petrov-Galerkin solution defined through* (2.10)*. Then, the error $\varepsilon_h = u - u_h$ satisfies*

$$|\varepsilon_h(\overline{x})| \leq C\left[h^2\|\varepsilon_h\|_{2,p} + h^4\|u\|_{6,p} + h\|e_h\|_{1,p}\right], \tag{5.6}$$

*where $\overline{x}$ is an arbitrary point in* $[0,1]$.

*Proof.* For a given $\overline{x} \in [0,1]$, let $\Phi$ be an element of $L_p(I) \bigcap C(I)$ satisfying the following auxiliary problem:

$$\begin{gathered}\Phi^{''} = 0, \quad x \in I - \{\overline{x}\}, \\ \Phi(0) = \Phi(1) = 0, \qquad \Phi'_{-}(\overline{x}) - \Phi'_{+}(\overline{x}) = -1.\end{gathered} \tag{5.7}$$

The above problem has a solution. For example,

$$\Phi(x) = \begin{cases}(\overline{x} - 1)x, & 0 \leq x \leq \overline{x}, \\ \overline{x}(x - 1), & \overline{x} \leq x \leq 1\end{cases} \tag{5.8}$$

satisfies the above differential equation, the boundary conditions, and the jump condition.

Let us define $\Psi$ as follows:

$$\Psi(x) = \begin{cases}\Phi^{''}, & x \in I - \{\overline{x}\}, \\ 0, & \text{at } x = \overline{x}.\end{cases} \tag{5.9}$$

Then, $\Psi = 0$ *a.e.* on $I$. We first multiply $\varepsilon_h$ with $\Psi$ and then integrate over $I$. On applying integration by parts, using the fact that $\varepsilon_h(0) = \varepsilon_h(1) = 0$ and the jump condition for $\Phi'$, we obtain

$$\begin{aligned}
0 = (\varepsilon_h, \Psi) &= \int_0^{\overline{x}} \varepsilon_h \Psi + \int_{\overline{x}}^1 \varepsilon_h \Psi = \int_0^{\overline{x}} \varepsilon_h \Phi'' + \int_{\overline{x}}^1 \varepsilon_h \Phi'' \\
&= \left[\varepsilon_h \Phi'\right]_0^{\overline{x}} - \int_0^{\overline{x}} \varepsilon_h' \Phi' + \left[\varepsilon_h \Phi'\right]_{\overline{x}}^1 - \int_{\overline{x}}^1 \varepsilon_h' \Phi' = \varepsilon_h(\overline{x}) \left[\Phi'_-(\overline{x}) - \Phi'_+(\overline{x})\right] - \int_0^{\overline{x}} \varepsilon_h' \Phi' - \int_{\overline{x}}^1 \varepsilon_h' \Phi' \\
&= -\varepsilon_h(\overline{x}) - \int_0^{\overline{x}} \varepsilon_h' \Phi' - \int_{\overline{x}}^1 \varepsilon_h' \Phi'.
\end{aligned} \tag{5.10}$$

Applying integration by parts once again, using boundary condition for $\Phi$ and the continuity of $\Phi$, we obtain

$$0 = -\varepsilon_h(\overline{x}) - \left\{ \left[\varepsilon_h' \Phi\right]_0^{\overline{x}} - \int_0^{\overline{x}} \varepsilon_h'' \Phi + \left[\varepsilon_h' \Phi\right]_{\overline{x}}^1 - \int_{\overline{x}}^1 \varepsilon_h'' \Phi \right\} = -\varepsilon_h(\overline{x}) + \left(\varepsilon_h'', \Phi\right), \tag{5.11}$$

that is, $\varepsilon_h(\overline{x}) = (\varepsilon_h'', \Phi)$. Let $\Phi_h$ be the linear interpolant of $\Phi$. Then, we have

$$\begin{aligned}
\varepsilon_h(\overline{x}) &= \left(\varepsilon_h'', \Phi - \Phi_h\right) + \left(\varepsilon_h'', \Phi_h\right) - \left\langle \varepsilon_h'', \Phi_h \right\rangle_h + \left\langle \varepsilon_h'', \Phi_h \right\rangle_h \\
|\varepsilon_h(\overline{x})| &\le \left|\left(\varepsilon_h'', \Phi - \Phi_h\right)\right| + \left|E_h\left(\varepsilon_h'' \Phi_h\right)\right| + \left|\left\langle \varepsilon_h'', \Phi_h \right\rangle_h\right| \le T_1 + T_2 + T_3.
\end{aligned} \tag{5.12}$$

We know that

$$\|\Phi_h\|_{1,q} \le \|\Phi - \Phi_h\|_{1,q} + \|\Phi\|_{1,q} \le Ch\|\Phi\|_{2,q} + \|\Phi\|_{2,q} \le C\|\Phi\|_{2,q}. \tag{5.13}$$

We now compute the estimates for the terms $T_1$, $T_2$, and $T_3$ as follows:

$$T_1 = \left|\left(\varepsilon_h'', \Phi - \Phi_h\right)\right| \le \left\|\varepsilon_h''\right\|_{0,p} \|\Phi - \Phi_h\|_{0,q} \le Ch^2 \|\varepsilon_h\|_{2,p} \|\Phi\|_{2,q}. \tag{5.14}$$

Using Lemma 5.1(c) and (5.13), we obtain

$$T_2 = \left|E_h\left(\varepsilon_h'' \Phi_h\right)\right| \le Ch^4 \|u\|_{6,p} \|\Phi\|_{2,q}. \tag{5.15}$$

Using (5.5), (2.3), and the Sobolev embedding theorem (4.7) locally on $I_k$ for both $\|e_h\|_{0,\infty,k}$ and $\|\Phi_h\|_{0,\infty,k}$, we have

$$T_3 = \left|\left\langle \varepsilon_h'', \Phi_h \right\rangle_h\right| = |\langle \alpha e_h, \Phi_h \rangle_h| \le C \sum_{k=1}^{n} \frac{h_k}{2} \|e_h\|_{0,\infty,k} \|\Phi_h\|_{0,\infty,k} \le C \sum_{k=1}^{n} \frac{h_k}{2} \|e_h\|_{1,p,k} \|\Phi_h\|_{1,q,k}. \tag{5.16}$$

Using Hölder's inequality for sums and (5.13), we have

$$T_3 \le Ch\|e_h\|_{1,p}\|\Phi_h\|_{1,q} \le Ch\|e_h\|_{1,p}\|\Phi\|_{2,q}. \tag{5.17}$$

For $\Phi$ satisfying the auxiliary problem, it is easy to verify that $\|\Phi\|_{2,q} \le K$, where $K$ is a constant not depending on $h$.

Using $T_1$, $T_2$, and $T_3$ in (5.12), we have

$$|\varepsilon_h(\overline{x})| \le C\left[h^2\|\varepsilon_h\|_{2,p} + h^4\|u\|_{6,p} + h\|e_h\|_{1,p}\right]. \tag{5.18}$$

This completes the proof. □

In the following lemma, we initially compute the error $(v - v_h)$ in terms of $(u - u_h)$, and then later on we establish an optimal estimate of error $(v - v_h)$ independent of $(u - u_h)$.

**Lemma 5.3.** *Let $u$ and $v$ be the weak solutions of the coupled equations* (1.4) *and* (1.5) *defined through* (2.7) *and* (2.8), *respectively. Further, let $u_h$ and $v_h$ be the corresponding discrete Petrov-Galerkin solutions defined through* (2.10) *and* (2.11), *respectively. Then the estimates of the errors $e_h = v - v_h$ in $L_p$, $W_p^1$, and $W_p^2$ norms are given as follows:*

$$\begin{aligned}
\|e_h\|_{0,p} &\le C\left[h^4\|v\|_{6,p} + h^5\|u\|_{6,p} + h^3\|\varepsilon_h\|_{2,p}\right],\\
\|e_h\|_{1,p} &\le C\left[h^3\|v\|_{6,p} + h^4\|u\|_{6,p} + h^2\|\varepsilon_h\|_{2,p}\right],\\
\|e_h\|_{2,p} &\le C\left[h^2\|v\|_{6,p} + h^4\|u\|_{6,p} + h^2\|\varepsilon_h\|_{2,p}\right].
\end{aligned} \tag{5.19}$$

*Proof.* Let $\eta$ be an arbitrary element of $L_q$, and let $\phi \in W_q^2$ be the solution of the auxiliary problem

$$\begin{gathered}
L^*\phi = \eta,\\
\phi(0) = \phi(1) = 0.
\end{gathered} \tag{5.20}$$

We now have

$$\begin{aligned}
(e_h, \eta) = (e_h, L^*\phi) = (Le_h, \phi) &= \left(e_h'', \phi - \phi_h\right) + \left(e_h'', \phi_h\right)\\
&= \left(e_h'', \phi - \phi_h\right) + \left(e_h'', \phi_h\right) - \left\langle e_h'', \phi_h\right\rangle_h + \left\langle e_h'', \phi_h\right\rangle_h\\
&= \left(e_h'', \phi - \phi_h\right) + E_h\left(e_h''\phi_h\right) + \left\langle e_h'', \phi_h\right\rangle_h,\\
\left|(e_h, \eta)\right| &\le \left|\left(e_h'', \phi - \phi_h\right)\right| + \left|E_h\left(e_h''\phi_h\right)\right| + \left|\left\langle e_h'', \phi_h\right\rangle_h\right|\\
&\le T_4 + T_5 + T_6,
\end{aligned} \tag{5.21}$$

where $\phi_h \in S_{h,1}$ is the linear interpolant of $\phi$.

We know that

$$\|\phi_h\|_{1,q} \le \|\phi - \phi_h\|_{1,q} + \|\phi\|_{1,q} \le Ch\|\phi\|_{2,q} + \|\phi\|_{2,q} \le C\|\phi\|_{2,q}. \tag{5.22}$$

We shall compute the estimates for the terms $T_4$, $T_5$, and $T_6$ as follows:

$$\begin{aligned} T_4 &= \left|\left(e_h'', \phi - \phi_h\right)\right| \le \left\|e_h''\right\|_{0,p} \|\phi - \phi_h\|_{0,q} \le Ch^2 \|e_h\|_{2,p} \|\phi\|_{2,q}, \\ T_5 &= \left|E_h\left(e_h''\phi_h\right)\right| \le Ch^4\|v\|_{6,p}\|\phi_h\|_{1,q} \le Ch^4\|v\|_{6,p}\|\phi\|_{2,q} \text{ by Lemma 5.1(a), (5.22).} \end{aligned} \tag{5.23}$$

Using (5.3), (2.3), and the Sobolev embedding theorem (4.7) locally on $I_k$ for $\|\phi_h\|_{0,\infty,k}$, we have

$$T_6 = \left|\left\langle e_h'', \phi_h\right\rangle_h\right| = \left|-\langle b\varepsilon_h, \phi_h\rangle_h\right| \le C\sum_{k=1}^{n} \frac{h_k}{2}\|\varepsilon_h\|_{0,\infty,k}\|\phi_h\|_{0,\infty,k} \le C\sum_{k=1}^{n}\frac{h_k}{2}\|\varepsilon_h\|_{0,\infty,k}\|\phi_h\|_{1,q,k}. \tag{5.24}$$

Using Hölder's inequality for sums, Lemma 5.2, and (5.22), we obtain

$$T_6 \le Ch\left[h^2\|\varepsilon_h\|_{2,p} + h^4\|u\|_{6,p} + h\|e_h\|_{1,p}\right]\|\phi_h\|_{1,q} \le C\left[h^3\|\varepsilon_h\|_{2,p} + h^5\|u\|_{6,p} + h^2\|e_h\|_{1,p}\right]\|\phi\|_{2,q}. \tag{5.25}$$

Substituting $T_4$, $T_5$, and $T_6$ in (5.21), we have

$$|(e_h, \eta)| \le C\left[h^2\|e_h\|_{2,p} + h^4\|v\|_{6,p} + h^3\|\varepsilon_h\|_{2,p} + h^5\|u\|_{6,p} + h^2\|e_h\|_{1,p}\right]\|\phi\|_{2,q}. \tag{5.26}$$

Using (4.6) and the regularity of the auxiliary problem, we have $\|\phi\|_{2,q} \le C\|\eta\|_{0,q}$. Since $\eta \in L_q$ is arbitrary, we have

$$\|e_h\|_{0,p} \le C\left(h^2\|e_h\|_{2,p} + h^3\|\varepsilon_h\|_{2,p} + h^4\|v\|_{6,p} + h^5\|u\|_{6,p}\right). \tag{5.27}$$

We now estimate $\|e_h''\|$ via a projection argument. Let $P_h$ be the orthogonal projection onto $S_{h,1}$ with respect to $L_2$ inner product defined by

$$\left(v'' - P_h v'', \psi_h\right) = 0, \quad \psi_h \in S_{h,1}. \tag{5.28}$$

The domain of $P_h$ may be taken to be $L_1$. From Crouzeix and Thomée [13] and de Boor [14], it is seen that the $L_2$ projection is stable. Thus,

$$\|P_h v\|_{0,p} \le C\|v\|_{0,p}. \tag{5.29}$$

Then the error $e_h''$ can be interpreted in terms of the error of the above projection:

$$\left\|e_h''\right\|_{0,p} = \left\|v'' - v_h''\right\|_{0,p} \leq \left\|v'' - P_h v''\right\|_{0,p} + \left\|P_h v'' - v_h''\right\|_{0,p}. \tag{5.30}$$

From the stability property (5.29), the error in the projection follows as in de Boor [14], that is,

$$\left\|v'' - P_h v''\right\|_{0,p} \leq Ch^2\left\|v''\right\|_{2,p} \leq Ch^2\|v\|_{4,p}. \tag{5.31}$$

Then the remaining task is to compute the estimate of $\|P_h v'' - v_h''\|_{0,p}$.

For $\psi_h \in S_{h,1}$,

$$\begin{aligned}
\left(P_h v'' - v_h'', \psi_h\right) &= \left(P_h v'' - v'' + v'' - v_h'', \psi_h\right) \\
&= \left(P_h v'' - v'', \psi_h\right) + \left(v'' - v_h'', \psi_h\right) \\
&= \left(v'' - v_h'', \psi_h\right) \text{ using (5.28)}, \\
\left(P_h v'' - v_h'', \psi_h\right) &= \left(e_h'', \psi_h\right) = \left(e_h'', \psi_h\right) - \left\langle e_h'', \psi_h\right\rangle_h + \left\langle e_h'', \psi_h\right\rangle_h \\
&= E_h\left(e_h''\psi_h\right) + \left\langle e_h'', \psi_h\right\rangle_h, \\
\left|\left(P_h v'' - v_h'', \psi_h\right)\right| &\leq \left|E_h\left(e_h''\psi_h\right)\right| + \left|\left\langle e_h'', \psi_h\right\rangle_h\right| \leq T_7 + T_8.
\end{aligned} \tag{5.32}$$

We shall compute the estimates for the terms $T_7$ and $T_8$

$$T_7 = \left|E_h\left(e_h''\psi_h\right)\right| \leq Ch^3\|v\|_{6,p}\left\|\psi_h\right\|_{0,q} \tag{5.33}$$

by Lemma 5.1(b).

Following the steps of computation involved in the term $T_6$, we obtain the estimate of $T_8$ as

$$T_8 = \left|\left\langle e_h'', \psi_h\right\rangle_h\right| \leq C\left[h^2\|\varepsilon_h\|_{2,p} + h^4\|u\|_{6,p} + h\|e_h\|_{1,p}\right]\left\|\psi_h\right\|_{0,q}, \tag{5.34}$$

where we have used the inverse inequality $\|\psi_h\|_{1,q,k} \leq h_k^{-1}\|\psi_h\|_{0,q,k}$ locally. Using $T_7$ and $T_8$ in (5.32), we get

$$\left|\left(P_h v'' - v_h'', \psi_h\right)\right| \leq C\left[h^3\|v\|_{6,p} + h^2\|\varepsilon_h\|_{2,p} + h^4\|u\|_{6,p} + h\|e_h\|_{1,p}\right]\left\|\psi_h\right\|_{0,q}. \tag{5.35}$$

We now show the above inequality for $\eta \in L_q$ to obtain $\|P_h v'' - v_h''\|_{0,p}$.

Now let $\eta$ be an arbitrary element of $L_q$. Then since $v_h^{''} \in S_{h,1}$, it follows from the definition of $P_h\eta$, (5.35), and (5.29) with $p$ replaced by $q$, that

$$\begin{aligned}
0 &= \left(P_h v^{''} - v_h^{''}, \eta - P_h\eta\right), \\
\left|\left(P_h v^{''} - v_h^{''}, \eta\right)\right| &= \left|\left(P_h v^{''} - v_h^{''}, P_h\eta\right)\right| \leq C\left[h^3\|v\|_{6,p} + h^2\|\varepsilon_h\|_{2,p} + h^4\|u\|_{6,p} + h\|e_h\|_{1,p}\right]\|P_h\eta\|_{0,q} \\
&\leq C\left[h^3\|v\|_{6,p} + h^2\|\varepsilon_h\|_{2,p} + h^4\|u\|_{6,p} + h\|e_h\|_{1,p}\right]\|\eta\|_{0,q}, \\
\left\|P_h v^{''} - v_h^{''}\right\|_{0,p} &\leq C\left[h^3\|v\|_{6,p} + h^2\|\varepsilon_h\|_{2,p} + h^4\|u\|_{6,p} + h\|e_h\|_{1,p}\right].
\end{aligned} \tag{5.36}$$

Now, from (5.30), (5.31), and (5.36), we conclude that

$$\begin{aligned}
\left\|e_h^{''}\right\|_{0,p} &\leq Ch^2\|v\|_{4,p} + C\left[h^3\|v\|_{6,p} + h^2\|\varepsilon_h\|_{2,p} + h^4\|u\|_{6,p} + h\|e_h\|_{1,p}\right] \\
&\leq C\left[h^2\|v\|_{6,p} + h^2\|\varepsilon_h\|_{2,p} + h^4\|u\|_{6,p} + h\|e_h\|_{1,p}\right].
\end{aligned} \tag{5.37}$$

Now, using the fact $\|e_h\|_{2,p} \leq \|e_h\|_{1,p} + \|e_h^{''}\|_{0,p}$ and the above estimate, we have

$$\begin{aligned}
\|e_h\|_{2,p} &\leq \|e_h\|_{1,p} + C\left[h^2\|v\|_{6,p} + h^2\|\varepsilon_h\|_{2,p} + h^4\|u\|_{6,p} + h\|e_h\|_{1,p}\right] \\
&\leq C\left[\|e_h\|_{1,p} + h^2\|v\|_{6,p} + h^2\|\varepsilon_h\|_{2,p} + h^4\|u\|_{6,p}\right] \\
&\leq C\left[\|e_h\|_{1,p} + h^2\|v\|_{6,p} + h^2\|\varepsilon_h\|_{2,p} + h^4\|u\|_{6,p}\right].
\end{aligned} \tag{5.38}$$

Now using (4.3) with $m = 2$ and $i = 1$, we have

$$\|e_h\|_{1,p} \leq C\left(h^{-1}\|e_h\|_{0,p} + h\|e_h\|_{2,p}\right). \tag{5.39}$$

Substituting (5.39) in the above expression, we obtain

$$\|e_h\|_{2,p} \leq C\left[\left(h^{-1}\|e_h\|_{0,p} + h\|e_h\|_{2,p}\right) + h^2\|v\|_{6,p} + h^2\|\varepsilon_h\|_{2,p} + h^4\|u\|_{6,p}\right]. \tag{5.40}$$

For sufficiently small $h$, we have

$$\|e_h\|_{2,p} \leq C\left[h^{-1}\|e_h\|_{0,p} + h^2\|v\|_{6,p} + h^2\|\varepsilon_h\|_{2,p} + h^4\|u\|_{6,p}\right]. \tag{5.41}$$

Using (5.41) in (5.27),

$$\|e_h\|_{0,p} \leq C\left[h^2\left(h^{-1}\|e_h\|_{0,p} + h^2\|v\|_{6,p} + h^2\|\varepsilon_h\|_{2,p} + h^4\|u\|_{6,p}\right) + h^4\|v\|_{6,p} + h^5\|u\|_{6,p} + h^3\|\varepsilon_h\|_{2,p}\right]. \tag{5.42}$$

For sufficiently small $h$, we get

$$\|e_h\|_{0,p} \leq C\left[h^4\|v\|_{6,p} + h^5\|u\|_{6,p} + h^3\|\varepsilon_h\|_{2,p}\right]. \tag{5.43}$$

Using (5.43) in (5.41), we have

$$\begin{aligned}\|e_h\|_{2,p} &\leq C\left[h^{-1}\left(h^4\|v\|_{6,p} + h^5\|u\|_{6,p} + h^3\|\varepsilon_h\|_{2,p}\right) + h^2\|v\|_{6,p} + h^2\|\varepsilon_h\|_{2,p} + h^4\|u\|_{6,p}\right] \\ &\leq C\left[h^2\|v\|_{6,p} + h^2\|\varepsilon_h\|_{2,p} + h^4\|u\|_{6,p}\right].\end{aligned} \tag{5.44}$$

Using (5.43) and (5.44) in (5.39), we have

$$\begin{aligned}\|e_h\|_{1,p} &\leq C\left[h^{-1}\left(h^4\|v\|_{6,p} + h^5\|u\|_{6,p} + h^3\|\varepsilon_h\|_{2,p}\right) + h\left(h^2\|v\|_{6,p} + h^2\|\varepsilon_h\|_{2,p} + h^4\|u\|_{6,p}\right)\right] \\ &\leq C\left[h^3\|v\|_{6,p} + h^4\|u\|_{6,p} + h^2\|\varepsilon_h\|_{2,p}\right].\end{aligned} \tag{5.45}$$

Equations (5.43), (5.44), and (5.45) give the required result. □

We now compute the error estimate of $\varepsilon_h$ in $L_p$, $W_p^1$, and $W_p^2$ norms as has been done in the previous case.

**Lemma 5.4.** *Let $u$ and $v$ be the weak solutions of the coupled equations* (1.4) *and* (1.5) *defined through* (2.7) *and* (2.8), *respectively. Further, let $u_h$ and $v_h$ be the corresponding discrete Petrov-Galerkin solutions defined through* (2.10) *and* (2.11), *respectively. Then the estimates of the errors $\varepsilon_h = u - u_h$ in $L_p$, $W_p^1$ and $W_p^2$ norms are given as follows:*

$$\begin{aligned}\|\varepsilon_h\|_{0,p} &\leq C\left[h^4\|u\|_{6,p} + h\|e_h\|_{1,p}\right], \\ \|\varepsilon_h\|_{1,p} &\leq C\left[h^3\|u\|_{6,p} + \|e_h\|_{1,p}\right], \\ \|\varepsilon_h\|_{2,p} &\leq C\left[h^2\|u\|_{6,p} + \|e_h\|_{1,p}\right].\end{aligned} \tag{5.46}$$

*Proof.* Let $\rho$ be an arbitrary element of $L_q$, and let $\phi \in W_q^2$ be the unique solution of the auxiliary problem

$$\begin{aligned}L^*\phi &= \rho, \\ \phi(0) &= \phi(1) = 0.\end{aligned} \tag{5.47}$$

Then we have

$$(\varepsilon_h, \rho) = (\varepsilon_h, L^*\phi) = (L\varepsilon_h, \phi) = \left(\varepsilon_h^{''}, \phi\right) = \left(\varepsilon_h^{''}, \phi - \phi_h\right) + \left(\varepsilon_h^{''}, \phi_h\right) - \left\langle \varepsilon_h^{''}, \phi_h \right\rangle_h + \left\langle \varepsilon_h^{''}, \phi_h \right\rangle_h, \tag{5.48}$$

where $\phi_h \in S_{h,1}$ is a linear interpolant of $\phi$,

$$\left|(\varepsilon_h,\rho)\right| \leq \left|\left(\varepsilon_h^{''}, \phi - \phi_h\right)\right| + \left|E_h\left(\varepsilon_h^{''}\phi_h\right)\right| + \left|\left\langle \varepsilon_h^{''}, \phi_h \right\rangle_h\right| \leq T_9 + T_{10} + T_{11}. \tag{5.49}$$

Following the steps involved in the computation of $T_4$ and $T_5$, we obtain the estimates of $T_9$ and $T_{10}$ as follows:

$$\begin{aligned} T_9 &\leq Ch^2\|\varepsilon_h\|_{2,p}\|\phi\|_{2,q}, \\ T_{10} &\leq Ch^4\|u\|_{6,p}\|\phi\|_{2,q}, \end{aligned} \tag{5.50}$$

by Lemma 5.1(c) and (5.22).

Using (5.5) and (2.3) first, then the Sobolev embedding theorem (4.7) locally on $I_k$ for $\|\phi_h\|_{0,\infty,k}$ and $\|e_h\|_{0,\infty,k}$ to estimate $T_{11}$, we have

$$\begin{aligned} T_{11} &= \left|\left\langle \varepsilon_h^{''}, \phi_h \right\rangle_h\right| = \left|\langle \alpha e_h, \phi_h\rangle_h\right| \leq C\sum_{k=1}^{n}\frac{h_k}{2}\|e_h\|_{0,\infty,k}\|\phi_h\|_{0,\infty,k} \leq C\sum_{k=1}^{n}\frac{h_k}{2}\|e_h\|_{0,\infty,k}\|\phi_h\|_{1,q,k} \\ &\leq C\sum_{k=1}^{n}\frac{h_k}{2}\|e_h\|_{1,p,k}\|\phi_h\|_{1,q,k}. \end{aligned} \tag{5.51}$$

Further, using Hölder's inequality for sums and (5.22), we obtain

$$T_{11} \leq Ch\|e_h\|_{1,p}\|\phi_h\|_{1,q} \leq Ch\|e_h\|_{1,p}\|\phi\|_{2,q}. \tag{5.52}$$

Substituting the estimates $T_9$, $T_{10}$, and $T_{11}$ in (5.49), we obtain

$$\left|(\varepsilon_h,\rho)\right| \leq C\left[h^2\|\varepsilon_h\|_{2,p} + h^4\|u\|_{6,p} + h\|e_h\|_{1,p}\right]\|\phi\|_{2,q}. \tag{5.53}$$

Using (4.6) and regularity of the auxiliary problem, we have $\|\phi\|_{2,q} \leq C\|\rho\|_{o,q}$. Since $\rho \in L_q$ is arbitrary, we have

$$\|\varepsilon_h\|_{0,p} \leq C\left[h^2\|\varepsilon_h\|_{2,p} + h^4\|u\|_{6,p} + h\|e_h\|_{1,p}\right]. \tag{5.54}$$

The estimate of $\|\varepsilon_h^{''}\|_{0,p}$ can be obtained through a projection argument as mentioned in Lemma 5.3 as

$$\left\|\varepsilon_h^{''}\right\|_{0,p} \leq C\left[h^2\|u\|_{6,p} + \|e_h\|_{1,p}\right], \tag{5.55}$$

where we have used Lemma 5.1(d). In a similar manner we can compute the estimates for $\|\varepsilon_h\|_{0,p}$, $\|\varepsilon_h\|_{1,p}$ and $\|\varepsilon_h\|_{2,p}$ as

$$\begin{aligned}\|\varepsilon_h\|_{0,p} &\leq C\left[h^4\|u\|_{6,p} + h\|e_h\|_{1,p}\right],\\ \|\varepsilon_h\|_{1,p} &\leq C\left[h^3\|u\|_{6,p} + \|e_h\|_{1,p}\right],\\ \|\varepsilon_h\|_{2,p} &\leq C\left[h^2\|u\|_{6,p} + \|e_h\|_{1,p}\right].\end{aligned} \tag{5.56}$$

Using all the estimates from Lemmas 5.3 and 5.4, we have the following main error estimates. □

**Theorem 5.5.** *Assume that $u$ and $v$ satisfy* (1.4) *and* (1.5)*, respectively, with* (4.1)*. Assume also that $u \in W_p^6$ and $v \in W_p^6$, where $p \in [1,\infty]$. Then* (2.10) *and* (2.11) *have unique solutions $u_h \in \overset{0}{S}_{h,3}$ and $v_h \in \overset{0}{S}_{h,3}$, respectively, and for $h$ sufficiently small, one has*

$$\begin{aligned}\|u-u_h\|_{i,p} &\leq Ch^{4-i}\left[\|u\|_{6,p} + \|v\|_{6,p}\right],\\ \|v-v_h\|_{i,p} &\leq Ch^{4-i}\left[\|u\|_{6,p} + \|v\|_{6,p}\right], \quad i = 0,1,2.\end{aligned} \tag{5.57}$$

*Proof.* Assume temporarily that solutions $u_h$ and $v_h$ of (2.10) and (2.11), respectively, exist. Using (5.46) in (5.45), we obtain

$$\|e_h\|_{1,p} \leq C\left[h^3\|v\|_{6,p} + h^4\|u\|_{6,p} + h^2\left(h^2\|u\|_{6,p} + \|e_h\|_{1,p}\right)\right]. \tag{5.58}$$

For sufficiently small $h$, we have

$$\|e_h\|_{1,p} \leq C\left(h^3\|v\|_{6,p} + h^4\|u\|_{6,p}\right). \tag{5.59}$$

An application of the above in (5.46), we get

$$\|\varepsilon_h\|_{2,p} \leq C\left[h^2\|u\|_{6,p} + h^3\|v\|_{6,p}\right]. \tag{5.60}$$

Apply (5.59) in (5.56) to have

$$\|\varepsilon_h\|_{0,p} \leq C\left[h^4\|u\|_{6,p} + h^4\|v\|_{6,p}\right]. \tag{5.61}$$

Use (5.60) in (5.43) to get

$$\|e_h\|_{0,p} \leq C\left[h^4\|v\|_{6,p} + h^5\|u\|_{6,p}\right]. \tag{5.62}$$

Using (5.60) in (5.44), we obtain

$$\|e_h\|_{2,p} \le C\left[h^2\|v\|_{6,p} + h^4\|u\|_{6,p}\right]. \tag{5.63}$$

Using (5.61) and (5.60) in (5.39) with $e_h$ replaced by $\varepsilon_h$, we have

$$\|\varepsilon_h\|_{1,p} \le C\left[h^3\|u\|_{6,p} + h^3\|v\|_{6,p}\right]. \tag{5.64}$$

The required result can be obtained from estimates (5.59) to (5.64). □

So far we have assumed temporarily that solutions $u_h$ and $v_h$ exist. We now discuss the existence and uniqueness of discrete Petrov-Galerkin approximation. Since the matrix corresponding to (2.10) and (2.11) with zero boundary conditions for $u_h$ and $v_h$ is square, existence of $u_h \in \overset{0}{S}_{h,3}$ and $v_h \in \overset{0}{S}_{h,3}$ for any $f \in C^0(I)$ will follow from uniqueness, that is, from the property that the corresponding homogeneous equations have only trivial solutions.

Suppose that $u_h$ and $v_h$ corresponding to $u$ and $v$ satisfy

$$\begin{aligned} \left\langle u_h'' - \alpha v_h, \chi_h \right\rangle &= 0, \\ \left\langle v_h'' + b u_h, \chi_h \right\rangle &= 0, \quad \chi_h \in S_{h,1}. \end{aligned} \tag{5.65}$$

It follows from (5.61) and (5.62) (with $u$ replaced by 0 and eventually $v \equiv 0$) that, for sufficiently small $h$,

$$\|u_h\|_{0,p} \le 0, \qquad \|v_h\|_{0,p} \le 0, \tag{5.66}$$

and hence $u_h \equiv 0$ and $v_h \equiv 0$. Thus, uniqueness is proved, and hence existence follows from uniqueness.

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