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Research Article

QR-Submanifolds of $(p-1)$ QR-Dimension in a Quaternionic Projective Space $QP^{(n+p)/4}$ under Some Curvature Conditions

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The purpose of this paper is to study n-dimensional QR-submanifolds of $(p-1)$ QR-dimension in a quaternionic projective space $QP^{(n+p)/4}$ and especially to determine such submanifolds under some curvature conditions.

1. Introduction

Let M be a connected real n-dimensional submanifold of real codimension p of a quaternionic Kähler manifold $\overline{M}$ with quaternionic Kähler structure $\{F, G, H\}$. If there exists an r-dimensional normal distribution ν of the normal bundle $TM^{\perp}$ such that

$$\begin{aligned} F\nu_x \subset \nu_x, \qquad G\nu_x \subset \nu_x, \qquad H\nu_x \subset \nu_x, \\ F\nu_x^{\perp} \subset T_xM, \qquad G\nu_x^{\perp} \subset T_xM, \qquad H\nu_x^{\perp} \subset T_xM \end{aligned} \tag{1}$$

at each point x in M, then M is called a QR-*submanifold of* r QR-*dimension*, where $\nu^{\perp}$ denotes the complementary orthogonal distribution to ν in $TM^{\perp}$ (cf. [1–3]). Real hypersurfaces, which are typical examples of QR-submanifold with $r = 0$, have been investigated by many authors (cf. [2–9]) in connection with the shape operator and the induced almost contact 3-structure (for definition, see [10–13]). In their paper [2, 3], Kwon and Pak had studied QR-submanifolds of $(p-1)$ QR-dimension isometrically immersed in a quaternionic projective space $QP^{(n+p)/4}$ and proved the following theorem as a quaternionic analogy to theorems given in [14, 15], which are natural extensions of theorems proved in [6] to the case of QR-submanifolds with $(p-1)$ QR-dimension and also extensions of theorems in [16].

Theorem K-P. *Let M be an n-dimensional QR-submanifold of $(p-1)$ QR-dimension isometrically immersed in a quaternionic projective space $QP^{(n+p)/4}$, and let the normal vector field N_1 be parallel with respect to the normal connection. If the shape operator A_1 corresponding to N_1 satisfies*

$$A_1\phi = \phi A_1, \qquad A_1\psi = \psi A_1, \qquad A_1\theta = \theta A_1, \tag{2}$$

then $\pi^{-1}(M)$ is locally a product of $M_1 \times M_2$ where M_1 and M_2 belong to some $(4n_1+3)$- and $(4n_2+3)$-dimensional spheres (π is the Hopf fibration $S^{n+p+3}(1) \to QP^{(n+p)/4}$).

On the other hand, when M is a real hypersurface of $QP^{(n+p)/4}$, if $\pi^{-1}(M)$ is (1) an Einstein space or (2) a locally symmetric space, then $\pi^{-1}(M)$ has a parallel second fundamental form (cf. [4, 6, 7, 9]). Projecting the quantities on $\pi^{-1}(M)$ onto M in $QP^{(n+p)/4}$, we can consider QR-submanifolds of $(p-1)$ QR-dimension with the conditions corresponding to (1) or (2). In this paper, we will study such QR-submanifolds isometrically immersed in $QP^{(n+p)/4}$ and obtain Theorem 3 and other results stated in the last Section 5 as quaternionic analogies to theorems given in [16, 17] and as the extensions of theorems given in [18] by using Theorem K-P.

2. Preliminaries

Let $\overline{M}$ be a real $(n+p)$-dimensional quaternionic Kähler manifold. Then, by definition, there is a 3-dimensional vector bundle V consisting of tensor fields of type $(1,1)$ over $\overline{M}$ satisfying the following conditions (a), (b), and (c).

(a) In any coordinate neighborhood $\overline{\mathcal{U}}$, there is a local basis $\{F, G, H\}$ of V such that

$$F^2 = -I, \qquad G^2 = -I, \qquad H^2 = -I,$$
$$FG = -GF = H, \qquad GH = -HG = F, \tag{3}$$
$$HF = -FH = G.$$

(b) There is a Riemannian metric g which is Hermite with respect to all of F, G, and H.

(c) For the Riemannian connection $\overline{\nabla}$ with respect to g,

$$\begin{pmatrix} \overline{\nabla}F \\ \overline{\nabla}G \\ \overline{\nabla}H \end{pmatrix} = \begin{pmatrix} 0 & r & -q \\ -r & 0 & p \\ q & -p & 0 \end{pmatrix} \begin{pmatrix} F \\ G \\ H \end{pmatrix}, \tag{4}$$

where p, q, and r are local 1-forms defined in $\overline{\mathcal{U}}$. Such a local basis $\{F, G, H\}$ is called a *canonical local basis* of the bundle V in $\overline{\mathcal{U}}$ (cf. [10, 19, 20]).

For canonical local bases $\{F, G, H\}$ and $\{{}'F, {}'G, {}'H\}$ of V in coordinate neighborhoods $\overline{\mathcal{U}}$ and ${}'\overline{\mathcal{U}}$, it follows that in $\overline{\mathcal{U}} \cap {}'\overline{\mathcal{U}}$

$$\begin{pmatrix} {}'F \\ {}'G \\ {}'H \end{pmatrix} = (s_{xy}) \begin{pmatrix} F \\ G \\ H \end{pmatrix} \quad (x, y = 1, 2, 3), \tag{5}$$

where s_{xy} are local differentiable functions with $(s_{xy}) \in SO(3)$ as a consequence of (3). As is well known (cf. [19]), every quaternionic Kähler manifold is orientable.

Now let M be an n-dimensional QR-submanifold of $(p-1)$ QR-dimension isometrically immersed in $\overline{M}$. Then by definition, there is a unit normal vector field N such that $\nu_x^{\perp} = \text{Span}\{N\}$ at each point x in M. We set

$$U = -FN, \qquad V = -GN, \qquad W = -HN. \tag{6}$$

Denoting by $\mathscr{D}_x$ the maximal quaternionic invariant subspace $T_xM \cap FT_xM \cap GT_xM \cap HT_xM$ of T_xM, we have $\mathscr{D}_x^{\perp} = \text{Span}\{U, V, W\}$, where $\mathscr{D}_x^{\perp}$ means the complementary orthogonal subspace to $\mathscr{D}_x$ in T_xM (cf. [1–3]). Thus, we have

$$T_xM = \mathscr{D}_x \oplus \text{Span}\{U, V, W\}, \quad x \in M, \tag{7}$$

which together with (3) and (6) implies

$$FT_xM,\ GT_xM,\ HT_xM \subset T_xM \oplus \text{Span}\{N\}. \tag{8}$$

Therefore, for any tangent vector field X and for a local orthonormal basis $\{N_\alpha\}_{\alpha=1,\ldots,p}$ $(N_1 := N)$ of normal vectors to M, we have

$$FX = \phi X + u(X)N,$$
$$GX = \psi X + v(X)N, \tag{9}$$
$$HX = \theta X + w(X)N,$$

$$FN_\alpha = -U_\alpha + P_1 N_\alpha,$$
$$GN_\alpha = -V_\alpha + P_2 N_\alpha, \tag{10}$$
$$HN_\alpha = -W_\alpha + P_3 N_\alpha,$$

$(\alpha = 1, \ldots, p)$. Then it is easily seen that $\{\phi, \psi, \theta\}$ and $\{P_1, P_2, P_3\}$ are skew-symmetric endomorphisms acting on T_xM and $T_xM^{\perp}$, respectively. Moreover, the Hermitian property of $\{F, G, H\}$ implies

$$g(X, \phi U_\alpha) = -u(X)\, g(N_1, P_1 N_\alpha),$$
$$g(X, \psi V_\alpha) = -v(X)\, g(N_1, P_2 N_\alpha), \quad \alpha = 1, \ldots, p, \tag{11}$$
$$g(X, \theta W_\alpha) = -w(X)\, g(N_1, P_3 N_\alpha),$$

$$g(U_\alpha, U_\beta) = \delta_{\alpha\beta} - g(P_1 N_\alpha, P_1 N_\beta),$$
$$g(V_\alpha, V_\beta) = \delta_{\alpha\beta} - g(P_2 N_\alpha, P_2 N_\beta), \quad \alpha, \beta = 1, \ldots, p, \tag{12}$$
$$g(W_\alpha, W_\beta) = \delta_{\alpha\beta} - g(P_3 N_\alpha, P_3 N_\beta).$$

Also, from the hermitian properties $g(FX, N_\alpha) = -g(X, FN_\alpha)$, $g(GX, N_\alpha) = -g(X, GN_\alpha)$, and $g(HX, N_\alpha) = -g(X, HN_\alpha)$, it follows that

$$g(X, U_\alpha) = u(X)\,\delta_{1\alpha}, \qquad g(X, V_\alpha) = v(X)\,\delta_{1\alpha},$$
$$g(X, W_\alpha) = w(X)\,\delta_{1\alpha}, \tag{13}$$

and hence,

$$g(U_1, X) = u(X), \qquad g(V_1, X) = v(X),$$
$$g(W_1, X) = w(X), \qquad U_\alpha = 0, \tag{14}$$
$$V_\alpha = 0, \qquad W_\alpha = 0, \quad \alpha = 2, \ldots, p.$$

On the other hand, comparing (6) and (10) with $\alpha = 1$, we have $U_1 = U$, $V_1 = V$, and $W_1 = W$, which together with (6) and (14) implies

$$g(U, X) = u(X), \qquad g(V, X) = v(X)$$
$$g(W, X) = w(X), \quad u(U) = 1, \quad v(V) = 1, \quad w(W) = 1. \tag{15}$$

In the sequel, we will use the notations U, V, and W instead of U_1, V_1, and W_1.

Next, applying F to the first equation of (9) and using (10), (14), and (15), we have

$$\phi^2 X = -X + u(X)U, \qquad u(X)P_1 N = -u(\phi X)N. \tag{16}$$

Similarly, we have

$$\phi^2 X = -X + u(X)U, \qquad \psi^2 X = -X + v(X)V, \qquad \theta^2 X = -X + w(X)W, \tag{17}$$

$$u(X)P_1 N = -u(\phi X)N, \qquad v(X)P_2 N = -v(\psi X)N, \qquad w(X)P_3 N = -w(\theta X)N, \tag{18}$$

from which, taking account of the skew symmetry of P_1, P_2, and P_3 and using (11), we also have

$$\begin{gathered} u(\phi X) = 0, \qquad v(\psi X) = 0, \qquad w(\theta X) = 0, \\ \phi U = 0, \quad \psi V = 0, \quad \theta W = 0, \quad P_1 N = 0, \\ P_2 N = 0, \qquad P_3 N = 0. \end{gathered} \tag{19}$$

So (10) can be rewritten in the form

$$\begin{gathered} FN = -U, \qquad GN = -V, \qquad HN = -W, \\ FN_\alpha = P_1 N_\alpha, \qquad GN_\alpha = P_2 N_\alpha, \qquad HN_\alpha = P_3 N_\alpha \end{gathered} \tag{20}$$

($\alpha = 2, \ldots, p$). Applying G and H to the first equation of (9) and using (3), (9), and (20), we have

$$\begin{gathered} \theta X + w(X)N = -\psi(\phi X) - v(\phi X)N + u(X)V, \\ \psi X + v(X)N = \theta(\phi X) + w(\phi X)N - u(X)W, \end{gathered} \tag{21}$$

and consequently,

$$\begin{gathered} \psi(\phi X) = -\theta X + u(X)V, \qquad v(\phi X) = -w(X), \\ \theta(\phi X) = \psi X + u(X)W, \quad w(\phi X) = v(X). \end{gathered} \tag{22}$$

Similarly, the other equations of (9) yield

$$\begin{gathered} \phi(\psi X) = \theta X + v(X)U, \qquad u(\psi X) = w(X), \\ \theta(\psi X) = -\phi X + v(X)W, \qquad w(\psi X) = -u(X), \\ \phi(\theta X) = -\psi X + w(X)U, \qquad u(\theta X) = -v(X), \\ \psi(\theta X) = \phi X + w(X)V, \qquad v(\theta X) = u(X). \end{gathered} \tag{23}$$

From the first three equations of (20), we also have

$$\begin{gathered} \psi U = -W, \qquad v(U) = 0, \qquad \theta U = V, \\ w(U) = 0, \qquad \phi V = W, \qquad u(V) = 0, \\ \theta V = -U, \qquad w(V) = 0, \qquad \phi W = -V, \\ u(W) = 0, \qquad \psi W = U, \qquad v(W) = 0. \end{gathered} \tag{24}$$

Equations (14)–(17), (19), and (22)–(24) tell us that M admits the so-called almost contact 3-structure and consequently $n = 4m + 3$ for some integer m (cf. [12]).

Now let ∇ be the Levi-Civita connection on M, and let $\nabla^\perp$ be the normal connection induced from $\overline{\nabla}$ in the normal bundle of M. Then Gauss and Weingarten formulae are given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{25}$$

$$\overline{\nabla}_X N_\alpha = -A_\alpha X + \nabla^\perp_X N_\alpha, \quad \alpha = 1, \ldots, p, \tag{26}$$

for X, Y tangent to M. Here h denotes the second fundamental form and A_α the shape operator corresponding to N_α. They are related by $h(X,Y) = \sum_{\alpha=1}^{p} g(A_\alpha X, Y)N_\alpha$. Furthermore, put

$$\nabla^\perp_X N_\alpha = \sum_{\beta=1}^{p} s_{\alpha\beta}(X)N_\beta, \tag{27}$$

where $(s_{\alpha\beta})$ is the skew-symmetric matrix of connection forms of $\nabla^\perp$.

Differentiating the first equation of (9) covariantly and using (4), (9), (10), (14) (25), and (26), we have

$$\begin{gathered} (\nabla_Y \phi)X = r(Y)\psi X - q(Y)\theta X + u(X)A_1 Y \\ - g(A_1 Y, X)U, \\ (\nabla_Y u)X = r(Y)v(X) - q(Y)w(X) + g(\phi A_1 Y, X). \end{gathered} \tag{28}$$

From the other equations of (9), we also have

$$\begin{gathered} (\nabla_Y \psi)X = -r(Y)\phi X + p(Y)\theta X + v(X)A_1 Y \\ - g(A_1 Y, X)V, \\ (\nabla_Y v)X = -r(Y)u(X) + p(Y)w(X) + g(\psi A_1 Y, X), \\ (\nabla_Y \theta)X = q(Y)\phi X - p(Y)\psi X + w(X)A_1 Y \\ - g(A_1 Y, X)W, \\ (\nabla_Y w)X = q(Y)u(X) - p(Y)v(X) + g(\theta A_1 Y, X). \end{gathered} \tag{29}$$

Next, differentiating the first equation of (20) covariantly and comparing the tangential and normal parts, we have

$$\begin{gathered} \nabla_Y U = r(Y)V - q(Y)W + \phi A_1 Y, \\ g(A_\alpha U, Y) = -\sum_{\beta=2}^{p} s_{1\beta}(Y)P_{1\beta\alpha}, \quad \alpha = 2, \ldots, p. \end{gathered} \tag{30}$$

From the other equations of (20), we have similarly

$$\begin{gathered} \nabla_Y V = -r(Y)U + p(Y)W + \psi A_1 Y, \\ g(A_\alpha V, Y) = -\sum_{\beta=2}^{p} s_{1\beta}(Y)P_{2\beta\alpha}, \quad \alpha = 2, \ldots, p, \\ \nabla_Y W = q(Y)U - p(Y)V + \theta A_1 Y, \\ g(A_\alpha W, Y) = -\sum_{\beta=2}^{p} s_{1\beta}(Y)P_{3\beta\alpha}, \quad \alpha = 2, \ldots, p. \end{gathered} \tag{31}$$

Finally the equation of Gauss is given as follows (cf. [21]):

$$\begin{aligned} g\left(\overline{R}(X,Y)Z,W\right) &= g(R(X,Y)Z,W) \\ &\quad + \sum_{\alpha}\{g(A_{\alpha}X,Z)\,g(A_{\alpha}Y,W) \\ &\qquad - g(A_{\alpha}Y,Z)\,g(A_{\alpha}X,W)\}, \end{aligned} \tag{32}$$

for X, Y, and Z tangent to M, where $\overline{R}$ and R denote the Riemannian curvature tensor of $\overline{M}$ and M, respectively.

In the rest of this paper *we assume that the distinguished normal vector field* $N_1 := N$ *is parallel with respect to the normal connection* $\nabla^{\perp}$. Then it follows from (27) that $s_{1\beta} = 0$, and consequently, (30)-(31) imply

$$A_{\alpha}U = 0, \qquad A_{\alpha}V = 0, \qquad A_{\alpha}W = 0, \quad \alpha = 2, \dots, p. \tag{33}$$

On the other hand, since the curvature tensor $\overline{R}$ of $QP^{(n+p)/4}$ is of the form

$$\begin{aligned} \overline{R}\left(\overline{X},\overline{Y}\right)\overline{Z} &= g\left(\overline{Y},\overline{Z}\right)\overline{X} - g\left(\overline{X},\overline{Z}\right)\overline{Y} \\ &\quad + g\left(F\overline{Y},\overline{Z}\right)F\overline{X} - g\left(F\overline{X},\overline{Z}\right)F\overline{Y} \\ &\quad - 2g\left(F\overline{X},\overline{Y}\right)F\overline{Z} + g\left(G\overline{Y},\overline{Z}\right)G\overline{X} \\ &\quad - g\left(G\overline{X},\overline{Z}\right)G\overline{Y} - 2g\left(G\overline{X},\overline{Y}\right)G\overline{Z} \\ &\quad + g\left(H\overline{Y},\overline{Z}\right)H\overline{X} - g\left(H\overline{X},\overline{Z}\right)H\overline{Y} \\ &\quad - 2g\left(H\overline{X},\overline{Y}\right)H\overline{Z} \end{aligned} \tag{34}$$

for $\overline{X}, \overline{Y}$, and $\overline{Z}$ tangent to $QP^{(n+p)/4}$, (32) reduces to

$$\begin{aligned} R(X,Y)Z &= g(Y,Z)X - g(X,Z)Y \\ &\quad + g(\phi Y,Z)\phi X - g(\phi X,Z)\phi Y \\ &\quad - 2g(\phi X,Y)\phi Z + g(\psi Y,Z)\psi X \\ &\quad - g(\psi X,Z)\psi Y - 2g(\psi X,Y)\psi Z \\ &\quad + g(\theta Y,Z)\theta X - g(\theta X,Z)\theta Y \\ &\quad - 2g(\theta X,Y)\theta Z \\ &\quad + \sum_{\alpha}\{g(A_{\alpha}Y,Z)A_{\alpha}X - g(A_{\alpha}X,Z)A_{\alpha}Y\}. \end{aligned} \tag{35}$$

3. Fibrations and Immersions

From now on n-dimensional QR-submanifolds of $(p-1)$ QR-dimension isometrically immersed in $QP^{(n+p)/4}$ only will be considered. Moreover, we will use the assumption and the notations as in Section 2.

Let $S^{n+p+3}(a)$ be the hypersphere of radius a (>0) in $Q^{(n+p+4)/4}$ the quaternionic space of quaternionic dimension $(n+p+4)/4$, which is identified with the Euclidean $(n+p+4)$-space $\mathbb{R}^{n+p+4}$. The unit sphere $S^{n+p+3}(1)$ will be briefly denoted by S^{n+p+3}. Let $\tilde{\pi} : S^{n+p+3} \to QP^{(n+p)/4}$ be the natural projection of S^{n+p+3} onto $QP^{(n+p)/4}$ defined by the Hopf fibration $S^3 \to S^{n+p+3} \to QP^{(n+p)/4}$. As is well known (cf. [10, 11, 20]), S^{n+p+3} admits a Sasakian 3-structure whereby $\tilde{\xi}, \tilde{\eta}$, and $\tilde{\zeta}$ are mutually orthogonal unit Killing vector fields. Thus it follows that

$$\begin{gathered} \tilde{\nabla}_{\tilde{\xi}}\tilde{\xi} = 0, \qquad \tilde{\nabla}_{\tilde{\eta}}\tilde{\eta} = 0, \qquad \tilde{\nabla}_{\tilde{\zeta}}\tilde{\zeta} = 0, \\ \tilde{\nabla}_{\tilde{\zeta}}\tilde{\eta} = -\tilde{\nabla}_{\tilde{\eta}}\tilde{\zeta} = \tilde{\xi}, \qquad \tilde{\nabla}_{\tilde{\xi}}\tilde{\zeta} = -\tilde{\nabla}_{\tilde{\zeta}}\tilde{\xi} = \tilde{\eta}, \\ \tilde{\nabla}_{\tilde{\eta}}\tilde{\xi} = -\tilde{\nabla}_{\tilde{\xi}}\tilde{\eta} = \tilde{\zeta}, \end{gathered} \tag{36}$$

where $\tilde{\nabla}$ denotes the Riemannian connection with respect to the canonical metric $\tilde{g}$ on S^{n+p+3} (cf. [6, 9–13]). Moreover, each fibre $\tilde{\pi}^{-1}(x)$ of x in $QP^{(n+p)/4}$ is a maximal integral submanifold of the distribution spanned by $\tilde{\xi}$, $\tilde{\eta}$, and $\tilde{\zeta}$. Thus the base space $QP^{(n+p)/4}$ admits the induced quaternionic Kähler structure of constant Q-sectional curvature 4 (cf. [10, 11]). We have especially a fibration $\pi : \pi^{-1}(M) \to M$ which is compatible with the Hopf fibration $\tilde{\pi}$. More precisely speaking, $\pi : \pi^{-1}(M) \to M$ is a fibration with totally geodesic fibers such that the following diagram is commutative:

$$\begin{array}{ccc} \pi^{-1}(M) & \xrightarrow{\;i'\;} & S^{n+p+3} \\ {\scriptstyle \pi}\big\downarrow & & \big\downarrow{\scriptstyle \tilde{\pi}} \\ M & \xrightarrow{\;i\;} & QP^{(n+p)/4} \end{array} \tag{37}$$

where $i' : \pi^{-1}(M) \to S^{n+p+3}$ and $i : M \to QP^{(n+p)/4}$ are isometric immersions.

Now, let ξ, η, and ζ be the unit vector fields tangent to the fibers of $\pi^{-1}(M)$ such that $i'_*\xi = \tilde{\xi}$, $i'_*\eta = \tilde{\eta}$, and $i'_*\zeta = \tilde{\zeta}$. (In what follows, we will again delete the i' and i'_* in our notation.) Furthermore, we denote by X^* the horizontal lift of a vector field X tangent to M. Then, the horizontal lifts N^*_{α} $(\alpha = 1, \dots, p)$ of the normal vectors N_{α} to M form an orthonormal basis of normal vectors to $\pi^{-1}(M)$ in S^{n+p+3}. Let A'_{α} and $s'_{\alpha\beta}$ be the corresponding shape operators and normal connection forms, respectively. Then, as shown in [3, 9, 22], the fundamental equations for the submersion π are given by

$$\begin{aligned} {}'\nabla_{X^*}Y^* &= (\nabla_X Y)^* + g'\left((\phi X)^*, Y^*\right)\xi + g'\left((\psi X)^*, Y^*\right)\eta \\ &\quad + g'\left((\theta X)^*, Y^*\right)\zeta, \end{aligned} \tag{38}$$

$$[X^*, Y^*] = [X,Y]^* + 2g'\left((\phi X)^*, Y^*\right)\xi + 2g'\left((\psi X)^*, Y^*\right)\eta + 2g'\left((\theta X)^*, Y^*\right)\zeta, \quad (39)$$

$$\begin{aligned} {}'\nabla_{X^*}\xi &= {}'\nabla_{\xi}X^* = -(\phi X)^*, \\ {}'\nabla_{X^*}\eta &= {}'\nabla_{\eta}X^* = -(\psi X)^*, \\ {}'\nabla_{X^*}\zeta &= {}'\nabla_{\zeta}X^* = -(\theta X)^*, \end{aligned} \quad (40)$$

$$[X^*, \xi] = 0, \qquad [X^*, \eta] = 0, \qquad [X^*, \zeta] = 0, \quad (41)$$

where g' denotes the Riemannian metric of $\pi^{-1}(M)$ induced from $\tilde{g}$ in S^{n+p+3} and ${}'\nabla$ the Levi-Civita connection with respect to g'. The same equations are valid for the submersion $\tilde{\pi}$ by replacing ϕ, ψ, and θ (resp., ξ, η, and ζ) with F, G, and H (resp., $\tilde{\xi}$, $\tilde{\eta}$, and $\tilde{\zeta}$), respectively. We denote by ${}'\nabla^{\perp}$ the normal connection of $\pi^{-1}(M)$ induced from $\tilde{\nabla}$. Since the diagram is commutative, $\tilde{\nabla}_{X^*}N^*_{\alpha}$ implies

$$\begin{aligned} {}'\nabla^{\perp}_{X^*}N^*_{\alpha} - A'_{\alpha}X^* &= \left(\overline{\nabla}_X N_{\alpha}\right)^* + \tilde{g}\left((FX)^*, N^*_{\alpha}\right)\tilde{\xi} \\ &\quad + \tilde{g}\left((GX)^*, N^*_{\alpha}\right)\tilde{\eta} + \tilde{g}\left((HX)^*, N^*_{\alpha}\right)\tilde{\zeta} \\ &= -\left(A_{\alpha}X\right)^* + g(U_{\alpha}, X)^*\xi + g(V_{\alpha}, X)^*\eta \\ &\quad + g(W_{\alpha}, X)^*\zeta + (\nabla^{\perp}_X N_{\alpha})^* \end{aligned} \quad (42)$$

because of (10), (26), and (38), from which, comparing the tangential part, we have

$$A'_{\alpha}X^* = (A_{\alpha}X)^* - g(U_{\alpha}, X)^*\xi - g(V_{\alpha}, X)^*\eta - g(W_{\alpha}, X)^*\zeta. \quad (43)$$

Next, calculating $\tilde{\nabla}_{\xi}N^*_{\alpha}$ and using (10), (26), and (40), we have

$${}'\nabla^{\perp}_{\xi}N^*_{\alpha} - A'_{\alpha}\xi = -(FN_{\alpha})^* = U^*_{\alpha} - (P_1N_{\alpha})^*, \quad (44)$$

which yields

$$A'_{\alpha}\xi = -U^*_{\alpha} \quad (45)$$

and similarly

$$A'_{\alpha}\xi = -U^*_{\alpha}, \qquad A'_{\alpha}\eta = -V^*_{\alpha}, \qquad A'_{\alpha}\zeta = -W^*_{\alpha}. \quad (46)$$

Hence, (43) and (46) with $\alpha = 1$ imply

$$\begin{aligned} A'_1X^* &= (A_1X)^* - g(U, X)^*\xi - g(V, X)^*\eta - g(W, X)^*\zeta, \\ A'_1\xi &= -U^*, \qquad A'_1\eta = -V^*, \qquad A'_1\zeta = -W^*. \end{aligned} \quad (47)$$

4. Co-Gauss Equations for the Submersion $\pi:\pi^{-1}(M) \rightarrow M$

In this section, we derive the co-Gauss and co-Codazzi equations of the submersion $\pi : \pi^{-1}(M) \rightarrow M$ for later use.

Differentiating (38) with $Y = U$ covariantly along $\pi^{-1}(M)$ and using (24), (38), and (39), we have

$$\begin{aligned} &{}'\nabla_{Y^*}{}'\nabla_{X^*}U^* \\ &\quad = (\nabla_Y\nabla_X U)^* + \{v(X)\theta Y - w(X)\psi Y\}^* \\ &\qquad + g(\phi Y, \nabla_X U)^*\xi \\ &\qquad + \{g(\psi Y, \nabla_X U) + g(\nabla_Y X, W) + g(X, \nabla_Y W)\}^*\eta \\ &\qquad + \{g(\theta Y, \nabla_X U) - g(\nabla_Y X, V) - g(X, \nabla_Y V)\}^*\zeta. \end{aligned} \quad (48)$$

Similarly (38) with $Y = V$ and (38) with $Y = W$ give

$$\begin{aligned} &{}'\nabla_{Y^*}{}'\nabla_{X^*}V^* \\ &\quad = (\nabla_Y\nabla_X V)^* + \{w(X)\phi Y - u(X)\theta Y\}^* \\ &\qquad + \{g(\phi Y, \nabla_X V) - g(\nabla_Y X, W) - g(X, \nabla_Y W)\}^*\xi \\ &\qquad + g(\psi Y, \nabla_X V)^*\eta \\ &\qquad + \{g(\theta Y, \nabla_X V) + g(\nabla_Y X, U) + g(X, \nabla_Y U)\}^*\zeta, \end{aligned} \quad (49)$$

$$\begin{aligned} &{}'\nabla_{Y^*}{}'\nabla_{X^*}W^* \\ &\quad = (\nabla_Y\nabla_X W)^* \\ &\qquad - \{v(X)\phi Y - u(X)\psi Y\}^* \\ &\qquad + \{g(\phi Y, \nabla_X W) + g(\nabla_Y X, V) + g(X, \nabla_Y V)\}^*\xi \\ &\qquad + \{g(\psi Y, \nabla_X W) - g(\nabla_Y X, U) - g(X, \nabla_Y U)\}^*\eta \\ &\qquad + g(\theta Y, \nabla_X W)^*\zeta, \end{aligned} \quad (50)$$

respectively. On the other hand, it follows from (19), (24), (38), and (39) that

$$\begin{aligned} {}'\nabla_{[Y^*, X^*]}U^* &= (\nabla_{[Y,X]}U)^* + 2g(\psi Y, X)^*W^* \\ &\quad - 2g(\theta Y, X)^*V^* + g([Y,X], W)^*\eta \\ &\quad - g([Y,X], V)^*\zeta, \end{aligned} \quad (51)$$

$$'\nabla_{[Y^*,X^*]}V^* = (\nabla_{[Y,X]}V)^* - 2g(\phi Y, X)^* W^* + 2g(\theta Y, X)^* U^* - g([Y,X], W)^* \xi + g([Y,X], U)^* \zeta, \quad (52)$$

$$'\nabla_{[Y^*,X^*]}W^* = (\nabla_{[Y,X]}W)^* + 2g(\phi Y, X)^* V^* - 2g(\psi Y, X)^* U^* + g([Y,X], V)^* \xi - g([Y,X], U)^* \eta. \quad (53)$$

By means of (48) and (51), we have

$$\begin{aligned} 'R(Y^*, X^*) U^* = {} & \{R(Y,X)U\}^* \\ & + \{w(Y)\psi X - w(X)\psi Y - v(Y)\theta X \\ & + v(X)\theta Y + 2g(\theta Y, X)V \\ & - 2g(\psi Y, X)W\}^* \\ & + \{g(\phi Y, \nabla_X U) - g(\phi X, \nabla_Y U)\}^* \xi \\ & + \{g(\psi Y, \nabla_X U) - g(\psi X, \nabla_Y U) \\ & + g(X, \nabla_Y W) - g(Y, \nabla_X W)\}^* \eta \\ & + \{g(\theta Y, \nabla_X U) - g(\theta X, \nabla_Y U) \\ & - g(X, \nabla_Y V) + g(Y, \nabla_X V)\}^* \zeta, \end{aligned} \quad (54)$$

where $'R$ denotes the curvature tensor of $\pi^{-1}(M)$ with respect to the connection $'\nabla$. Using (30), (31), (33), and (35), we can easily see that

$$\begin{aligned} 'R(Y^*, X^*) U^* = {} & \{u(X)Y - u(Y)X \\ & + u(A_1X)A_1Y - u(A_1Y)A_1X\}^* \\ & + \{r(Y)w(X) - r(X)w(Y) \\ & + q(Y)v(X) - q(X)v(Y) \\ & + u(X)u(A_1Y) - u(Y)u(A_1X)\}^* \xi \\ & + \{p(X)v(Y) - p(Y)v(X) \\ & + v(X)u(A_1Y) - v(Y)u(A_1X)\}^* \eta \\ & + \{p(X)w(Y) - p(Y)w(X) \\ & + w(X)u(A_1Y) - w(Y)u(A_1X)\}^* \zeta. \end{aligned} \quad (55)$$

By the same method, we can easily verify that (49), (50), (52), and (53) yield

$$\begin{aligned} 'R(Y^*, X^*) V^* = {} & \{v(X)Y - v(Y)X + v(A_1X)A_1Y \\ & - v(A_1Y)A_1X\}^* \\ & + \{q(X)u(Y) - q(Y)u(X) \\ & - u(Y)v(A_1X) + u(X)v(A_1Y)\}^* \xi \\ & + \{r(Y)w(X) - r(X)w(Y) \\ & + p(Y)u(X) - p(X)u(Y) \\ & + v(X)v(A_1Y) - v(Y)v(A_1X)\}^* \eta \\ & + \{q(X)w(Y) - q(Y)w(X) \\ & - w(Y)v(A_1X) + w(X)v(A_1Y)\}^* \zeta, \\ 'R(Y^*, X^*) W^* = {} & \{w(X)Y - w(Y)X + w(A_1X)A_1Y \\ & - w(A_1Y)A_1X\}^* \\ & + \{r(X)u(Y) - r(Y)u(X) \\ & - u(Y)w(A_1X) + u(X)w(A_1Y)\}^* \xi \\ & + \{r(X)v(Y) - r(Y)v(X) \\ & - v(Y)w(A_1X) + v(X)w(A_1Y)\}^* \eta \\ & + \{q(Y)v(X) - q(X)v(Y) \\ & + p(Y)u(X) - p(X)u(Y) \\ & + w(X)w(A_1Y) - w(Y)w(A_1X)\}^* \zeta. \end{aligned} \quad (56)$$

Differentiating (38) with $X = U$ covariantly along $\pi^{-1}(M)$ and using (24), we have

$$\begin{aligned} & '\nabla_{Y^*} {}'\nabla_{U^*} X^* \\ & = (\nabla_Y \nabla_U X)^* + \{w(X)\psi Y - v(X)\theta Y\}^* \\ & \quad + g(\phi Y, \nabla_U X)^* \xi \\ & \quad + \{g(\psi Y, \nabla_U X) - g(\nabla_Y W, X) - g(W, \nabla_Y X)\}^* \eta \\ & \quad + \{g(\theta Y, \nabla_U X) + g(\nabla_Y V, X) + g(V, \nabla_Y X)\}^* \zeta. \end{aligned} \quad (57)$$

Similarly, (38) with $X = V$ and (38) with $X = W$, respectively, give

$$\begin{aligned}
&{}'\nabla_{Y^*}{}'\nabla_{V^*}X^* \\
&\quad = (\nabla_Y\nabla_V X)^* - \{w(X)\phi Y - u(X)\theta Y\}^* \\
&\qquad + g(\psi Y, \nabla_V X)^*\eta \\
&\qquad + \{g(\phi Y, \nabla_V X) + g(\nabla_Y W, X) + g(W, \nabla_Y X)\}^*\xi \\
&\qquad + \{g(\theta Y, \nabla_V X) - g(\nabla_Y U, X) - g(U, \nabla_Y X)\}^*\zeta,
\end{aligned} \tag{58}$$

$$\begin{aligned}
&{}'\nabla_{Y^*}{}'\nabla_{W^*}X^* \\
&\quad = (\nabla_Y\nabla_W X)^* + \{v(X)\phi Y - u(X)\psi Y\}^* \\
&\qquad + g(\theta Y, \nabla_W X)^*\zeta \\
&\qquad + \{g(\phi Y, \nabla_W X) - g(\nabla_Y V, X) - g(V, \nabla_Y X)\}^*\xi \\
&\qquad + \{g(\psi Y, \nabla_W X) + g(\nabla_Y U, X) + g(U, \nabla_Y X)\}^*\eta.
\end{aligned} \tag{59}$$

Differentiating (38) also covariantly in the direction of U^* and using (24), we have

$$\begin{aligned}
&{}'\nabla_{U^*}{}'\nabla_{Y^*}X^* \\
&\quad = (\nabla_U\nabla_Y X)^* \\
&\qquad + g(\psi Y, X)^*W^* - g(\theta Y, X)^*V^* \\
&\qquad + \{g(\nabla_U(\phi Y), X) + g(\phi Y, \nabla_U X)\}^*\xi \\
&\qquad + \{g(\nabla_U(\psi Y), X) + g(\psi Y, \nabla_U X) - g(W, \nabla_Y X)\}^*\eta \\
&\qquad + \{g(\nabla_U(\theta Y), X) + g(\theta Y, \nabla_U X) + g(V, \nabla_Y X)\}^*\zeta.
\end{aligned} \tag{60}$$

Similarly, differentiating (38) covariantly in the direction of V^* and W^*, respectively, we have

$$\begin{aligned}
&{}'\nabla_{V^*}{}'\nabla_{Y^*}X^* \\
&\quad = (\nabla_V\nabla_Y X)^* \\
&\qquad - g(\phi Y, X)^*W^* - g(\theta Y, X)^*U^* \\
&\qquad + \{g(\nabla_V(\phi Y), X) + g(\phi Y, \nabla_V X) + g(W, \nabla_Y X)\}^*\xi \\
&\qquad + \{g(\nabla_V(\psi Y), X) + g(\psi Y, \nabla_V X)\}^*\eta \\
&\qquad + \{g(\nabla_V(\theta Y), X) + g(\theta Y, \nabla_V X) - g(U, \nabla_Y X)\}^*\zeta,
\end{aligned} \tag{61}$$

$$\begin{aligned}
&{}'\nabla_{W^*}{}'\nabla_{Y^*}X^* \\
&\quad = (\nabla_W\nabla_Y X)^* \\
&\qquad - g(\psi Y, X)^*U^* + g(\phi Y, X)^*V^* \\
&\qquad + \{g(\nabla_W(\phi Y), X) + g(\phi Y, \nabla_W X) - g(V, \nabla_Y X)\}^*\xi \\
&\qquad + \{g(\nabla_W(\psi Y), X) + g(\psi Y, \nabla_W X) + g(U, \nabla_Y X)\}^*\eta \\
&\qquad + \{g(\nabla_W(\theta Y), X) + g(\theta Y, \nabla_W X)\}^*\zeta.
\end{aligned} \tag{62}$$

On the other hand, (38) and (39) with $X = U$ imply

$$\begin{aligned}
{}'\nabla_{[Y^*,U^*]}X^* &= (\nabla_{[Y,U]}X)^* - 2\{w(Y)\psi X - v(Y)\theta X\}^* \\
&\quad + g(\phi[Y,U], X)^*\xi + g(\psi[Y,U], X)^*\eta \\
&\quad + g(\theta[Y,U], X)^*\zeta.
\end{aligned} \tag{63}$$

Similarly, from (39) with $X = V$ and (39) with $X = W$, respectively, we find that

$$\begin{aligned}
{}'\nabla_{[Y^*,V^*]}X^* &= (\nabla_{[Y,V]}X)^* + 2\{w(Y)\phi X - u(Y)\theta X\}^* \\
&\quad + g(\phi[Y,V], X)^*\xi + g(\psi[Y,V], X)^*\eta \\
&\quad + g(\theta[Y,V], X)^*\zeta,
\end{aligned} \tag{64}$$

$$\begin{aligned}
{}'\nabla_{[Y^*,W^*]}X^* &= (\nabla_{[Y,W]}X)^* - 2\{v(Y)\phi X - u(Y)\psi X\}^* \\
&\quad + g(\phi[Y,W], X)^*\xi + g(\psi[Y,W], X)^*\eta \\
&\quad + g(\theta[Y,W], X)^*\zeta.
\end{aligned} \tag{65}$$

Using (28)–(31), it follows from (57), (60), and (63) that

$$\begin{aligned}
{}'R(Y^*, U^*)X^* &= \{R(Y,U)X\}^* \\
&\quad + \{w(X)\psi Y - v(X)\theta Y \\
&\qquad - g(\psi Y, X)W + g(\theta Y, X)V \\
&\qquad + 2w(Y)\psi X - 2v(Y)\theta X\}^* \\
&\quad - \{r(U)g(\psi Y, X) - q(U)g(\theta Y, X) \\
&\qquad + u(Y)u(A_1X) + r(Y)w(X) \\
&\qquad + q(Y)v(X) - g(A_1Y, X)\}^*\xi \\
&\quad - \{-p(Y)v(X) - r(U)g(\phi Y, X) \\
&\qquad + p(U)g(\theta Y, X) + v(Y)u(A_1X)\}^*\eta \\
&\quad - \{-p(Y)w(X) + q(U)g(\phi Y, X) \\
&\qquad - p(U)g(\psi Y, X) + w(Y)u(A_1X)\}^*\zeta,
\end{aligned} \tag{66}$$

from which, taking account of (35) and using (24) and (33), we obtain

$$\begin{aligned}{}'R(Y^*,U^*)X^* = {} & \{u(X)Y - g(Y,X)U \\ & + u(A_1X)A_1Y - g(A_1Y,X)A_1U\}^* \\ & - \{r(U)g(\psi Y,X) - q(U)g(\theta Y,X) \\ & + u(Y)u(A_1X) + r(Y)w(X) \\ & + q(Y)v(X) - g(A_1Y,X)\}^*\xi \\ & - \{-p(Y)v(X) - r(U)g(\phi Y,X) \\ & + p(U)g(\theta Y,X) + v(Y)u(A_1X)\}^*\eta \\ & - \{-p(Y)w(X) + q(U)g(\phi Y,X) \\ & - p(U)g(\psi Y,X) + w(Y)u(A_1X)\}^*\zeta. \end{aligned} \tag{67}$$

Similarly, by using (58), (59), (61), (62), (64), and (65), we can easily obtain

$$\begin{aligned}{}'R(Y^*,V^*)X^* = {} & \{v(X)Y - g(Y,X)V \\ & + v(A_1X)A_1Y - g(A_1Y,X)A_1V\}^* \\ & - \{-r(V)g(\phi Y,X) + p(V)g(\theta Y,X) \\ & + v(Y)v(A_1X) + r(Y)w(X) \\ & + p(Y)u(X) - g(A_1Y,X)\}^*\eta \\ & + \{q(Y)u(X) - r(V)g(\psi Y,X) \\ & + q(V)g(\theta Y,X) - u(Y)v(A_1X)\}^*\xi \\ & - \{-q(Y)w(X) + q(V)g(\phi Y,X) \\ & - p(V)g(\psi Y,X) + w(Y)v(A_1X)\}^*\zeta, \\ {}'R(Y^*,W^*)X^* = {} & \{w(X)Y - g(Y,X)W \\ & + w(A_1X)A_1Y - g(A_1Y,X)A_1W\}^* \\ & - \{q(W)g(\phi Y,X) - p(W)g(\psi Y,X) \\ & + w(Y)w(A_1X) + q(Y)v(X) \\ & + p(Y)u(X) - g(A_1Y,X)\}^*\zeta \\ & - \{-r(Y)u(X) + r(W)g(\psi Y,X) \\ & - q(W)g(\theta Y,X) + u(Y)w(A_1X)\}^*\xi \\ & + \{r(Y)v(X) + r(W)g(\phi Y,X) \\ & - p(W)g(\theta Y,X) - v(Y)w(A_1X)\}^*\eta. \end{aligned} \tag{68}$$

5. Main Results

It is well known [3] that if $\pi^{-1}(M)$ is locally symmetric then ${}'\nabla A_1 = 0$ which implies identities (2) in Theorem K-P. In this point of view, we consider the following assumptions in (69) which are weaker conditions than the locally symmetry of $\pi^{-1}(M)$.

In order to obtain our main results, let M be n-dimensional QR-submanifolds of $(p-1)$ QR-dimension in $QP^{(n+p)/4}$ with the assumptions

$$\begin{gathered} ({}'\nabla_\xi {}'R)(Y^*,X^*)U^* = 0, \qquad ({}'\nabla_\eta {}'R)(Y^*,X^*)V^* = 0, \\ ({}'\nabla_\zeta {}'R)(Y^*,X^*)W^* = 0, \\ ({}'\nabla_\xi {}'R)(Y^*,U^*)X^* = 0, \qquad ({}'\nabla_\eta {}'R)(Y^*,V^*)X^* = 0, \\ ({}'\nabla_\zeta {}'R)(Y^*,W^*)X^* = 0. \end{gathered} \tag{69}$$

We notice here that the above curvature conditions in (69) are different from those in [18] due to Pak and Sohn.

We first consider the assumption

$$({}'\nabla_\xi {}'R)(Y^*,X^*)U^* = 0. \tag{70}$$

Differentiating (55) covariantly in the direction of ξ and using (19), (36), and (40), and the assumption $({}'\nabla_\xi {}'R)(Y^*,X^*)U^* = 0$, we have

$$\begin{aligned} & -{}'R((\phi Y)^*,X^*)U^* - {}'R(Y^*,(\phi X)^*)U^* \\ & \quad = \{-u(X)\phi Y + u(Y)\phi X \\ & \qquad - u(A_1X)\phi A_1Y + u(A_1Y)\phi A_1X\}^* \\ & \qquad + \{p(X)w(Y) - p(Y)w(X) \\ & \qquad + w(X)u(A_1Y) - w(Y)u(A_1X)\}^*\eta \\ & \qquad - \{p(X)v(Y) - p(Y)v(X) \\ & \qquad + v(X)u(A_1Y) - v(Y)u(A_1X)\}^*\zeta, \end{aligned} \tag{71}$$

from which, taking the vertical component of ξ, η, and ζ, respectively, and using (22)–(24) and (55) itself, we can get

$$\begin{aligned} & r(X)v(Y) - r(Y)v(X) - q(X)w(Y) \\ & \quad + q(Y)w(X) - r(\phi Y)w(X) + r(\phi X)w(Y) \\ & \quad - q(\phi Y)v(X) + q(\phi X)v(Y) \\ & \quad - u(X)u(A_1\phi Y) + u(Y)u(A_1\phi X) = 0, \end{aligned} \tag{72}$$

$$\begin{aligned} & -u(A_1\phi Y)v(X) + u(A_1\phi X)v(Y) + p(\phi Y)v(X) \\ & \quad - p(\phi X)v(Y) = 0, \end{aligned} \tag{73}$$

$$\begin{aligned} & -u(A_1\phi Y)w(X) + u(A_1\phi X)w(Y) + p(\phi Y)w(X) \\ & \quad - p(\phi X)w(Y) = 0. \end{aligned} \tag{74}$$

Putting $Y = U$ in (72) and using (19) and (24), we have

$$\phi A_1 U + r(U) V - q(U) W = 0, \tag{75}$$

and consequently,

$$r(U) = w(A_1 U) = u(A_1 W), \quad q(U) = v(A_1 U) = u(A_1 V). \tag{76}$$

Putting $Y = W$ and $X = V$ in (73) and using (15) and (24) yield

$$p(V) = v(A_1 U) = u(A_1 V). \tag{77}$$

Also, putting $Y = V$ and $X = W$ in (74) and using (15) and (24), we have

$$p(W) = w(A_1 U) = u(A_1 W). \tag{78}$$

Summing up, we have

$$\begin{aligned} A_1 U &= u(A_1 U) U + p(V) V + p(W) W, \\ p(V) &= v(A_1 U) = u(A_1 V) = q(U), \\ p(W) &= w(A_1 U) = u(A_1 W) = r(U). \end{aligned} \tag{79}$$

Thus we get the following lemma.

Lemma 1. *Let M be an n-dimensional QR-submanifold of $(p-1)$ QR-dimension in a quaternionic projective space $QP^{(n+p)/4}$, and let the normal vector field N_1 be parallel with respect to the normal connection. If the equalities in (69) are established, then*

$$\begin{aligned} A_1 U &= u(A_1 U) U + p(V) V + p(W) W, \\ A_1 V &= q(U) U + v(A_1 V) V + q(W) W, \\ A_1 W &= r(U) U + r(V) V + w(A_1 W) W, \\ p(V) &= v(A_1 U) = u(A_1 V) = q(U), \\ p(W) &= w(A_1 U) = u(A_1 W) = r(U), \\ q(W) &= w(A_1 V) = v(A_1 W) = r(V). \end{aligned} \tag{80}$$

Next, we assume the additional condition

$$({}'\nabla_\xi\, {}'R)(Y^*, U^*) X^* = 0. \tag{81}$$

Differentiating (67) covariantly in the direction of ξ and using (36), (40) and the assumption $({}'\nabla_\xi\, {}'R)\,(Y^*, U^*)X^* = 0$, we have

$$\begin{aligned} &- {}'R((\phi Y)^*, U^*) X^* - {}'R(Y^*, U^*)(\phi X)^* \\ &= \{-u(X)\phi Y - u(A_1 X)\phi A_1 Y \\ &\quad + g(A_1 Y, X)\phi A_1 U\}^* \\ &\quad - \{-p(Y) w(X) + q(U) g(\phi Y, X) \\ &\quad - p(U) g(\psi Y, X) + w(Y) u(A_1 X)\}^* \eta \\ &\quad + \{-p(Y) v(X) - r(U) g(\phi Y, X) \\ &\quad + p(U) g(\theta Y, X) + v(Y) u(A_1 X)\}^* \zeta, \end{aligned} \tag{82}$$

from which, taking the vertical component of ξ, η, and ζ, respectively, and using (22)–(24) and (67) itself, we can find

$$\begin{aligned} &r(U)\{-2g(\theta Y, X) + u(Y) v(X) - v(Y) u(X)\} \\ &\quad - q(U)\{2g(\psi Y, X) + u(Y) w(X) - w(Y) u(X)\} \\ &\quad + u(Y) u(A_1 \phi X) + r(\phi Y) w(X) + r(Y) v(X) \\ &\quad + q(\phi Y) v(X) - q(Y) w(X) \\ &\quad + g(\phi A_1 Y - A_1 \phi Y, X) = 0, \end{aligned} \tag{83}$$

$$\begin{aligned} &- q(U) g(\phi Y, X) + p(\phi Y) v(X) - v(Y) u(A_1 \phi X) \\ &\quad - p(U)\{g(\psi Y, X) + u(Y) w(X) \\ &\quad - w(Y) u(X)\} = 0, \end{aligned} \tag{84}$$

$$\begin{aligned} &- r(U) g(\phi Y, X) + p(\phi Y) w(X) - w(Y) u(A_1 \phi X) \\ &\quad - p(U)\{g(\theta Y, X) + v(Y) u(X) - u(Y) v(X)\} = 0. \end{aligned} \tag{85}$$

Taking the skew-symmetric part of (83) with respect to X and Y, we have

$$\begin{aligned} &2r(U)\{-2g(\theta Y, X) + u(Y) v(X) - v(Y) u(X)\} \\ &\quad - 2q(U)\{2g(\psi Y, X) + u(Y) w(X) - w(Y) u(X)\} \\ &\quad + u(Y) u(A_1 \phi X) - u(X) u(A_1 \phi Y) + r(\phi Y) w(X) \\ &\quad - r(\phi X) w(Y) + r(Y) v(X) - r(X) v(Y) \\ &\quad + q(\phi Y) v(X) - q(\phi X) v(Y) \\ &\quad - q(Y) w(X) + q(X) w(Y) = 0. \end{aligned} \tag{86}$$

Replacing Y with θY in (86) and using (19) and (22)–(24), we have

$$\begin{aligned} &r(U)\{4g(Y, X) - 3w(Y) w(X) \\ &\quad - 2u(Y) u(X) - 2v(Y) v(X)\} \\ &\quad - q(U)\{4g(\phi Y, X) + 3w(Y) v(X) - 2w(X) v(Y)\} \\ &\quad - q(\theta Y) w(X) - q(\psi Y) v(X) - q(\phi X) u(Y) \\ &\quad + r(\theta Y) v(X) - r(X) u(Y) - r(\psi Y) w(X) \\ &\quad - v(Y) u(A_1 \phi X) + u(X) u(A_1 \psi Y) \\ &\quad - u(A_1 U) u(X) w(Y) = 0. \end{aligned} \tag{87}$$

Now we consider the following orthonormal basis:

$$\begin{aligned} &\{U, V, W, e_1, \ldots, e_m, \phi(e_1), \ldots, \phi(e_m), \\ &\quad \psi(e_1), \ldots, \psi(e_m), \theta(e_1), \ldots, \theta(e_m)\}, \end{aligned} \tag{88}$$

which will be called Q-basis, where $4m + 3 = \dim M$. Taking the trace of the above equation with respect to the Q-basis and using (76), we can easily see $4mr(U) = 0$; that is,

$$r(U) = 0. \tag{89}$$

Replacing also Y with ψY in (86) and using (19) and (22)–(24), we have

$$q(U) = 0. \tag{90}$$

Substituting (89) and (90) into (75), we have $\phi A_1 U = 0$ and hence

$$A_1 U = u(A_1 U) U. \tag{91}$$

On the other hand, replacing Y with ψY in (84) and using (19), (22), (23), and (90), we obtain

$$\begin{aligned} &p(U)\{g(Y,X) - u(Y)u(X) - w(Y)w(X)\} \\ &\quad + p(\theta Y)v(X) = 0, \end{aligned} \tag{92}$$

from which, taking the trace with respect to the Q-basis and using (15) and (24), we find $4mp(U) = 0$; that is,

$$p(U) = 0 \tag{93}$$

which together with (84) and (90) implies

$$p(\phi Y) = 0. \tag{94}$$

Replacing Y with ϕY in the above equation and using (17) and (93), we can easily see that

$$p(Y) = 0 \tag{95}$$

for any vector Y tangent to M.

Summing up, we have the following lemma.

Lemma 2. *Let M be as in Lemma 1, and let the normal vector field N_1 be parallel with respect to the normal connection. If the equalities in* (69) *and (5.2) are established, then*

$$\begin{aligned} A_1 U = u(A_1 U)U, \quad A_1 V = v(A_1 V)V, \\ A_1 W = w(A_1 W)W, \quad p = q = r = 0. \end{aligned} \tag{96}$$

Finally, we will prove our main theorem.

Theorem 3. *Let M be an n-dimensional QR-submanifold of $(p-1)$ QR-dimension in a quaternionic projective space $QP^{(n+p)/4}$, and let the normal vector field N_1 be parallel with respect to the normal connection. If the equalities in* (69) *and (5.2) are established, then $\pi^{-1}(M)$ is locally a product of $M_1 \times M_2$ where M_1 and M_2 belong to some $(4n_1+3)$- and $(4n_2+3)$-dimensional spheres (π is the Hopf fibration $S^{n+p+3}(1) \to QP^{(n+p)/4}$).*

Proof. By means of (96), it follows easily from (83) that

$$\phi A_1 = A_1 \phi. \tag{97}$$

By the quite same method, we can obtain

$$\psi A_1 = A_1 \psi, \qquad \theta A_1 = A_1 \theta. \tag{98}$$

Combining with those equalities and Theorem K-P, we complete the proof. □

Corollary 4. *Let M be an n-dimensional QR-submanifold of $(p-1)$ QR-dimension in a quaternionic projective space $QP^{(n+p)/4}$, and let the normal vector field N_1 be parallel with respect to the normal connection. If the following equalities:*

$${}'\nabla_{\xi}{}'R = 0, \qquad {}'\nabla_{\eta}{}'R = 0, \qquad {}'\nabla_{\zeta}{}'R = 0 \tag{99}$$

are established, then $\pi^{-1}(M)$ is locally a product of $M_1 \times M_2$ where M_1 and M_2 belong to some $(4n_1+3)$- and $(4n_2+3)$-dimensional spheres.

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