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Research Article

Nonderogatory Directed Webgraph

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By assigning a certain direction to the webgraphs, which are defined as the Cartesian product of cycles and paths, we prove that they are nonderogatory.

1. Introduction

Let G be a digraph with vertices labelled $\{1, 2, \ldots, n\}$ and its adjacency matrix $A(G)$ the $n \times n$ matrix whose ijth entry is the number of arcs joining vertex i to vertex j. A digraph is nonderogatory if the characteristic polynomial and minimal polynomial of its adjacency matrix are equal. Computation of the minimal polynomial of a matrix is harder than the characteristic polynomial especially when the matrix is large. That is, why it is important to know when the matrix is nonderogatory. The ladder graphs are examples of nonderogatory graphs first studied by Lim and Lam [1]. Later, difans were added to this family by Deng and Gan [2]. After that, Gan [3], proved that the complete product of difans and diwheels is also nonderogatory [3]. Bravo and Rada [4], found a characterization of nonderogatory unicyclic digraphs in terms of Hamiltonicity conditions. In another article, Rada [5], showed that directed windmills $M_h(r)$ where $r \geq 2$, $h \geq 3$, are nonderogatory if and only if $r = 2$.

All graphs considered in the paper are directed, finite, loopless, and without multiple arcs.

A *dipath* P_n is a digraph (directed graph) with vertex set $\{v_1, \ldots, v_n\}$ and arcs (v_i, v_{i+1}) for $i = 1, \ldots, n-1$.

A *dicycle* C_n is a digraph with vertex set $\{v_1, \ldots, v_n\}$ having arcs (v_i, v_{i+1}) for $i = 1, \ldots, n-1$ and (v_n, v_1).

A *Cartesian product* $G_1 \times G_2$ of graphs G_1 and G_2 with disjoint point sets V_1 and V_2 and edge sets X_1 and X_2 is the graph with point set $V_1 \times V_2$ and $u = (u_1, u_2)$ adjacent with $v = (v_1, v_2)$ whenever $u_1 = v_1$ and u_2 adjacent with v_2 or $u_2 = v_2$ and u_1 adjacent with v_1.

We use the definition as in [6]; for any arbitrary $n \times n$ matrix A, form the characteristic matrix $xI_n - A$ and let $d_j(x)$ denote the greatest common divisor (gcd) of all minors of order j of $xI_n - A$, $j = 1, 2, \ldots, n$. These polynomials are called the determinantal divisor of $xI_n - A$, and it follows that the quotients $i_j(x) = d_j(x)/d_{j-1}(x)$ for $j = 1, 2, \ldots, n$ ($d_0 \equiv 1$) are also polynomials, called the similarity invariants of A.

We use the following theorem from [6] to prove our main result.

Theorem 1. *A matrix $A(G)$ is nonderogatory if and only if its first $n-1$ similarity invariants are unity.*

2. Diwebgraph

The *diwebgraph* (m, n) denoted shortly by $\vec{W}(m, n)$ is the digraph obtained by taking the Cartesian product of C_m and P_n.

Without loss of generality, we assume that the arcs of C_m have clockwise orientation and the arcs of P_n have inward orientation as in Figure 1. For the algorithms described below, we used the prescribed labelling in the figure.

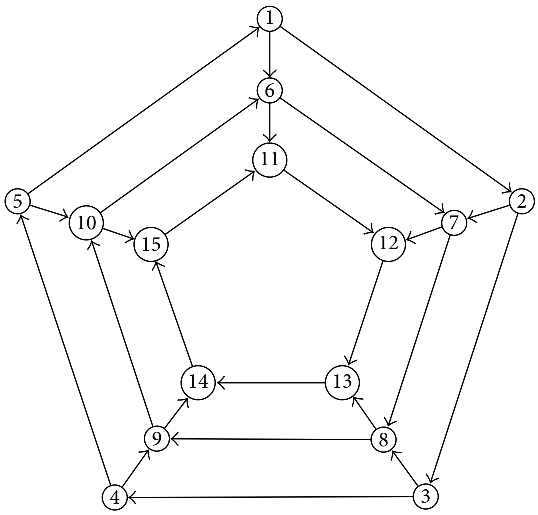

Figure 1: $C_5 \times P_3 = \vec{W}(5,3)$.

The adjacency matrix $A_{mn\times mn}$ of $\vec{W}(m,n)$ can be put in a block matrix form with blocks:

(1) $A_{ii} = A(C_m)$ for $i = 1, 2, \ldots, n$,

(2) $A_{i,i+1} = I_m$ for $i = 1, 2, \ldots, n-1$, where I_m is $m\times m$ identity matrix,

(3) all the remaining blocks are zero matrices and if we write them explicitly, $A_{ij} = \mathbf{0}$ for $i = 1, 2, \ldots, n-2$, $j = i+2, i+3, \ldots, n$ and $A_{ij} = \mathbf{0}$ for $i = 2, 3, \ldots, n$, $j = 1, 2, \ldots, i-1$, where $\mathbf{0}$ is $m\times m$ zero matrix.

For example, the adjacency matrix of the graph in Figure 1 is shown below:

$$\left[\begin{array}{ccccc|ccccc|ccccc}
0&1&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&1&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&1&0&0&0&0&0\\
\hline
0&0&0&0&0&0&1&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&1&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&1\\
\hline
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0
\end{array}\right]. \quad (1)$$

We compute the invariant factors of characteristic matrix $M = xI - A$ by using the following algorithms. In all the algorithms r_i and c_i denote the ith row and column, respectively, and $r_i \leftrightarrow r_j$ ($c_i \leftrightarrow c_j$) means interchange of row (column) i with j.

After Algorithm 1, we obtain a block matrix with n blocks on the diagonal of the form

$$D_{m\times m} = \operatorname{diag}\left[(-1)^{m-1}(x^m - 1)^1\right], \quad (2)$$

and all the other nonzero entries are

$$M_{ij} = (-1)^{r+s}\binom{m}{s-r}x^{m-(s-r)}, \quad (3)$$

where $i = rm$, $r = 1, 2, \ldots, n$ and $j = sm$, $s = r+1, r+2, \ldots, n$.

For example, applying Algorithm 1 to $M = xI - A(\vec{W}(5,3))$, we get

$$M = \left[\begin{array}{ccccc|ccccc|ccccc}
-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&x^5-1&0&0&0&0&-5x^4&0&0&0&0&10x^3\\
\hline
0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&x^5-1&0&0&0&0&-5x^4\\
\hline
0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&-1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&x^5-1
\end{array}\right]. \quad (4)$$

Now, we apply the next algorithm to put the matrix M in a simpler form.

By applying Algorithm 2, the matrix M whose nonzero entries are the first $mn - n$ diagonal entries that are all 1 and

Input: m, n and $M = xI - A(\vec{W}(m,n))$
for $k = 0 \rightarrow mn - m$ **by** m **do**
  for $i = m \rightarrow 2$ **by** (-1) **do**
    $c_{k+i-1} \leftarrow x \times c_{k+i} + c_{k+i-1}$
  end for
  for $i = 1 \rightarrow m - 1$ **do**
    $r_{k+m} \leftarrow M_{k+m,k+i+1} \times r_{k+i} + r_{k+m}$
  end for
  if $(k < mn - m)$ **then**
    for $i = k + 2 \rightarrow k + m$ **do**
      $c_{i+m-1} \leftarrow (-1) \times c_i + c_{i+m-1}$
    end for
  end if
  for $i = 1 \rightarrow m - 1$ **do**
    $c_{k+i} \leftrightarrow c_{k+i+1}$
  end for
end for
for $k = mn - m \rightarrow m$ **by** $(-m)$ **do**
  for $i = mn - 1 \rightarrow m + 1$ **by** (-1) **do**
    if $(i \bmod m \neq 0)$ **then**
      $r_k \leftarrow M_{ki} \times r_i + r_k$
    end if
  end for
end for
Output: M

Algorithm 1

Input: m, n and M (output of the Algorithm 1)
$s \leftarrow 0$
for $k = m \rightarrow mn$ **by** m **do**
  for $i = k - s \rightarrow mn - 1$ **do**
    $r_i \leftrightarrow r_{i+1}$
    $c_i \leftrightarrow c_{i+1}$
  end for
  $s \leftarrow s + 1$
end for
for $i = 1 \rightarrow mn - n$ **do**
  $r_i \leftarrow (-1) \times r_i$
end for
Output: M

Algorithm 2

Input: m, n and N
$U \leftarrow$ minor N_{n1}
for $i = 1 \rightarrow n - 2$ **do**
  $r_{i+1} \leftarrow \dfrac{-(x^m - 1)}{U_{ii}} \times r_i + r_{i+1}$
end for
Output: U

Algorithm 3

the remaining $n \times n$ block N on the diagonal has the following form.

The entry $N_{11} = x^m - 1$ and the remaining entries on the first row are given by $(-1)^i \binom{m}{i} x^{m-i}$, where $i = 1, 2, \ldots, n-1$. The next row of N is obtained by deleting the last entry of the previous row, then cyclic shifting of it, and so on.

For example, applying Algorithm 2 to M (output of the Algorithm 1 applied to $M = xI - A(\vec{W}(5,3))$), we get

$$M = \left[\begin{array}{cccccccccccc|ccc}
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0\\
\hline
0&0&0&0&0&0&0&0&0&0&0&0&x^5-1&-5x^4&10x^3\\
0&0&0&0&0&0&0&0&0&0&0&0&0&x^5-1&-5x^4\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&x^5-1
\end{array}\right], \tag{5}$$

$$N = \begin{bmatrix} x^5-1 & -5x^4 & 10x^3 \\ 0 & x^5-1 & -5x^4 \\ 0 & 0 & x^5-1 \end{bmatrix}. \tag{6}$$

Now, we have to get the relationship between the invariant factors of N.

Lemma 2. *We have the following relationship between the determinant of some of the minors of (N);*

$$\gcd\left(\det\left(\text{minor } N_{11}\right), \det\left(\text{minor } N_{n1}\right)\right) = 1. \tag{7}$$

Proof. $\det(\text{minor } N_{11}) = (x^m - 1)^{n-1}$ and $\det(\text{minor } N_{n1})$ can be computed by Algorithm 3, which turns our matrix into a diagonal one. Since the entries on the diagonal are not a factor of $x^m - 1$ the result follows. ☐

For example, applying Algorithm 3 to the 3 by 3 matrix N in the example, we get

$$U = \begin{bmatrix} -5x^4 & 10x^3 \\ 0 & -\dfrac{3x^5+2}{x} \end{bmatrix}. \tag{8}$$

Theorem 3. *$\vec{W}(m,n)$ is nonderogatory.*

Proof. By applying Lemma 2, we get that the

$$\gcd\left(\det\left(\text{minor } N_{11}\right), \det\left(\text{minor } N_{n1}\right)\right) = 1. \tag{9}$$

Now, by Theorem 1, the result follows. ☐

Remark 4. By changing the orientation of the arcs of C_m to be counterclockwise and the arcs of P_n to be outward and applying similar algorithms shown above, we can show that the formed new diwebgraphs are still nonderogatory.

Further topics for research: are there any other digraphs formed by a Cartesian product that are nonderogatory?

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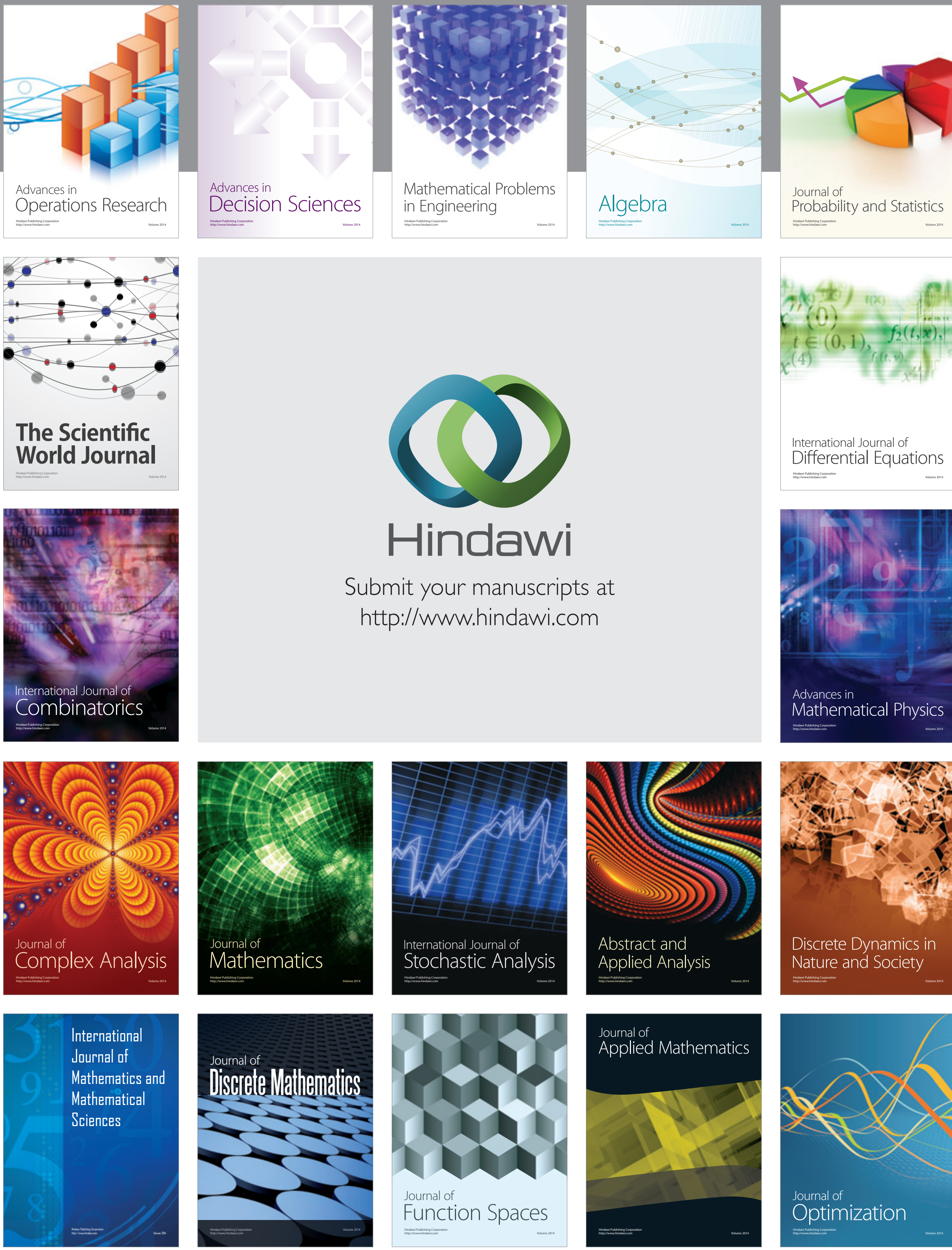
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