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Research Article

A Bordism Viewpoint of Fiberwise Intersections

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We use the geometric data to define a bordism invariant for the fiberwise intersection theory. Under some certain conditions, this invariant is an obstruction for the theory. Moreover, we prove the converse of fiberwise Lefschetz fixed point theorem.

1. Introduction

In topological fixed point theory, there is a classical questions that people may ask; given a smooth self-map $f : M \to M$ of a smooth compact manifold M, when is f homotopic to a fixed point free map?

The famous theorem, Lefschetz fixed point theorem, gave a sufficient condition to answer the above question as follows.

Theorem 1 (Lefschetz fixed point theorem). *Let $f : M \to M$ be a smooth self-map of a compact smooth manifold M.*

If f has no fixed point, then the Lefschetz number $L(f) = 0$, where $L(f) = \sum_{k\geq 0}(-1)^k \operatorname{Tr}(f_{*|}H_k(M;\mathbb{Q}))$.

In general, the converse of the above theorem does not hold. It requires a more refined invariant than the Lefschetz number to make the converse hold (see [1–3]).

For this work, we focus on the similar arguments as above for the family of smooth maps over a compact base space B. The proof of the main theorem depends heavily on the intersection problem as follows.

From now on, the notations X^a means the smooth manifold X of dimension a and I means the unit interval $[0, 1]$. If Y^b is a submanifold of X^a, $f : X \to Z$, and η is a bundle over Z, then $\nu_{Y\subseteq X}$ means the normal bundle of Y in X and $f^*(\eta)$ is a pull-back bundle of η along the map f.

Later we define "framed bordism with coefficient in a bundle" as follows. Let X be a smooth manifold with a bundle ξ over it. Define $\Omega_*^{fr}(X;\xi)$ to be the bordism groups of manifolds mapping to X, together with a stable isomorphism of the normal bundle with the pullback of ξ. This framed bordism group will be a home for our invariants $L^{\text{bord}}(f)$ (described in the last section) which detects more fixed point information than the regular Lefschetz number.

(I) Suppose that E_P^{p+k}, E_Q^{q+k}, and E_M^{m+k} are smooth fiber bundles over a compact manifold B^k. Let $f : E_P^{p+k} \to E_M^{m+k}$ be a bundle map, and let E_Q be a subbundle of E_M with the inclusion bundle map $i_Q : E_Q \hookrightarrow E_M$.

We have a commutative diagram

$$\begin{array}{ccccc} P^p & & M^m & & Q^q \\ \downarrow & & \downarrow & & \downarrow \\ E_P & \xrightarrow{f} & E_M & \xleftarrow{i_Q} & E_Q \\ & \searrow^{\mathrm{pr}_P} & \downarrow{\scriptstyle \mathrm{pr}_M} & \swarrow_{\mathrm{pr}_Q} & \\ & & B & & \end{array} \qquad (1)$$

where P, Q, and M are the fibers of pr_P, pr_Q, and pr_M, respectively.

We may assume that $f \pitchfork E_Q$ in E_M (see [4]).

The homotopy pullback is

$$E(f, i_Q) := \left\{(x, \lambda, y) \in E_P \times E_M^I \times E_Q \mid \lambda(0) = f(x), \lambda(1) = y\right\}. \qquad (2)$$

We have a diagram which commutes up to homotopy

$$\begin{CD} E(f,i_Q) @>{\pi_Q}>> E_Q \\ @V{\pi_P}VV @VV{i_Q}V \\ E_P @>{f}>> E_M \end{CD} \tag{3}$$

where π_P and π_Q are the trivial projections; that is, we have a homotopy $E(f,i_Q)\times I \xrightarrow{K} E_M$ defined by $K(x,\lambda,y,t)=\lambda(t)$, $t\in I$.

We also have a map $c: f^{-1}(E_Q) \to E(f,i_Q)$ defined by $x \mapsto (x, c_{f(x)}, f(x))$ where $c_{f(x)}$ = constant path in E_M at $f(x)$.

Transversality yields a bundle map

$$\begin{CD} \nu_{f^{-1}(E_Q)\subseteq E_P} @>>> \nu_{E_Q\subseteq E_M} \\ @VVV @VVV \\ f^{-1}(E_Q) @>{f_{|f^{-1}(E_Q)}}>> E_Q \end{CD} \tag{4}$$

Choose an embedding $E_P^{p+k} \subseteq S^L$, for sufficiently large L, where S^L is a sphere of dimension L. Then, we have $f^{-1}(E_Q) \overset{i}{\hookrightarrow} E_P^{p+k} \subseteq S^L$. So $\nu_{f^{-1}(E_Q)\subseteq S^L} \cong \nu_{f^{-1}(E_Q)\subseteq E_P} \oplus i^*\nu_{E_P\subseteq S^L}$.

The commutative diagram

$$\begin{array}{ccccc} f^{-1}(E_Q) & & \xrightarrow{\quad f_{|f^{-1}(E_Q)}\quad} & & \\ & \searrow^{c} & & & \\ & & E(f,i_Q) & \xrightarrow{\pi_Q} & E_Q \\ & \searrow_{i} & \downarrow{\scriptstyle \pi_P} & & \\ & & E_P & & \end{array} \tag{5}$$

yields a bundle map

$$\begin{CD} \nu_{f^{-1}(E_Q)\subseteq S^L} @>{\widehat{c}}>> \pi^*(\nu_{E_Q\subseteq E_M}) \oplus \pi^*(\nu_{E_P\subseteq S^L}) := \zeta \\ @VVV @VVV \\ f^{-1}(E_Q) @>{c}>> E(f,i_Q) \end{CD} \tag{6}$$

Thus, $(c,\widehat{c})$ determines an element $[c,\widehat{c}] \in \Omega^{fr}_{p+q+k-m}(E(f,i_Q);\xi)$.

(II) Suppose that $(N \xrightarrow{c_1} E(f,i_Q), \nu_{N\subseteq S^L} \xrightarrow{\widehat{c}_1} \xi)$ is another representative of $[c,\widehat{c}]$, where $N^{p+q+k-m} \subseteq E_P^{p+k} \subseteq S^L$. This means we have a normal bordism $(W \xrightarrow{\mathscr{C}} E(f,i_Q), \nu_W \xrightarrow{\widehat{\mathscr{C}}} \xi)$ between $(c,\widehat{c})$ and $(c_1,\widehat{c}_1)$; that is

(i) $W^{p+q+k-m+1} \subseteq (S^L\times I)$,

(ii) $\partial W \subseteq (S^L \times \partial I)$,

(iii) $W \pitchfork (S^L\times \partial I)$,

(iv) $W\cap(S^L\times 0) = f^{-1}(E_Q)$ and $W\cap (S^L\times 1) = N$

such that

$$\begin{gathered} \mathscr{C}_{|f^{-1}(E_Q)} = c : f^{-1}(E_Q) \longrightarrow E(f,i_Q), \\ \mathscr{C}_{|N} = c_1 : N \longrightarrow E(f,i_Q), \\ \widehat{\mathscr{C}}_{|\nu_{f^{-1}(E_Q)\subseteq S^L}} = \widehat{c}, \qquad \widehat{\mathscr{C}}_{|\nu_{N\subseteq S^L}} = \widehat{c}_1. \end{gathered} \tag{7}$$

Theorem 2 (main theorem). *Assume $m > q + ((p+k)/2)+1$. Then, there exists a smooth fiber-preserving map over B*

$$\Psi : E_P \times I \longrightarrow E_M \tag{8}$$

such that $\Psi_{|E_P\times\{0\}} = f$, $\Psi \pitchfork E_Q$ and $\Psi^{-1}_{|E_P\times\{1\}}(E_Q) = N$.

Note that if we let $g = \Psi_{|E_P\times\{1\}}$, then g is fiber-preserving homotopic to f and $g^{-1}(E_Q)=N$.

In 1974, Hatcher and Quinn [5] showed an interesting result as follows.

Theorem 3 (Hatcher-Quinn). *Given smooth bundle maps $f: E_P \to E_M$ and $i_Q : E_Q \to E_M$ over a compact base space B which are immersions in each fiber, assume $m > q + ((p+k)/2)+1$ and $m > p + ((q+k)/2)+1$. Then there is a fiber-preserving map $g: E_P \to E_M$ over B homotopic to f such that $g^{-1}(E_Q) = N$.*

Their theorem required a map f to be an immersion on each fiber which cannot be applied to our main theorem. We decided to give an alternative technique to proof the main theorem by constructing the required homotopy step by step. This techniques could be used as a model to achieve the similar result for the equivariant setting (see [6]).

2. Proof of the Main Theorem

Theorem 4 (Whitney embedding theorem). *Let M^m and N^n be smooth manifolds, and let $f: M \to N$ be a smooth map. If $n > 2m$, then f is homotopic to an embedding $g: M\to N$.*

Lemma 5. *Let $f: M^m \to N^n$ be a map between two smooth manifolds. Let A be a closed submanifold of M. Assume that $f_{|A}$ is an embedding.*

If $n>2m$, then f is homotopic to an embedding g relative to A.

Proof. Let T be a tubular neighborhood of A in M.

Step 1. Extend the embedding $f_{|A}: A \to N$ to an embedding $f_T : T \to N$.

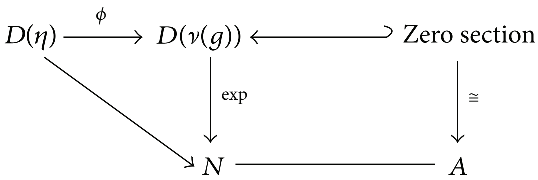
Let $\nu_{A\subseteq M}$ be the normal bundle of A in M and $D(\nu)$ denote the disc bundle of ν. Then the tubular neighborhood theorem implies $D(\nu_{A\subseteq M}) \cong T$.

Claim. For any given embedding $A \overset{g}{\hookrightarrow} N$ and vector bundle η over A, then

$$\begin{aligned} &(g \text{ extends to an embedding of } D(\eta) \text{ into } N) \\ &\Longleftrightarrow (\text{there exists a bundle monomorphism} \\ &\quad \phi : \eta \longrightarrow \nu(g)), \end{aligned} \tag{9}$$

where $\nu(g)$ is the normal bundle of A in N via g.

Proof of Claim. ($\Leftarrow$) We have a diagram

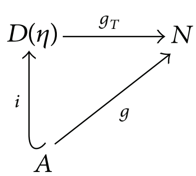


(10)

where exp is the exponential map.

Note that $\exp(D(\nu(g)) \cong$ tubular neighborhood of A in N via g. Then $\exp \circ \phi : D(\eta) \hookrightarrow N$ is a desired embedding.

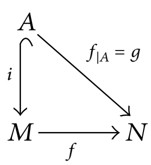
($\Rightarrow$) Assume there exists an embedding g_T so that the following diagram commutes



(11)

Then $\nu(g) \cong \eta \oplus i^* \nu(g_T)$.

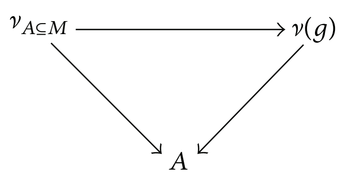
We are in the situation where we have a commutative diagram



(12)

Let $\nu(f) := f^*\tau_N - \tau_M$. Then $i^*(\nu(f)) \oplus \nu_{A\subseteq M} \overset{\text{stable}}{\cong} \nu(g)$, where $\overset{\text{stable}}{\cong}$ denotes the stable isomorphism between 2 vector bundles.

If $n - a > a$, then $i^*(\nu(f)) \oplus \nu_{A\subseteq M} \cong \nu(g)$, so there exists a bundle monomorphism



(13)

Apply the Claim when $g = f_{|A}$ and $\eta = \nu_{A\subseteq M}$. Then we have an extension embedding of g from $D(\nu_{A\subseteq M}) \cong T \overset{f_T}{\hookrightarrow} N$.

Step 2. We have a map $f_{|M-\operatorname{int} T} : M \setminus \operatorname{int}(T) \to N$ and $\partial(M \setminus \operatorname{int}(T)) = \partial T$.

The condition $b > (c + a/2 + 1)$ and Theorem 4 imply that $f_{|M-\operatorname{int}(T)}$ is homotopic to an embedding $g_{M-\operatorname{int}(T)}$.

Define $g : M \to N$ by

$$g(x) = \begin{cases} g_{M-\operatorname{int}(T)}(x) & \text{if } x \in M \setminus \operatorname{int}(T), \\ g_T(x) & \text{if } x \in T. \end{cases} \tag{14}$$

Then f is homotopic to g relative to A. □

2.1. *Proof of the Main Theorem.*

We divide the proof into 3 steps.

Step 1. Goal: fiber-preserving homotoped the map $W \xrightarrow{a:=\pi_P\circ\mathscr{C}} E_P$ to an embedding over B.

By assumption, we have

$$W \overset{\mathscr{C}}{\longrightarrow} E(f, i_Q) \subseteq E_P \times E_M^I \times E_Q, \tag{15}$$

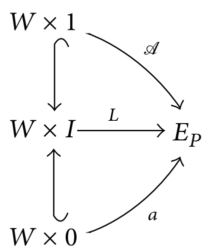
and we also have maps

$$W \xrightarrow{a:=\pi_P\circ\mathscr{C}} E_P, \qquad W \xrightarrow{b:=\pi_Q\circ\mathscr{C}} E_Q, \qquad W \times I \overset{H}{\longrightarrow} E_M, \tag{16}$$

where $H := K \circ (\mathscr{C} \times \mathrm{id}_I)$, so $H_{|W\times 0} = f \circ a$, $H_{|W\times 1} = b$.

Recall that $\partial W = f^{-1}(E_Q) \sqcup N$, $a_{|\partial W}$ is just the inclusion of $f^{-1}(E_Q)$ and N into E_P.

Apply the condition $m > q + ((p+k)/2) + 1$ to Lemma 5, there exists an embedding $\mathscr{A} \simeq a$ (rel ∂W). That is, we have a commutative diagram

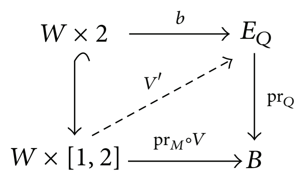


(17)

We have a map $W \xrightarrow{b:=\pi_Q\circ\mathscr{C}} E_Q$. By concatenating the homotopy H and $f \circ L$ together, we have a homotopy $V : W \times [1,2] \to E_M$ such that $V(\_, 1) = \mathscr{A}$ and $V(\_, 2) = b$. Hence, $i_Q \circ b$ is homotopic to $f \circ \mathscr{A}$.

Next, we want to modify the homotopy V such that it is fiber preserving with respect to pr_M.

Note that we have a commutative diagram



(18)

We can apply the homotopy lifting property for pr_Q to get a homotopy of b to b' through V' such that the following diagram commute:

$$\begin{array}{ccccc} W\times 2 & \xrightarrow{b' := V'_{W\times 1}} & E_Q & & \\ \big\downarrow & & \big\downarrow i_Q & \searrow \mathrm{pr}_Q & \\ W\times[1,2] & \xrightarrow{V'} & E_Q\subseteq E_M & \xrightarrow{\mathrm{pr}_M} & B \\ \big\uparrow & & \big\uparrow f & \nearrow \mathrm{pr}_P & \\ W\times 1 & \xrightarrow{\mathscr{A}} & E_P & & \end{array} \tag{19}$$

Let $\Psi_W := b' : W \to E_Q$. Then Ψ_W is a fiber-preserving map over B through the lifting V'.

Step 2. Goal: construct a bundle isomorphism

$$\nu(\mathscr{A})\oplus\epsilon^1 \cong b'^*\left(\nu_{E_Q\subseteq E_M}\right), \tag{20}$$

where ϵ^1 is the trivial bundle.

Since $\dim W < \operatorname{rank}\nu(\mathscr{A})$, it is enough to give a stable equivalence between such bundles.

Now, we have

$$W \overset{\mathscr{A}}{\hookrightarrow} E_P \subseteq S^L \Longrightarrow \nu_{W\subseteq S^L} \cong \nu(\mathscr{A})\oplus\mathscr{A}^*\left(\nu_{E_P\subseteq S^L}\right). \tag{21}$$

We also have a commutative diagram

$$\begin{array}{ccc} W & \xrightarrow{\quad b' \quad} & E_Q \\ \mathscr{C}\searrow & & \\ & E(f,i_Q) \xrightarrow{\pi_Q} & E_Q \\ \mathscr{A}\ \big\downarrow & \big\downarrow \pi_P & \\ & E_P & \end{array} \tag{22}$$

$$\mathscr{A}\simeq a = \pi_P\circ\mathscr{C} \Longrightarrow \mathscr{A}^*\left(\nu_{E_P\subseteq S^L}\right) \cong \left(\pi_P\circ\mathscr{C}\right)^*\left(\nu_{E_P\subseteq S^L}\right),$$
$$b'\simeq b = \pi_Q\circ\mathscr{C} \Longrightarrow b'^*\left(\nu_{E_Q\subseteq E_M}\right) \cong \left(\pi_Q\circ\mathscr{C}\right)^*\left(\nu_{E_Q\subseteq E_M}\right). \tag{23}$$

Thus, the bundle map $\widehat{\mathscr{C}} : \nu_{W\subseteq S^L\times I} \to \xi = \pi_P^*(\nu_{E_P\subseteq S^L}) \oplus \pi_Q^*(\nu_{E_Q\subseteq E_M})$ yields the following stable isomorphism

$$\nu_{W\subseteq S^L}\oplus\epsilon^1 \overset{\text{stable}}{\cong} \left(\pi_P\circ\mathscr{C}\right)^*\left(\nu_{E_P\subseteq S^L}\right)\oplus\left(\pi_Q\circ\mathscr{C}\right)^*\left(\nu_{E_Q\subseteq E_M}\right). \tag{24}$$

Putting (21), (23), and (24) together, we get

$$\nu(\mathscr{A})\oplus\left(\pi_P\circ\mathscr{C}\right)^*\left(\nu_{E_P\subseteq S^L}\right)\oplus\epsilon^1 \overset{\text{stable}}{\cong} \left(\pi_P\circ\mathscr{C}\right)^*\left(\nu_{E_P\subseteq S^L}\right) \oplus b'^*\left(\nu_{E_Q\subseteq E_M}\right). \tag{25}$$

Consequently, we have

$$\nu(\mathscr{A})\oplus\epsilon^1 \cong b'^*\left(\nu_{E_Q\subseteq E_M}\right). \tag{26}$$

This implies that we did construct a bundle map

$$\begin{array}{ccc} \nu(\mathscr{A})\oplus\epsilon^1 & \xrightarrow{\widehat{b'}} & \nu_{E_Q\subseteq E_M} \\ \big\downarrow & & \big\downarrow \\ W & \xrightarrow{b'} & E_Q \end{array} \tag{27}$$

which gives us the extension of map b' to the tubular neighborhood of W in E_P. More precisely,

$$\Psi_T : D(\nu(\mathscr{A})) \hookrightarrow D\left(\nu(\mathscr{A})\oplus\epsilon^1\right) \xrightarrow{\widehat{b'}} D\left(\nu_{E_Q\subseteq E_M}\right), \tag{28}$$

where D denotes the disc bundle. Note that $\Psi_T \pitchfork E_Q$ and $\Psi_T(\partial D(\eta_1)) \subseteq E_M \setminus E_Q$.

Since $\nu(\mathscr{A})\oplus\epsilon^1 \cong b'^*(\nu_{E_Q\subseteq E_M})$, we can find a subbundle η_2 of $b'^*(\nu_{E_Q\subseteq E_M})$ such that $\eta_2 \cong \nu(\mathscr{A})$. For simplicity, let $\eta_1 := \nu(\mathscr{A})$.

Step 3. Goal: construct the fiber-preserving smooth map $\Psi : E_P\times I \to E_M$ over B.

Recall that we have

$$W \hookrightarrow D(\nu(\mathscr{A})) \simeq D\left(\nu(\mathscr{A})\oplus\epsilon^1\right) \cong D\left(b'^*\left(\nu_{E_Q\subseteq E_M}\right)\right). \tag{29}$$

Then there exists a neighborhood $\overline{D}$ of W in $D(b'^*(\nu_{E_Q\subseteq E_M}))$ such that $\overline{D} \simeq D(b'^*(\nu_{E_Q\subseteq E_M}))$ and $\overline{D} \cong D(\nu(\mathscr{A}))$.

According to (19), we have a commutative diagram

$$\begin{array}{ccc} W\times 2 & \xrightarrow{V'_{|W\times 2}} & E_M\times 2 \\ \big\downarrow & & \big\downarrow \\ W\times[1,2] & \xrightarrow{\Psi_1} & E_M\times[1,2] \\ \big\uparrow & & \big\uparrow \\ W\times 1 & \xrightarrow{V'_{|W\times 1}} & E_M\times 1 \end{array} \tag{30}$$

where $\Psi_1(w,t) = (V'(w,t),t)$.

Let $\eta_1 = \nu(\mathscr{A}\oplus\epsilon^1)$, $\eta_2 = b^*(\nu_{E_Q\subseteq E_M})$.

According to (27), there exists a bundle η over $W\times I$ such that $\eta_{|W\times i} = \eta_i$ for $i = 1, 2$.

Let $D_1 := D(\nu(\mathscr{A}))$, $D_2 := \overline{D}$. Then $D_i \hookrightarrow D(\eta_i)$ is a homotopy equivalence for $i = 1, 2$ and also $D_1 \cong D_2$.

Since $D_1 \cup W\times[1,2] \cup D_2 \hookrightarrow D_1\times[1,2]$ is a cofibration and a homotopy equivalence, there exist an extension

$D_1 \times [1,2] \xrightarrow{\widehat{\Psi}_1} E_M \times [1,2]$ such that the following diagram commutes

$$\begin{array}{ccccc} D_2 & \longrightarrow & D(\nu_{E_Q \subseteq E_M}) & \xhookrightarrow{\exp} & E_M \times 2 \\ \Big\downarrow & & & & \Big\downarrow \\ D_1 \times [1,2] & & \xrightarrow{\qquad \widehat{\Psi}_1 \qquad} & & E_M \times [1,2] \\ \Big\uparrow & & & & \Big\uparrow \\ D_1 & & \xrightarrow{\qquad f_{|D_1} \qquad} & & E_M \cong E_M \times 1 \end{array} \tag{31}$$

Next we want to construct an embedding $W \overset{\mathscr{A}'}{\hookrightarrow} D_2 \times [2,3]$ such that the following hold:

(i) $\mathscr{A}'(W) \cap \{D_2 \times 2\} = f^{-1}(E_Q)$,

(ii) $\mathscr{A}'(W) \cap \{D_2 \times 3\} = N$,

(iii) $\mathscr{A}' \pitchfork D_2 \times \partial[2,3]$.

We start by letting $\alpha : W \to [2,3]$ be a smooth map such that $\alpha \pitchfork \partial[2,3]$, $\alpha^{-1}(2) = f^{-1}(E_Q)$ and $\alpha^{-1}(3) = N$, and we also have an inclusion $W \overset{i_W}{\hookrightarrow} D_2$.

Let $\mathscr{A}' := i_W \times \alpha$. Then $\mathscr{A}'$ is such a required map.

By the construction, we have

$$D\left(\nu\left(\mathscr{A}'\right)\right) \cong D_2 \times [2,3]. \tag{32}$$

Let ψ_2 be the composition of the maps

$$W \overset{\mathscr{A}'}{\hookrightarrow} D_2 \times [2,3] \xrightarrow{\Psi_{T|D_2} \times \mathrm{id}_{[2,3]}} M \times [2,3] \xrightarrow{\text{proj}} M. \tag{33}$$

Define a map $\Psi_2 := \psi_2 \times \alpha : W \to E_M \times [2,3]$.

Using the fact that $D(\nu(\mathscr{A}')) \cong D_2 \times [2,3]$, then $D_2 \times \partial[2,3] \cup \mathscr{A}'(W) \hookrightarrow D_2 \times [2,3]$ is a cofibration and homotopy equivalence. Hence there exists an extension $D_2 \times [2,3] \xrightarrow{\widehat{\Psi}_2} E_M \times [2,3]$.

Note that for $(x,3) \in D_2 \times 3$ such that $\widehat{\Psi}_2(x,3) \in E_Q \times 3$, the map $\widehat{\Psi}_{2|D_2 \times 3} = \Psi_T$ forces that x has to be in W, so the definition of $\widehat{\Psi}_2$ implies $x \in N$. Thus

$$\widehat{\Psi}^{-1}_{2|D_2 \times 3}\left(E_Q\right) = N. \tag{34}$$

We define a map $\widetilde{\Psi} : \{E_P \times I\} \cup \{D_1 \times [1,2]\} \cup \{D_2 \times [2,3]\} \to E_M \times [0,3]$ by

$$\widetilde{\Psi}(p,t) = \begin{cases} (f(p), t) & \text{if } t \in [0,1], \\ \left(\widehat{\Psi}_1(p), t\right) & \text{if } t \in [1,2], \\ \left(\widehat{\Psi}_2(p), t\right) & \text{if } t \in [2,3]. \end{cases} \tag{35}$$

Then $\widetilde{\Psi}$ is well-defined map over B by the construction.

It's not hard to see that $\{E_P \times I\} \cup \{D_1 \times [1,2]\} \cup \{D_2 \times [2,3]\}$ is diffeomorphic to $E_P \times I$. Define the map Ψ to be the composition of maps

$$\begin{aligned} E_P \times I &\xrightarrow{\cong} \{E_P \times I\} \cup \{D_1 \times [1,2]\} \cup \{D_2 \times [2,3]\} \\ &\xrightarrow{\widetilde{\Psi}} E_M \times [0,3] \xrightarrow{\text{proj}} E_M, \end{aligned} \tag{36}$$

where proj is the projection to the first factor.

Thus, we get a map $\Psi : E_P \times I \to E_M$ over B so that $\Psi_{|E_P \times 0} = f$. By construction, $\Psi \pitchfork E_Q$ and $\Psi^{-1}_{|E_P \times 1}(E_Q) = N$ as required.

Corollary 6. *Assume $m > q + ((p+k)/2) + 1$. Then the map f can be fiber-preserving homotoped to a map whose image does not intersect E_Q if and only if $[c, \widehat{c}] = 0 \in \Omega^{fr}_{p+q+k-m}(E(f, i_Q); \xi)$.*

3. Application to Fixed Point Theory

Let $p : M^{m+k} \to B^k$ be a smooth fiber bundle with compact fibers and $k > 2$. Assume that B is a closed manifold. Let $f : M \to M$ be a smooth map over B. That is, $p \circ f = p$.

The fixed point set of f is

$$\mathrm{Fix}(f) := \{x \in M \mid f(x) = x\}. \tag{37}$$

We have a homotopy pull-back diagram

$$\begin{array}{ccc} \mathscr{L}_f M & \xrightarrow{\quad ev_0 \quad} & M \\ {\scriptstyle ev_1}\Big\downarrow & & \Big\downarrow{\scriptstyle \Delta} \\ M & \xrightarrow[\quad \Delta_f \quad]{} & M \times_B M \end{array} \tag{38}$$

where

(i) $\mathscr{L}_f M := \{\alpha \in M^I \mid f(\alpha(0)) = \alpha(1)\}$,

(ii) ev_0 and ev_1 are the evaluation map at 0 and 1 respectively,

(iii) $\Delta :=$ the diagonal map, defined by $x \mapsto (x, x)$,

(iv) $\Delta_f :=$ the twisted diagonal map, defined by $x \mapsto (x, f(x))$,

(v) $M \times_B M :=$ the fiber bundle over B with fiber over $b \in B$, given by $F_b \times F_b$ where F_b is the fiber of p over b.

Proposition 7. *There exists a homotopy from f to f_1 such that $\Delta_{f_1} \pitchfork \Delta$.*

The proof relies on the work of Koźniewski [4], relating to B-manifolds. Let B be a smooth manifold. A B-manifold is a manifold X together with a locally trivial submersion $p : X \to B$. A B-map is a smooth fiber-preserving map.

Lemma 8. *Let X and Y be B-manifolds, and let Z be a B-submanifold of Y. Let $g : X \to Y$ be a B-map. Then, there* *is a fiber-preserving smooth B-homotopy* $H_t : X \to Y$ *such that* $H_0 = g$ *and* $H_1 \pitchfork Z$.

Proof. See [7] for the proof. □

We have a transversal (pull-back) square

$$\begin{array}{ccc} \mathrm{Fix}(f_1) & \xrightarrow{\ i\ } & M \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \Delta} \\ M & \xrightarrow[\Delta_{f_1}]{} & M \times M \end{array} \tag{39}$$

where i is the inclusion. Transversality yields that $\nu(i) \cong i^*(\nu(\Delta)) \cong i^*(\tau M)$.

Choose an embedding $M \hookrightarrow S^{m+k}$ for sufficiently large k. Then we have

$$\begin{aligned} \nu_{\mathrm{Fix}(f_1)\subseteq S^{m+k}} &\cong \nu(i) \oplus i^*\left(\nu_{M\subseteq S^{m+k}}\right) \\ &\cong i^*(\tau M) \oplus i^*\left(\nu_{M\subseteq S^{p+k}}\right) \cong \epsilon. \end{aligned} \tag{40}$$

We denote this bundle isomorphism by $\nu_{\mathrm{Fix}(f_1)\subseteq S^{m+k}} \xrightarrow{\widehat{g}} \epsilon$. We also have a map $\mathrm{Fix}(f_1) \xrightarrow{g} \mathscr{L}_f M$ defined by $x \mapsto c_x$, where c_x is the constant map at x. Thus $L^{\mathrm{bord}}(f) := [\mathrm{Fix}(f_1), g, \widehat{g}]$ determines the element in $\Omega_k^{fr}(\mathscr{L}_f M; \epsilon)$.

Applying Theorem 2, We obtain the following corollary.

Corollary 9 (converse of fiberwise Lefschetz fixed point theorem). *Let* $f : M^{m+k} \to M^{m+k}$ *be a smooth bundle map over the closed manifold* B^k. *Assume that* $m \geq k+3$. *Then,* f *is fiber homotopic to a fixed point free map if and only if* $L^{bord}(f) = 0 \in \Omega_k^{fr}(\mathscr{L}_f M; \epsilon)$.

Corollary 10. *Let* $f : M^{m+k} \to M^{m+k}$ *be a smooth bundle map over the closed manifold* B^k. *Assume that* $m \geq k+3$. *If there is* $[N, N \xrightarrow{g} \mathscr{L}_f M, \nu_{N\subseteq S^{m+k}} \xrightarrow{\widehat{g}} \epsilon] \in \Omega_k^{fr}(\mathscr{L}_f M; \epsilon) f$ *where* N *is a finite subset of* M *such that* $L^{bord}(f) = [N, g, \widehat{g}]$, *then the map* f *can be fiber-preserving homotoped to a map* h *such that* $\mathrm{Fix}(h) = N$.

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References

[1] F. Wecken, "Fixpunktklassen. I," *Mathematische Annalen*, vol. 117, pp. 659–671, 1941 (German).

[2] J. Nielsen, "Über die Minimalzahl der Fixpunkte bei den Abbildungstypen der Ringflächen," *Mathematische Annalen*, vol. 82, no. 1-2, pp. 83–93, 1920 (German).

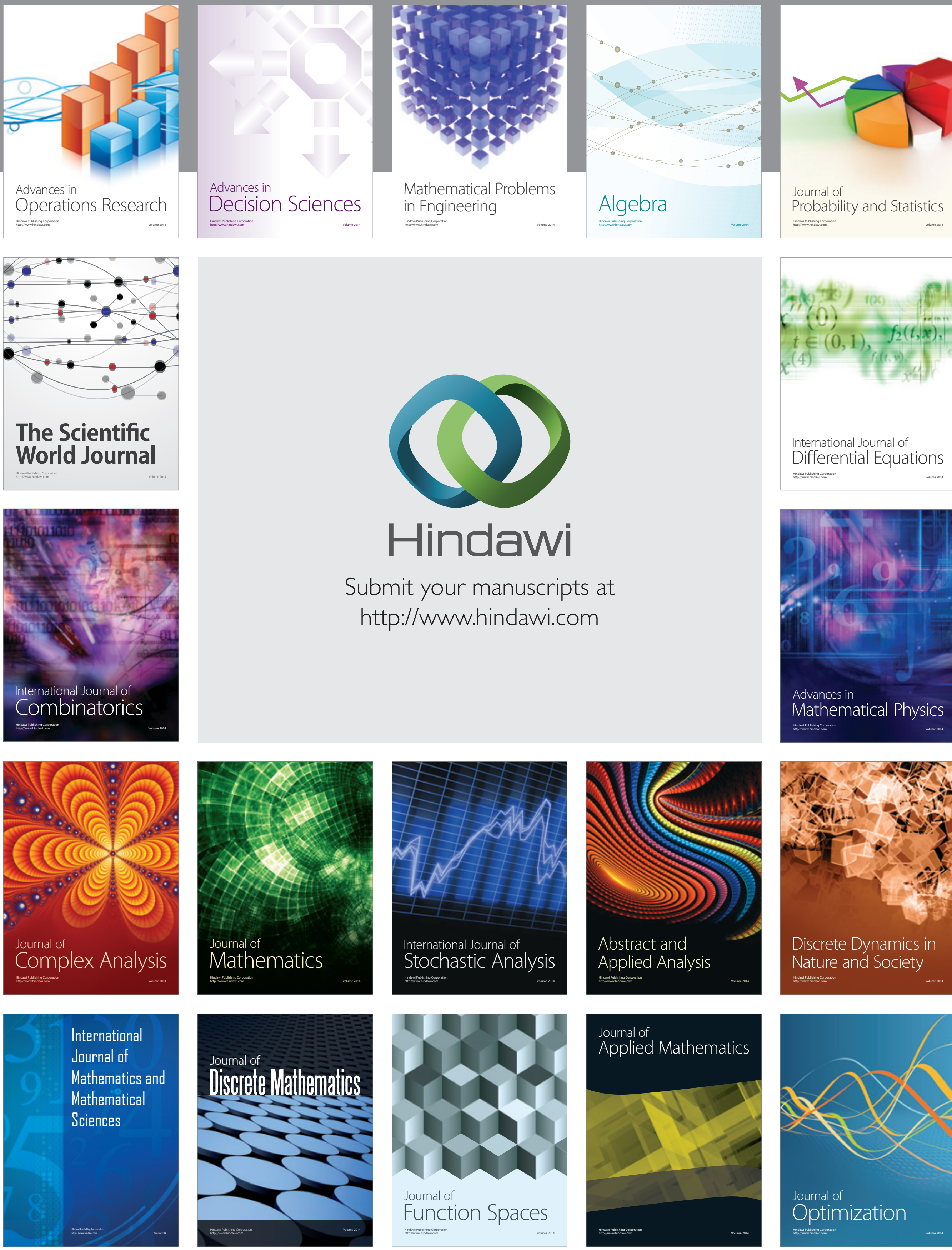
[3] B. J. Jiang, "Fixed point classes from a differential viewpoint," in *Fixed Point Theory (Sherbrooke, Que., 1980)*, vol. 886 of *Lecture Notes in Mathematics*, pp. 163–170, Springer, Berlin, Germany, 1981.

[4] T. Koźniewski, "The category of submersions," *L'Académie Polonaise des Sciences. Bulletin. Série des Sciences Mathématiques*, vol. 27, no. 3-4, pp. 321–326, 1979.

[5] A. Hatcher and F. Quinn, "Bordism invariants of intersections of submanifolds," *Transactions of the American Mathematical Society*, vol. 200, pp. 327–344, 1974.

[6] G. Sunyeekhan, *Equivariant intersection theory [Ph.D. thesis]*, University of Notre Dame, Notre Dame, Indiana, 2010.

[7] V. Coufal, *A family version of Lefschetz-Nielsen fixed point theory [Ph.D. thesis]*, University of Notre Dame, Notre Dame, Indiana, 2004.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