Internat. J. Math. & Math. Sci.
VOL. 20 NO. 1 (1997) 51-60

ASYMPTOTIC BEHAVIOR OF ALMOST-ORBITS OF REVERSIBLE SEMIGROUPS OF NON-LIPSCHITZIAN MAPPINGS IN BANACH SPACES

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(Received February 15, 1994 and in revised form October 25, 1995)

ABSTRACT. Let C be a nonempty closed convex subset of a uniformly convex Banach space E with a Fréchet differentiable norm, G a right reversible semitopological semigroup, and $\mathcal{S} = \{S(t) : t \in G\}$ a continuous representation of G as mappings of asymptotically nonexpansive type of C into itself The weak convergence of an almost-orbit $\{u(t) : t \in G\}$ of $\mathcal{S} = \{S(t) : t \in G\}$ on C is established. Furthermore, it is shown that if P is the metric projection of E onto set $F(\mathcal{S})$ of all common fixed points of $\mathcal{S} = \{S(t) : t \in G\}$, then the strong limit of the net $\{Pu(t) : t \in G\}$ exists.

KEY WORDS AND PHRASES: Almost-orbit, fixed point, reversible semitopological semigroup, semigroup of asymptotically nonexpansive type, uniformly convex Banach space
1991 AMS SUBJECT CLASSIFICATION CODES: 47H20, 47H10, 47H09

1. INTRODUCTION

Let C be a nonempty closed convex subset of a real Banach space E and let $\mathcal{S} = \{S(t) : t \geq 0\}$ be a family of mappings from C into itself such that $S(0) = I$, $S(t+s) = S(t)S(s)$ for all $t, s \in [0, \infty)$ and $S(t)x$ is continuous in $t \in [0, \infty)$ for each $x \in C$. $\mathcal{S}$ is said to be

(a) nonexpansive semigroup on C if $\|S(t)x - S(t)y\| \leq \|x - y\|$ for all $x, y \in C$ and $t \geq 0$,

(b) asymptotically nonexpansive semigroup on C [1] if there is a function $k: [0, \infty) \to [0, \infty)$ with $\limsup_{t\to\infty} k(t) \leq 1$ such that $\|S(t)x - S(t)y\| \leq k(t)\|x - y\|$ for all $x, y \in C$ and $t \geq 0$,

(c) semigroup of asymptotically nonexpansive type on C [1] if for each $x \in C$,

$$\limsup_{t\to\infty} \left\{ \sup_{y\in C} [\|S(t)x - S(t)y\| - \|x - y\|] \right\} \leq 0;$$

see [2] for mappings of asymptotically nonexpansive type. It is easily seen that (a) $\Rightarrow$ (b) $\Rightarrow$ (c) and that both the inclusions are proper (cf. [1, p. 112]).

In [3], Myadera and Kobayashi introduced the notion of almost-orbits of nonexpansive semigroups on C and provided the weak and strong almost convergences of such an almost-orbit in a uniformly convex Banach space; see also [4] for almost-orbits of nonexpansive mappings. Recently, Tan and Xu [5] extended this notion to semigroups of asymptotic nonexpansive type in Hilbert spaces The case of general commutative nonexpansive semigroups in uniformly convex Banach spaces was studied by Takahashi and Park [11]. Oka [6] gave the results for the case of commutative asymptotically nonexpansive semigroups in uniformly convex Banach spaces. In particular, Takahashi and Zhang [7] established the convergences of almost-orbits of noncommutative asymptotically nonexpansive semigroups in the same Banach spaces, see [8] for the case of Hilbert spaces

The purpose of this paper is to generalize their results to the case of noncommutative semigroups of asymptotically nonexpansive type Section 2 is a preliminary part In Section 3, we prove several lemmas which are crucial for our discussion. Main results are given in Section 4 First, we establish the weak convergence (Theorem 1) of an almost-orbit $\{u(t) : t \in G\}$ of a semigroup $\mathcal{S} = \{S(t) : t \in G\}$ of asymptotically nonexpansive type on C in a uniformly convex Banach space with a Fréchet differentiable norm, where G is a right reversible semitopological semigroup. Next, we show that if P is the metric projection of E onto set $F(\mathcal{S})$ of all common fixed points of $\mathcal{S} = \{S(t) : t \in G\}$, then the strong limit of the net $\{Pu(t) : t \in G\}$ exists (Theorem 2). Our proofs employ the methods of Hirano and Takahashi [9], Ishihara and Takahashi [10], Takahashi and park [11], and Takahashi and Zhang [7,8] The results are generalizations of the corresponding results in [5], [7], [8], [11], [12] and [13].

2. PRELIMINARIES

Let E be a real Banach space and let E^* be its dual. The value of $f \in E^*$ at $x \in E$ will be denoted by (x, f) With each $x \in E$, we associate the set

$$J(x) = \{f \in E^* : (x, f) = \|x\|^2 = \|f\|^2\}.$$

Using the Hahn-Banach theorem, it is readily verified that $J(x) \neq \emptyset$ The multivalued mapping $J : E \to E^*$ is called the duality mapping of E. Let $U = \{x \in E : \|x\| = 1\}$ be the unit sphere of E Then a Banach space E is said to be smooth provided the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2 1}$$

exists for each $x, y \in U$. In this case, the norm of E is said to be Gâteaux differentiable. It is said to be Fréchet differentiable if for each $x \in U$, the limit (2.1) is attained uniformly for $y \in U$ It is also known that if E is smooth, then the duality mapping J is single valued. It is easy to see that the norm of E is Fréchet differentiable if and only if for any bounded set $B \subset E$ and any $x \in E$, $\lim_{t \to 0}(2t)^{-1}(\|x + ty\|^2 - \|x\|^2) = (y, J(x))$ uniformly in $y \in E$; see [14].

A Banach space E is called uniformly convex if the modulus of convexity

$$\delta(\epsilon) = \inf\left\{1 - \frac{1}{2}\|x + y\| : \|x\|, \|y\| \leq 1, \|x - y\| \geq \epsilon\right\}$$

is positive in its domain of definition $\{\epsilon : 0 < \epsilon \leq 2\}$. For the properties of $\delta(\epsilon)$, see [15].

For a subset D of E, $\overline{D}$ denotes the closure of E, coD the convex hull of D, and $\overline{co}D$ the closed convex hull of E, respectively.

Let G be a semitopological semigroup, i.e., G is a semigroup with a Hausdorff topology such that for each $a \in G$ the mappings $g \to a \cdot g$ and $g \to g \cdot a$ from G to G are continuous. G is said to be right reversible if any two closed left ideals of G have nonempty intersection. If G is right reversible, $(G, \preceq)$ is a directed system when the binary relation " $\preceq$ " on G is defined by $a \preceq b$ if and only if $\{a\} \cup \overline{Ga} \supseteq \{b\} \cup \overline{Gb}$.

Let C be a nonempty closed convex subset of a Banach space E and let G be a semitopological semigroup. A family $\mathcal{S} = \{S(t) : t \in G\}$ of mappings from C into itself is said to be a (continuous) representation of G on C if $\mathcal{S}$ satisfies the following:

(i) $S(ts)x = S(t)S(s)x$ for all $t, s \in G$ and $x \in C$

(ii) for every $x \in C$, the mappings $s \to S(s)x$ from G into C is continuous.

DEFINITION 1. A representation $\mathcal{S} = \{S(t) : t \in G\}$ of G on C is said to be a semigroup of asymptotically nonexpansive type on C if for each $x \in C$,

$$\inf_{s\in G} \sup_{s\preceq t} \sup_{y\in C} (\|S(t)x - S(t)y\| - \|x - y\|) \leq 0. \tag{2 2}$$

Let G be right reversible and let $\mathcal{S} = \{S(t) : t \in G\}$ be a representation of G on C A function $u : G \to C$ is called an almost-orbit of $\mathcal{S} = \{S(t) : t \in G\}$ if

$$\lim_{s\in G} \left(\sup_{t\in G} \|u(ts) - S(t)u(s)\| \right) = 0. \tag{2 3}$$

$\omega(u)$ denotes the set of all weak limit points of subnets of the net $\{u(t) : t \in G\}$, and $F(\mathcal{S}) = \bigcap_{t\in G} F(S(t))$ the set of all common fixed points of mappings $S(t)$, $t \in G$ in C

3. LEMMAS

In this section, we prove several lemmas which are crucial in convergence of almost-orbits.

LEMMA 1. Let C be a nonempty closed convex subset of a uniformly convex Banach space E and let $\mathcal{S} = \{S(t) : t \in G\}$ be a semigroup of asymptotically nonexpansive type of a right reversible semitopological semigroup G on C. Then $F(\mathcal{S})$ is a closed and convex subset of C.

PROOF. The closedness of $F(\mathcal{S})$ is obvious. To show convexity, it is sufficient to show that $z = \frac{x+y}{2} \in F(\mathcal{S})$ for all $x, y \in F(\mathcal{S})$. Let $x, y \in F(\mathcal{S})$, $x \neq y$ If $\lim_{t\in G} S(t)z = z$, then for any $s \in G$,

$$S(s)z = \lim_{t\in G} S(s)S(t)z = \lim_{t\in G} S(st)z = \lim_{t\in G} S(t)z = z,$$

i.e., $z \in F(\mathcal{S})$. Hence it suffices to prove that $\lim_{t\in G} S(t)z = z$. If not, there exists $\epsilon > 0$ such that for any $t \in G$, there is $t' \in G$ with $t' \succeq t$ and

$$4\|S(t')z - z\| = \|2(S(t')z - x) - 2(y - S(t')z)\| \geq \epsilon.$$

Choose $d > 0$ so small that

$$(R + d)\left(1 - \delta\left(\frac{\epsilon}{R + d}\right)\right) < R,$$

where $R = \|x - y\| > 0$ and δ is the modulus of convexity of E Since $\mathcal{S} = \{S(t) : t \in G\}$ is asymptotically nonexpansive type on C, there is $t_0 \in G$ such that

$$\sup_{t_0\preceq t} \sup_{w\in C} (\|S(t)z - S(t)w\| - \|z - w\|) \leq \frac{d}{2}.$$

Put $u = 2(S(t_0')z - x)$, $v = 2(y - S(t_0')z)$. Then $\|u - v\| = 4\|S(t_0')z - z\| \geq \epsilon$. Furthermore, since $t_0 \preceq t_0'$, we have

$$\begin{aligned}
\|u\| &= 2\|S(t_0')z - x\| \\
&= 2(\|S(t_0')z - S(t_0')x\| - \|z - x\|) + 2\|z - x\| \\
&\leq 2 \sup_{t_0\preceq t} \sup_{w\in C} (\|S(t)z - S(t)w\| - \|z - w\|) + \|x - y\| < R + d
\end{aligned}$$

and

$$\begin{aligned}
\|v\| &= 2\|y - S(t_0')z\| \\
&= 2(\|S(t_0')z - S(t_0')y\| - \|z - y\|) + 2\|z - y\| \\
&\leq 2 \sup_{t_0\preceq t} \sup_{w\in C} (\|S(t)z - S(t)w\| - \|z - w\|) + \|x - y\| < R + d.
\end{aligned}$$

So we have

$$\left\|\frac{u + v}{2}\right\| \leq (R + d)\left(1 - \delta\left(\frac{\epsilon}{R + d}\right)\right),$$

and hence

$$\|x-y\| = \left\|\frac{u+v}{2}\right\| \le (R+d)\Big(1-\delta\Big(\frac{\epsilon}{R-d}\Big)\Big) < R = \|x-y\|.$$

This is a contraction. Therefore, $\lim_{t\in G} S(t)z = z$,which completes the proof

LEMMA 2. Let C be a nonempty closed convex subset of Banach space E Let G be a right reversible semitopological semigroup and let $\mathcal{S} = \{S(t) : t \in G\}$ be a semigroup of asymptotically nonexpansive type on C. If $\{u(t) : t \in G\}$ and $\{v(t) : t \in G\}$ are almost-orbits of $\mathcal{S} = \{S(t) : t \in G\}$, then $\lim_{t\in G}\|u(t) - v(t)\|$ exists. In particular, for every $z \in F(\mathcal{S})$, $\lim_{t\in G}\|u(t) - z\|$ exists

PROOF. Put

$$\phi(s) = \sup_{t\in G}\|u(ts) - S(t)u(s)\|, \quad \psi(s) = \sup_{t\in G}\|v(ts) - S(t)v(s)\|$$

for $s \in G$ Then $\lim_{s\in G}\phi(s) = \lim_{s\in G}\psi(s) = 0$. Let $\epsilon > 0$. Since $\mathcal{S} = \{S(t) : t \in G\}$ is of asymptotically nonexpansive type on C, there exists $t_0 \in G$ such that

$$\sup_{t_0\preceq t}\sup_{w\in C}(\|S(t)u(s) - S(t)w\| - \|u(s) - w\|) < \epsilon$$

for all $s \in G$. On the other hand, since, for any $s, t \in G$,

$$\begin{aligned}\|u(ts) - v(ts)\| &\le \phi(s) + \psi(s) + (\|S(t)u(s) - S(t)v(s)\| - \|u(s) - v(s)\|) + \|u(s) - v(s)\| \\ &\le \phi(s) + \psi(s) + \sup_{w\in C}(\|S(t)u(s) - S(t)w\| - \|u(s) - w\|) + \|u(s) - v(s)\|,\end{aligned}$$

we have

$$\begin{aligned}\inf_{t\in G}\sup_{t\preceq\tau}\|u(\tau) - v(\tau)\| &\le \phi(s) + \psi(s) + \sup_{t_0\preceq t}\sup_{w\in C}(\|S(t)u(s) - S(t)w\| - \|u(s) - w\|) + \|u(s) - v(s)\| \\ &\le \phi(s) + \psi(s) + \epsilon + \|u(s) - v(s)\|,\end{aligned}$$

and then $\inf_{t\in G}\sup_{t\preceq\tau}\|u(\tau) - v(\tau)\| \le \sup_{t\in G}\inf_{t\preceq s}\|u(s) - v(s)\|$ Thus $\lim_{t\in G}\|u(t) - v(t)\|$ exists. Let $z \in F(\mathcal{S})$ and put $v(t) = z$. Then $v(t)$ is an almost-orbit and hence $\lim_{t\in G}\|u(t) - z\|$ exists

LEMMA 3. Let C be a nonempty closed convex subset of Banach space E. Let G be a right reversible semitopological semigroup and let $\mathcal{S} = \{S(t) : t \in G\}$ be a semigroup of asymptotically nonexpansive type on C. Let $\{u(t) : t \in G\}$ be an almost-orbit of $\mathcal{S} = \{S(t) : t \in G\}$ If $F(\mathcal{S}) \ne \emptyset$, then there exists $t_0 \in G$ such that $\{u(t) : t \succeq t_0\}$ is bounded.

PROOF. Let $z \in F(\mathcal{S})$. Then, since $\lim_{t\in G}\|u(t) - z\|$ exists by Lemma 2, there is $t_0 \in G$ such that $\{\|u(t) - z\| : t \succeq t_0\}$ is bounded. Hence $\{u(t) : \succeq t_0\}$ is bounded.

LEMMA 4. Let C be a nonempty closed convex subset of a uniformly convex Banach space E Let G be a right reversible semitopological semigroup and let $\mathcal{S} = \{S(t) : t \in G\}$ be a semigroup of asymptotically nonexpansive type on C. Let $\{u(t) : t \in G\}$ be an almost-orbit of $\mathcal{S} = \{S(t) : t \in G\}$ Suppose that $F(\mathcal{S}) \ne \emptyset$. Let $y \in F(\mathcal{S})$ and $0 < \alpha \le \beta < 1$. Then for any $\epsilon > 0$, there is $t_0 \in G$ such that

$$\|S(t)(\lambda u(s) + (1-\lambda)y) - (\lambda S(t)u(s) + (1-\lambda)y)\| < \epsilon$$

for all $t, s \succeq t_0$ and $\lambda \in [\alpha, \beta]$.

PROOF. By Lemma 2, $\lim_{t\in G}\|u(t) - y\|$ exists. Let $\epsilon > 0$ and

$$r = \lim_{t\in G}\|u(t) - y\|.$$

If $r = 0$, since $\mathcal{S} = \{S(t) : t \in G\}$ is of asymptotically nonexpansive type on C, there exists $t_0 \in G$ such that

$$\sup_{t_0\preceq t}\sup_{w\in C}(\|S(t)(\lambda u(s) + (1-\lambda)y) - S(t)w\| - \|\lambda u(s) + (1-\lambda)y - w\|) < \frac{\epsilon}{2},$$

and

$$\|u(t) - y\| < \frac{\epsilon}{4}$$

for $t \succeq t_0$, $0 < \lambda < 1$ and $s \in G$. Hence for $s, t \succeq t_0$, $0 < \lambda < 1$ and $s \in G$,

$$\begin{aligned}
&\|S(t)(\lambda u(s) + (1-\lambda)y) - (\lambda S(t)u(s) + (1-\lambda)y)\| \\
&\quad \le \lambda\|S(t)(\lambda u(s) + (1-\lambda)y) - S(t)u(s)\| + (1-\lambda)\|S(t)(\lambda u(s) + (1-\lambda)y) - y\| \\
&\quad \le \lambda\Big(\sup_{t_0 \preceq t}\sup_{w \in C}(\|S(t)(\lambda u(s) + (1-\lambda)y) - S(t)w\| - \|\lambda u(s) + (1-\lambda)y - w\|)\Big) \\
&\qquad + \lambda\|\lambda u(s) + (1-\lambda)y - u(s)\| + (1-\lambda)\Big(\sup_{t_0 \preceq t}\sup_{w \in C}(\|S(t)(\lambda u(s) + (1-\lambda)y) - \\
&\qquad S(t)w\| - \|\lambda u(s) + (1-\lambda)y - w\|)\Big) + (1-\lambda)\|\lambda u(s) + (1-\lambda)y - y\| \\
&\quad < \lambda\frac{\epsilon}{2} + (1-\lambda)\frac{\epsilon}{2} + 2\lambda(1-\lambda)\|u(s) - y\| \\
&\quad < \frac{\epsilon}{2} + \frac{\epsilon}{2}(\lambda(1-\lambda)) < \epsilon.
\end{aligned}$$

Now, let $r > 0$. Then we can choose $d > 0$ so small that

$$(r+d)\Big(1 - c\delta\Big(\frac{\epsilon}{r+d}\Big)\Big) = r_0 < r,$$

where δ is the modulus of convexity of E and

$$c = \min\{2\lambda(1-\lambda) : \alpha \le \lambda \le \beta\}.$$

Let $a > 0$ with $2a + r_0 < r$. Then there is $t_0 \in G$ such that

$$r - a < \|u(s) - y\| \le r + \frac{d}{2} \quad \text{for} \quad s \succeq t_0,$$

$$\|S(s)u(t) - u(st)\| < a \quad \text{for} \quad t \succeq t_0 \quad \text{and} \quad s \in G,$$

$$\sup_{t_0 \preceq t}\sup_{w \in C}(\|S(t)z - S(t)w\| - \|z - w\|) < \frac{c}{4}d \quad \text{for} \quad z \in C,$$

and

$$\sup_{t_0 \preceq t}\sup_{w \in C}(\|S(t)u(s) - S(t)w\| - \|u(s) - w\|) < \frac{c}{4}d \quad \text{for} \quad s \in G.$$

Suppose that $\|S(t)(\lambda u(s) + (1-\lambda)y) - (\lambda S(t)u(s) + (1-\lambda)y)\| \ge \epsilon$ for some $s, t \succeq t_0$ and $\lambda \in [\alpha, \beta]$ Put $z = \lambda u(s) + (1-\lambda)y$, $u = (1-\lambda)(S(t)z - y)$ and $v = \lambda(S(t)u(s) - S(t)z)$ Then we have

$$\begin{aligned}
\|u\| &= (1-\lambda)(\|S(t)z - S(t)y\| - \|z - y\|) + (1-\lambda)\|z - y\| \\
&\le (1-\lambda)\sup_{t_0 \preceq t}\sup_{w \in C}(\|S(t)z - S(t)w\| - \|z - w\|) + (1-\lambda)\|\lambda u(s) + (1-\lambda)y - y\| \\
&< (1-\lambda)\frac{c}{4}d + \lambda(1-\lambda)\|u(s) - y\| \\
&\le \lambda(1-\lambda)\Big((1-\lambda)\frac{d}{2} + r + \frac{d}{2}\Big) < \lambda(1-\lambda)(r+d)
\end{aligned}$$

and

$$\begin{aligned}
\|v\| &= \lambda(\|S(t)u(s) - S(t)z\| - \|u(s) - z\|) + \lambda\|u(s) - z\| \\
&\le \lambda\sup_{t_0 \preceq t}\sup_{w \in C}(\|S(t)u(s) - S(t)w\| - \|u(s) - w\|) + \lambda(1-\lambda)\|u(s) - y\| \\
&< \lambda\frac{c}{4}d + \lambda(1-\lambda)\Big(r + \frac{d}{2}\Big) \\
&\le \lambda(1-\lambda)\Big(\lambda\frac{d}{2} + r + \frac{d}{2}\Big) < \lambda(1-\lambda)(r+d).
\end{aligned}$$

We also have that

$$\|u - v\| = \|S(t)z - (\lambda S(t)u(s) + (1-\lambda)y)\| \ge \epsilon$$

and

$$\lambda u + (1-\lambda)v = \lambda(1-\lambda)(S(t)z - y) + (1-\lambda)\lambda(S(t)u(s) - S(t)z)$$
$$= \lambda(1-\lambda)(S(t)u(s) - y).$$

By the Lemma in [16], we have

$$\begin{aligned}\lambda(1-\lambda)\|S(t)u(s) - y\| &= \|\lambda u + (1-\lambda)v\| \\ &\le \lambda(1-\lambda)(r+d)\left(1 - 2\lambda(1-\lambda)\delta\left(\frac{\epsilon}{r+d}\right)\right) \\ &\le \lambda(1-\lambda)(r+d)\left(1 - c\delta\left(\frac{\epsilon}{r+d}\right)\right) = \lambda(1-\lambda)r_0\end{aligned}$$

and hence $\|S(t)u(s) - y\| \le r_0$ This implies that

$$\|u(ts) - y\| \le \|u(ts) - S(t)u(s)\| + \|S(t)u(s) - y\| < a + r_0 < r - a.$$

This contradicts the fact $\|u(s) - y\| > r - a$ for $s \succeq t_0$. The proof is complete

LEMMA 5. Let C be a nonempty closed convex subset of a uniformly convex Banach space E Let G be a right reversible semitopological semigroup and let $\mathcal{S} = \{S(t) : t \in G\}$ be a semigroup of asymptotically nonexpansive type on C. Let $\{u(t) : t \in G\}$ be an almost-orbit of $\mathcal{S} = \{S(t) : t \in G\}$. Suppose that $F(\mathcal{S}) \neq \emptyset$. Then $\lim_{t\in G}\|\lambda u(t) + (1-\lambda)x - y\|$ exists for every $x, y \in F(\mathcal{S})$

PROOF. Let $\lambda \in (0,1)$ and $x, y \in F(\mathcal{S})$. By (2.2), (2.3), and Lemma 4, for any $\epsilon > 0$, there exists $t_0 \in G$ such that

$$\|S(t)(\lambda u(s) + (1-\lambda)x) - (\lambda S(t)u(s) + (1-\lambda)x)\| \le \frac{\epsilon}{3} \quad \text{for} \quad t, s \succeq t_0,$$
$$\sup_{t\in G}\|u(ts) - S(t)u(s)\| < \frac{\epsilon}{3} \quad \text{for} \quad s \succeq t_0,$$
$$\sup_{t_0 \preceq t}\sup_{w\in C}(\|S(t)(\lambda u(s) + (1-\lambda)x) - S(t)w\| - \|\lambda u(s) + (1-\lambda)x - w\|) < \frac{\epsilon}{3} \quad \text{for} \quad s \in G.$$

Since

$$\begin{aligned}&\|\lambda u(ts) + (1-\lambda)x - y\| \\ &\quad \le \lambda\|u(ts) - S(t)u(s)\| + \|\lambda S(t)u(s) + (1-\lambda)x - S(t)(\lambda u(s) + (1-\lambda)x)\| \\ &\qquad + \sup_{w\in C}(\|S(t)(\lambda u(s) + (1-\lambda)x) - S(t)w\| - \|\lambda u(s) + (1-\lambda)x - w\|) \\ &\qquad + \|\lambda u(s) + (1-\lambda)x - y\| \\ &\quad < \epsilon + \|\lambda u(s) + (1-\lambda)x - y\|\end{aligned}$$

for all $t, s \in G$, we have

$$\begin{aligned}\inf_{t\in G}\sup_{t\preceq\tau}\|\lambda u(\tau) + (1-\lambda)x - y\| &\le \sup_{t_0\preceq t}\|\lambda(u(ts) + (1-\lambda)x - y\| \\ &\le \epsilon + \|\lambda u(s) + (1-\lambda)x - y\|\end{aligned}$$

for all $s \succeq t_0$, and then

$$\inf_{t\in G}\sup_{t\preceq\tau}\|\lambda u(\tau) + (1-\lambda)x - y\| \le \sup_{t\in G}\inf_{t\preceq s}\|\lambda u(s) + (1-\lambda)x - y\|.$$

Thus $\lim_{t\in G}\|\lambda u(t) + (1-\lambda)x - y\|$ exists.

LEMMA 6. Let C be a nonempty closed convex subset of a uniformly convex Banach space E with a Fréchet differentiable norm. Let G be a right reversible semitopological semigroup and let $\mathcal{S} = \{S(t) : t \in G\}$ be a semigroup of asymptotically nonexpansive type on C. Let $\{u(t) : t \in G\}$ be an almost-orbit of $\mathcal{S} = \{S(t) : t \in G\}$. Then

$$F(\mathcal{S}) \cap \bigcap_{s\in G} \overline{co}\,\{u(t) : t \succeq s\}$$

is at most a singleton.

PROOF. Note that $\bigcap_{s\in G}\overline{co}\{u(t) : t \succeq s\} = \overline{co}\,\omega(u)$, see [17] Let $x, y \in F(\mathcal{S})$ Since E has a Fréchet differentiable norm, there exists an increasing function $\gamma : R^+ \to R^+$ such that $\gamma(t)/t \to 0$ as $t \to 0^+$, and

$$\begin{aligned}\frac{1}{2}\|x-y\|^2 + (h, J(x-y)) &\le \frac{1}{2}\|x-y+h\|^2 \\ &\le \frac{1}{2}\|x-y\|^2 + (h, J(x-y)) + \gamma(\|h\|)\end{aligned}$$

for all $h \in E$. Take $h = \lambda(u(t) - x)$ Then

$$\begin{aligned}\frac{1}{2}\|x-y\|^2 + \lambda(u(t)-x, J(x-y)) &\le \frac{1}{2}\|\lambda u(t) + (1-\lambda)x - y\|^2 \\ &\le \frac{1}{2}\|x-y\|^2 + \lambda(u(t)-x, J(x-y)) + \gamma(\lambda\|u(t)-x\|).\end{aligned}$$

Using Lemma 5, we have

$$\begin{aligned}&\frac{1}{2}\|x-y\|^2 + \lambda \inf_{t\in G}\sup_{t\preceq\tau}(u(\tau)-x, J(x-y)) \\ &\le \frac{1}{2}\lim_{t\in G}\|\lambda u(t) + (1-\lambda)x - y\|^2 \\ &\le \frac{1}{2}\|x-y\|^2 + \lambda \sup_{t\in G}\inf_{t\preceq\tau}(u(\tau)-x, J(x-y)) + \gamma(\lambda M),\end{aligned}$$

where $\sup_{t\in G}\|u(t)-x\| = M$. Dividing by λ and letting $\lambda \to 0^+$, we have $\lim_{t\in G}(u(t), J(x-y)) = r$ exists. Of course $r = (v, J(x-y))$ for all $v \in \omega(u)$ and hence for all $v \in \overline{co}\ \omega(u)$ Therefore $(v - w, J(x-y)) = 0$ for all $v, w \in \overline{co}\,\omega(u)$, and it readily follows that $F(\mathcal{S}) \cap \bigcap_{s\in G}\{u(t) : t \succeq s\} = F(\mathcal{S}) \cap \overline{co}\,\omega(u)$ is at most a singleton.

4. MAIN RESULTS

In this section, we study the convergence of an almost-orbit $\{u(t) : t \in G\}$ of $\mathcal{S} = \{S(t) : t \in G\}$

THEOREM 1. Let E be a uniformly convex Banach space with a Fréchet differentiable norm and let C be a nonempty closed convex subset of E Let F be a subset of C and let G be a right reversible semitopological semigroup. Let $\mathcal{S} = \{S(t) : t \in G\}$ be a semigroup of asymptotically nonexpansive type on C and let $\{u(t) : t \in G\}$ be an almost-orbit of $\mathcal{S} = \{S(t) : t \in G\}$. Assume that

(a) $F \subset F(\mathcal{S})$.

Assume also that

(b) if a subnet $\{u(t_\alpha)\}$ of the net $\{u(t) : t \in G\}$ converges weakly to z, then $z \in F$.

Then either (i) $F = \emptyset$ and $\|u(t)\| \to \infty$ or (ii) $F \neq \emptyset$ and the net $\{u(t) : t \in G\}$ converges weakly to some $z \in F(\mathcal{S})$.

PROOF. Suppose that some subnet $\{u(t_\alpha)\}$ of $\{u(t) : t \in G\}$ is bounded. Since E is reflexive, a subnet of $\{u(t_\alpha)\}$ must converge weakly to an element $z \in E$, which is in F by (b). Thus $F = \emptyset$ implies $\|u(t)\| = \infty$.

If, on the other hand, $F \neq \emptyset$, then by Lemma 3, $\{u(t) : t \in G\}$ is bounded. So $\{u(t) : t \in G\}$ must contain a subnet $\{u(t_\alpha)\}$ which converges to some $z \in F$ by (b) Since $F \subset F(\mathcal{S})$ and $z \in \overline{co}\ \omega(u) = \bigcap_{s\in G}\overline{co}\{u(t) : t \in G\}$, we have

$$z \in F \cap \bigcap_{s\in G}\overline{co}\{u(t) : t \succeq s\} \subset F(\mathcal{S}) \cap \bigcap_{s\in G}\overline{co}\{u(t) : t \succeq s\}.$$

Therefore it follows from Lemma 6 that $\{u(t) : t \in G\}$ converges weakly to $z \in F(\mathcal{S})$.

As a direct consequence, we have the following corollary, which is a generalization of a result in [5], [7], [8], [11], [12] and [13]

COROLLARY 1. Let E be a uniformly convex Banach space with a Fréchet differentiable norm and let C be a nonempty closed convex subset of E Let G be a right reversible semitopological semigroup and let $\mathcal{S} = \{S(t) : t \in G\}$ be a semigroup of asymptotically nonexpansive type on C Suppose that $F(\mathcal{S}) \neq \emptyset$ and let $\{u(t) : t \in G\}$ be an almost-orbit of $\mathcal{S} = \{S(t) : t \in G\}$ If $\omega(u) \subset F(\mathcal{S})$, then the net $\{u(t) : t \in G\}$ converges weakly to some $z \in F(\mathcal{S})$

PROOF. The result follows by putting $F = \omega(u)$ in Theorem 1

The following theorem is also a generalization of [7, Theorem 4]

THEOREM 2. Let C be a nonempty closed convex subset of a uniformly convex Banach space E Let G be a right reversible semitopological semigroup and let $\mathcal{S} = \{S(t) : t \in G\}$ be a semigroup of asymptotically nonexpansive type on C Suppose that $F(\mathcal{S}) \neq \emptyset$ and let $\{u(t) : t \in G\}$ be an almost-orbit of $\mathcal{S} = \{S(t) : t \in G\}$ Let P denote the metric projection of E onto $F(\mathcal{S})$ Then the strong limit of the net $\{Pu(t) : t \in G\}$ exists and $\lim_{t\in G} Pu(t) = z_0$, where z_0 is a unique element of $F(\mathcal{S})$ such that

$$\lim_{t\in G} \|u(t) - z_0\| = \min\left\{\lim_{t\in G} \|u(t) - z\| : z \in F(\mathcal{S})\right\}.$$

PROOF. Since $F(\mathcal{S}) \neq \emptyset$, we know that $\{u(t) : t \in G\}$ is bounded and $\lim_{t\in G}\|u(t) - z\| = g(z)$ exists for each $z \in F(\mathcal{S})$. Let $R = \inf\{g(z) : z \in F(\mathcal{S})\}$ and $M = \{u \in F(\mathcal{S}) : g(u) = R\}$ Then, since $g(z)$ is convex and continuous on $F(\mathcal{S})$ and $g(z) \to \infty$ as $\|z\| \to \infty$, M is a nonempty closed convex bounded subset of $F(\mathcal{S})$. Fix $z_0 \in M$ with $g(z_0) = R$. Since P is the metric projection of E ontp $F(\mathcal{S})$, we have $\|u(t) - Pu(t)\| \leq \|u(t) - y\|$ for all $t \in G$ and $y \in F(\mathcal{S})$, and hence

$$\inf_{t\in G} \sup_{t\preceq s} \|u(s) - Pu(s)\| \leq R.$$

Suppose that $\inf_{t\in G} \sup_{t\preceq s} \|u(s) - Pu(s)\| < R$. Then we may choose $\epsilon > 0$ and $t_0 \in G$ such that

$$\|u(s) - Pu(s)\| \leq R - \epsilon$$

$$\sup_{t_0\preceq t} \sup_{w\in C} (\|S(t)u(s) - S(t)w\| - \|u(t) - w\|) < \frac{\epsilon}{4},$$

and

$$\sup_{t\in G} \|u(ts) - S(t)u(s)\| < \frac{\epsilon}{4}$$

for all $s \succeq t_0$. Since

$$\begin{aligned}\|u(ts) - Pu(s)\| &\leq \|u(ts) - S(t)u(s)\| + \|S(t)u(s) - S(t)Pu(s)\| \\ &\quad - \|u(s) - Pu(s)\| + \|u(s) - Pu(s)\| \\ &\leq \phi(s) + \sup_{w\in C} (\|S(t)u(s) - S(t)w\| - \|u(s) - w\|) + \|u(s) - Pu(s)\|\end{aligned}$$

for all s, $t \in G$ and $\lim_{s\in G} \phi(s) = 0$, where $\phi(s) = \sup_{t\in G} \|u(ts) - S(t)u(s)\|$, we have

$$\|u(ts) - Pu(s)\| < \frac{\epsilon}{4} + \frac{\epsilon}{4} + r - \epsilon = R - \frac{\epsilon}{2}$$

for $s \succeq t_0$ and all $t \in G$. Therefore, we obtain

$$\lim_{t\in G} \|u(t) - Pu(s)\| = \inf_{t\in g} \sup_{t\preceq\tau} \|u(\tau) - Pu(s)\| \leq R - \frac{\epsilon}{2} < R.$$

This is a contradiction. So we conclude that

$$\inf_{t\in G} \sup_{t\preceq s} \|u(s) - Pu(s)\| = R.$$

Now we claim that $\lim_{t\in G} Pu(t) = z_0$. If not, then there exists $\epsilon > 0$ such that for any $t \in G$, $\|Pu(t') - z_0\| \geq \epsilon$ for some $t' \succeq t$. Choose $a > 0$ so small that

$$(R + a)\left(1 - \delta\left(\frac{\epsilon}{R + a}\right)\right) = R_1 < R,$$

where δ is the modulus of convexity of the norm of E We have $\|u(t') - Pu(t')\| \leq R + a$ and $\|u(t') - z_0\| \leq R + a$ for large enough t'. Therefore

$$\left\| u(t') - \frac{Pu(t') + z_0}{2} \right\| \leq (R+a)\left(1 - \delta\left(\frac{\epsilon}{R+a}\right)\right) = R_1.$$

Since, by Lemma 1, the point $w_{t'} = \frac{Pu(t')+z_0}{2}$ belongs to $F(\mathcal{S})$, as in the above,

$$\|u(tt') - w_{t'}\| \leq \phi(t') + \sup_{w \in C} (\|S(t)u(t') - S(t)w\| - \|u(t') - w\|) + \|u(t') - w_{t'}\|.$$

Since $\lim_{s \in G} \phi(s) = 0$, there is $t' \in G$ such that

$$\phi(t') < \frac{R - R_1}{4}$$

and

$$\sup_{t' \preceq t} \sup_{w \in C} (\|S(t)u(t') - S(t)w\| - \|u(t') - w\|) < \frac{R - R_1}{4},$$

and hence

$$\lim_{t \in G} \|u(t) - w_{t'}\| = \inf_{\tau \in G} \sup_{t \preceq \tau} \|u(\tau) - w_{t'}\| < \frac{R - R_1}{2} + R_1 = \frac{R + R_1}{2} < R.$$

This contradicts the fact $R = \inf\{g(z) : z \in F(\mathcal{S})\}$ Thus we have $\lim_{t \in G} Pu(t) = z_0$ Consequently, it follows that the element $z_0 \in F(\mathcal{S})$ with $g(z_0) = \min\{g(z) : z \in F(\mathcal{S})\}$ is unique. The proof is complete

By Corollary 1 and Theorem 2, we have the following, which is an improvement of [8, Theorem 3] and [5, Theorem 3.3].

COROLLARY 2. Let C be a nonempty closed convex subset of a real Hilbert space H Let G be a right reversible semitopological semigroup and $\mathcal{S} = \{S(t) : t \in G\}$ be a semigroup of asymptotically nonexpansive type on C. Suppose that $F(\mathcal{S}) \neq \emptyset$. Let $\{u(t) : t \in G\}$ be an almost-orbit of $\mathcal{S} = \{S(t) : t \in G\}$. Then $\{u(t) : t \in G\}$ converges weakly to some $z \in C$ if and only if $u(ht) - u(t)$ converges weakly to 0 for all $h \in G$. In this case, $z \in F(\mathcal{S})$ and $\lim_{t \in G} Pu(t) = z$

PROOF. We need only prove the "if" part. By Corollary 1, it suffices to show that $\omega(u) \subset F(\mathcal{S})$ Let $\{u(t_\alpha)\}$ be a subnet of $\{u(t) : t \in G\}$ converging weakly to $y \in C$ Given $\epsilon > 0$ Since $\mathcal{S}$ is of asymptotically nonexpansive type and $\{u(t_\alpha)\}$ is bounded, there exists $t_0 \in G$ such that for any α,

$$\sup_{t_0 \preceq t} \sup_{w \in C} (\|S(t)u(t_\alpha) - S(t)w\| - \|u(t_\alpha) - w\|) < \epsilon.$$

So we have, for $t \succeq t_0$ and any α,

$$\begin{aligned} &\|S(t)u(t_\alpha) - S(t)y\|^2 - \|u(t_\alpha) - y\|^2 \\ &\quad = (\|S(t)u(t_\alpha) - S(t)y\| - \|u(t_\alpha) - y\|)(\|S(t)u(t_\alpha) - S(t)y\| + \|u(t_\alpha) - y\|) \\ &\quad \leq \sup_{t_0 \preceq t} \sup_{w \in C} (\|S(t)u(t_\alpha) - S(t)w\| - \|u(t_\alpha) - w\|)\Big(\sup_{t_0 \preceq t} \sup_{w \in C} (\|S(t)u(t_\alpha) - S(t)w\| \\ &\qquad\qquad - \|u(t_\alpha) - w\|) + 2\|u(t_\alpha) - y\|\Big) \\ &\quad < \epsilon(\epsilon + 2M), \end{aligned}$$

where $M = \sup_\alpha \|u(t_\alpha) - y\|$. Let $u \in F(\mathcal{S})$ and $\epsilon' = \epsilon(\epsilon + 2M)$. Then we have, for $t \succeq t_0$ and all α,

$$\begin{aligned} -\epsilon' &< \|u(t_\alpha) - y^2\| - \| - S(t)u(t_\alpha)y\|^2 \\ &= \|u(t_\alpha) - u\|^2 + 2(u(t_\alpha) - u, u - y) + \|u - y\|^2 \\ &\quad - \|S(t)u(t_\alpha) - u\|^2 - 2(S(t)u(t_\alpha) - u, u - S(t)y) - \|u - S(t)y\|^2 \\ &= \|u(t_\alpha) - u\|^2 - \|S(t)u(t_\alpha) - u\|^2 + \|u - y\|^2 - \|u - S(t)y\|^2 \\ &\quad + 2(u(t_\alpha) - u, S(t)y - y) + 2(u(t_\alpha) - S(t)u(t_\alpha), u - S(t)y). \end{aligned}$$

Since $\{u(t) : t \in G\}$is an almost-orbit of $\mathcal{S} = \{S(t) : t \in G\}$ and $u(hs) - u(s)$ converges weakly to 0 for all $h \in G$, it follows that

$$\lim_\alpha \|S(t)u(t_\alpha) - u\|^2 = \lim_\alpha \|u(tt_\alpha) - u\|^2 = \lim_\alpha \|u(t_\alpha) - u\|^2$$

and

$$u(t_\alpha) - S(t)u(t_\alpha) = u(t_\alpha) - u(tt_\alpha) \to 0 \text{ weakly.}$$

Thus we have

$$-\epsilon' \le 2(y-u, S(t)y-y) + \|y-u\|^2 - \|u-S(t)y\|^2 = -\|y-S(t)y\|^2$$

for $t \succeq t_0$, and hence $\limsup_{t\in G}\|S(t)y-y\| \le \epsilon'$. Since ϵ' is arbitrary, we have $\lim_{t\in G} S(t)y = y$ Now, for $s \in G$,

$$S(s)y = \lim_{t\in G} S(s)S(t)y = \lim_{t\in G} S(st)y = \lim_{t\in G} S(t)y = y,$$

i e, $y \in F(\mathcal{S})$ and hence $\omega(u) \subset F(\mathcal{S})$ By Corollary 1, the net $\{u(t) : t \in G\}$ converges weakly to some $z \in F(\mathcal{S})$ On the other hand, since P is the metric projection of H onto $F(\mathcal{S})$, we know that

$$(u(t) - Pu(t), Pu(t) - y) \ge 0$$

for all $y \in F(\mathcal{S})$. So, if $Pu(t) \to u$ by Theorem 2, we have $(z-u, u-y) \ge 0$ for all $y \in F(\mathcal{S})$ Putting $z = y$, we obtain $-\|z-u\|^2 \ge 0$ and hence $z = u$.

As a direct consequence, we have the following·

COROLLARY 3. Let C be a nonempty closed convex subset of a real Hilbert space H Let G be a right reversible semitopological semigroup and let $\mathcal{S} = \{S(t) : t \in G\}$ be a semigroup of asymptotically nonexpansive type on C. Suppose that $F(\mathcal{S}) \neq \emptyset$ Let $\{u(t) : t \in G\}$ be an almost-orbit of $\mathcal{S} = \{S(t) : t \in G\}$. If $\lim_{t\in G}\|u(ht) - u(t)\| = 0$ for all $h \in G$, then the net $\{u(t) : t \in G\}$ converges weakly to some $z \in F(\mathcal{S})$.

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