Internat. J. Math. & Math. Sci.
VOL. 20 NO. 1 (1997) 137-146

NONLINEAR DELAY-DIFFERENTIAL EQUATIONS WITH SMALL LAG

MANUEL PINTO
Departamento de Matemáticas
Facultad de Ciencias
Universidad de Chile
Casilla 653
Santiago, CHILE

(Received October 4, 1994 and in revised form April 6,1995)

ABSTRACT. Asymptotic formulae for the solutions of nonlinear functional differential system are obtained.

KEY WORDS AND PHRASES. Asymptotic behavior, functional differential equations.
1991 AMS SUBJECT CLASSIFICATION CODES, 34K10.

1 Introduction

Let $q \geq 0$ be a constant and let $C_0 = C([-q,0],\mathbf{R}^n)$ be the Banach space of continuous functions $\varphi : [-q,0] \to \mathbf{R}^n$ equipped with the norm

$$\|\varphi\| = sup_{-q\leq s\leq 0}|\varphi(s)|.$$

For $y \in C([t-q,t],\mathbf{R}^n)$, we denote by y_t the element of C_0 defined by

$$y_t(s) = y(t+s), \quad -q \leq s \leq 0.$$

We will also denote, for $y \in C([t-2q,t],\mathbf{R}^n), y^t$ the functional defined by

$$y^t(s) = y(t+s), \quad -2q \leq s \leq 0$$

for which we consider the norm:

$$\|\varphi\|_2 = sup_{-2q\leq s\leq 0}|\varphi(s)|.$$

Consider $F : [0,\infty) \times C_0 \to \mathbf{R}^n$ and $g : [0,\infty)\times\mathbf{R}^n \to \mathbf{R}^n$ two continuous functions satisfying the "closeness" condition

(C) There exists a continuous function $\lambda : [0,\infty) \to [0,\infty)$ such that

$$|F(t,\varphi) - g(t,\varphi(0))| \leq \lambda(t)\|\varphi'\| \tag{1.1}$$

for any continuously differentiable function $\varphi : [-q,0] \to \mathbf{R}^n$.

We Remark that (1.1) holds with $\|\varphi'\|$ and not with $\|\varphi\|$. See [15].

We wish to study the relation between the solutions of the functional differential system

$$y'(t) = F(t, y_t) \tag{1.2}$$

and the solutions of the ordinary differential system

$$x'(t) = g(t, x(t)) \tag{1.3}$$

For system (1.3) we suppose that the following condition is fulfilled:

(G) The derivative of $g : g_x = g_x(t,x)$ exists and is continuous on $[0,\infty) \times \mathbf{R}^n$. System (1.3) is an h-system in variation with radius of attraction δ, where $h : [0,\infty) \to (0,\infty)$ is a continuous function.

We recall that a system (1.3) or its null-solution is an h-system in variation [5, 6] with radius of attraction δ, if there exist a continuous function $h : [0,\infty) \to (0,\infty)$ and constants $K \geq 1$ and $\delta > 0$ such that for $0 \leq |x_0| < \delta$ we have

$$|\Phi(t, t_0, x_0)| \leq K h(t) h(t_0)^{-1} \qquad (t \geq t_0 \geq 0),$$

where $\Phi(t, t_0, x_0)$ is the fundamental matrix of the variational system

$$z'(t) = g_x(t, x(t, t_0, x_0)) z(t)$$

such that $\Phi(t, t_0, x_0) = Id$ (the identity matrix). Here $x = x(t, t_0, x_0)$ represents the solution x passing throught the point (t_0, x_0).

This problem appears in Bellman [1] who proposed to investigate conditions on the lag r to know the behavior of solutions of the functional differential equation

$$u'(t) + a u(t - r(t)) = 0, \qquad a \text{ constant} \tag{1.4}$$

when $r(t) \to 0$ as $t \to \infty$. In [2], Cooke proves that if $a > 0$ and $r \in L_1([0,\infty))$ then any solution u of (1.1) satisfies

$$u(t) = e^{-at}[c + o(1)], \qquad t \to \infty$$

for some constant c. In [3], Cooke generalizes this result to linear systems of functional differential equations asymptotically autonomous. Grossman and Yorke [4] consider the one-dimensional functional differential equation

$$u'(t) = a(t) u(t - r(t)).$$

In [10] we have extended some of these results to the scalar functional equation

$$u'(t) = a(t) u(t - r(t, u))$$

with a lag of implicit type, generalizing the case

$$u'(t) = -au(t - r(u(t)))$$

studied by Cooke [4]. See also [12, 14]. We note that in all of these cases the estimate (1.1) does not hold with $\|\varphi\|$ instead of $\|\varphi'\|$.

In this paper, for the nonlinear problem (1.2), we obtain the relation

$$y = x + h \cdot \tilde{o}(1),$$

between the solutions y of (1.2) and x of (1.3), where $\tilde{o}(1)$ is a convergent function as $t \to \infty$. We will prove also that the nonlinear functional system (1.2) is an h-system (see Remark 1).

As an application we get asymptotic formulae of the solutions of second order delay equation [11, 13]

$$y'' + c(t)y(t - r(t)) = 0 \tag{1.5}$$

in terms of the solutions of

$$z'' + c(t)z = 0, \tag{1.6}$$

extending ordinary results [7, 8].

2 Main Results

In this section we get asymptotic formulae for the solutions of system (1.2). We denote by $y = y(t; t_0, y_{t_0})$ a solution y of Eq. (1.2) with initial function $y_{t_0} \in C_0$.

Theorem 1 *In addition to conditions (C) and (G),assume:*

(i) There exists a continuous and nonnegative function $c(t)$ such that

$$|F(t, \varphi)| \leq c(t)\|\varphi\|$$

for all $t \geq 0$ and all $\varphi \in C_0$.

(ii) $\beta(t)\lambda(t)\|c_t\| \in L_1([0, \infty))$, where $\beta(t) = h(t)^{-1}\|h^t\|_2$.

Then for any solution $y = y(t; t_o, y_{t_0})$ of (1.2) with $\|y_{t_0}\| \leq \delta$ there exists a solution x of (1.3) such that

$$y = x + h \cdot \tilde{o}(1),$$

where $\tilde{o}(1)$ is a function defined on $[t_0, \infty)$ which converges as $t \to \infty$.

Proof. By condition (G), for $|y(t_0)| \leq \delta$, the solution $x = x(t; t_0, y(t_0))$ of the ordinary system (1.3) is well defined and satisfies $|x(t; t_0, y(t_0))| \leq K|y(t_0)|h(t)h(t_0)^{-1}$ for $t \geq t_0 \geq 0$ and $K \geq 1$ a constant. Now, by (i), the solution $y = y(t, t_0, y_{t_0})$ of system (1.2) is defined on $[t_0 - q, \infty)$. By the formula of variation of the constants, we have for $t \geq t_1 \geq t_0$

$$y(t) = x(t; t_1, y(t_1)) + \int_{t_1}^{t} \Phi(t, s, y(s))[F(s, y_s) - g(s, y(s))]ds \tag{2.1}$$

Then, by (C) and (G)

$$|y(t)| \leq K|y(t_1)|h(t)h(t_1)^{-1} + Kh(t)\int_{t_1}^{t} h(s)^{-1}\lambda(s)\|y_s'\|ds$$

or

$$h(t)^{-1}|y(t)| \leq Kh(t_1)^{-1}|y(t_1)| + K\int_{t_1}^{t} \lambda(s)h(s)^{-1}\|y_s'\|ds.$$

Thus $z(t) = h(t)^{-1}|y(t)|$ satisfies

$$z(t) \leq Kz(t_1) + \int_{t_1}^{t} K\lambda(s)h(s)^{-1}\|y_s'\|ds \tag{2.2}$$

For $u \in [-q, 0]$ and $s \geq t_1$, by (i), we have

$$|y_s'(u)| = |F(s+u, y_{s+u})| \leq c_s(u)\|y_{s+u}\| = c_s(u)|y(v)|$$

for some $v = v(s) \in [s - 2q, s]$. Further

$$h(s)^{-1}|y(v)| = h(s)^{-1}h(v)z(v) \leq \beta(s)z(v).$$

Thus

$$h(s)^{-1}\|y_s'\| \leq \beta(s)\|c_s\|m(s), \tag{2.3}$$

where $m(t) = \max_{t-2q \leq s \leq t} |z(s)|$. Substituting this into (2.2) we obtain

$$z(t) \leq Kz(t_1) + \int_{t_1}^{t} K\lambda(s)\beta(s)\|c_s\|m(s)ds. \tag{2.4}$$

Since the right member of (2.4) is increasing as a function in t, for $t \geq t_1 + 2q$ we have $m(t) \leq Kz(t_1) + \int_{t_1}^{t} Kr(s)\beta(s)\|c_s\|m(s)ds$. Then by (ii), Gronwall's inequality implies that m and hence z are bounded. Moreover, for any t fixed $\Phi(t, s, y(s))[F(s, y_s) - g(s, y(s))] \in L_1([0, \infty))$ as a function of s because from (C), (G), (ii) and (2.3) we get

$$\begin{aligned} |\Phi(t, s, y(s))[F(s, y_s) - g(s, y(s))]| &\leq Kh(t)h(s)^{-1}\lambda(s)\|y_s'\| \\ \leq K_1h(t)\|c_s\|\lambda(s)\beta(s)m(s) &\leq K_2h(t)\lambda(s)\beta(s)\|c_s\| \in L_1([0, \infty)). \end{aligned}$$

Then the integral in (2.1) can be written as $h(t) \cdot \tilde{o}(1)$, where $\tilde{o}(1)$ denotes a function of t which has a limit as $t \to \infty$. The proof is complete.

Remark 1. Since we have proved $h(t)^{-1}|y(t)| \leq m(t) \leq KK_1|z(t_1)| = KK_1h(t_1)^{-1}|y(t_1)|$ for $t \geq t_1 \geq t_0$ and K_1 a positive constant, we have also established

$$|y(t)| \leq K_2h(t)h(t_1)^{-1}|y(t_1)|, (t \geq t_1 \geq t_0), K_2 \text{ constant}$$

that is, the nonlinear functional system (1.2) is also an h-system.

Theorem 1 includes the interesting type of equations as:

$$y' = F(t, y(t) - y(t - r(t))), \tag{2.5}$$

where $r : [0, \infty) \to [0, q]$ is a continuous function.

For this equation, system (1.3) becomes $x' = 0$ and (1.1) becomes

$$|F(t, \varphi)| \leq r(t)\|\varphi'\| \tag{2.6}$$

Thus here $h \equiv 1$, $\beta \equiv 1$ and we have

Corollary 1 *Assume that (i) of Theorem 1 and (2.6) hold. If* $r(t) \cdot \|c_t\| \in L_1([0, \infty))$, *then any solution* $y = y(t; t_0, y_{t_0})$ *of (2.5) there exists a constant vector* v *such that*

$$y = y(t_0) + v + o(1)$$

as $t \to \infty$. *In particular, any solution of (2.5) is asymptotically constant.*

Proceeding as in the proof of the Theorem 1, with a Bihari's inequality, Corollary 1 can be obtained for the nonlinear equations

$$y' = y^3(t) - y^3(t - r(t)) \quad \text{or} \quad y' = [y(t) - y(t - r(t))]^3$$

since in this case we have an estimate of the type:

$$|F(t, g)| \leq Kr(t)w(\|g'\|), \tag{2.7}$$

where $w : (0, \infty) \to (0, \infty)$ is a continuous, nondecreasing function satisfying $w(0) \geq 0$ and

$$\int_{0+}^{1} \frac{ds}{w(s)} = \infty \tag{2.8}$$

Thus from lemma 1, [6] we obtain:

Corollary 2 *Under the conditions of Corollary 1 with (2.7-2.8) instead of (2.6), there exists a constant* $\rho > 0$ *such that any solution* $y = y(t; t_0, y_{t_0})$ *with* $\|y_{t_0}\| \leq \rho$ *is defined on* $[t_0 - q, \infty)$ *and*

$$y = y(t_0) + v(t_0) + o(1), \qquad t \to \infty \tag{2.9}$$

where $v = v(t_0)$ *is a constant vector such that* $v(t_0) \to 0$ *as* $t_0 \to \infty$. *Moreover,* $\rho = \rho(t_0)$ *verifies* $\rho(t_0) \to \infty$ *as* $t_0 \to \infty$. *Then if* t_0 *is chosen large enough for any initial function* φ *there exists* t_0 *large enough such that the solution* $y = y(t, t_0, \varphi)$ *verifies the above asymptotic formulae.*

Some simple consequences are the following:

Corollary 3 *If, for* $h(t) = exp(\int_0^t a(s)ds)$, $a\|a_t\|h(t)^{-1}\|h^t\|_2 r \in L_1([0,\infty))$, *then the solutions of the scalar equation*

$$y'(t) = a(t)y(t - r(t)),$$

satisfy

$$y(t) = exp(\int_0^t a(s)ds)[c + o(1)], \qquad c \text{ constant.}$$

Thus, in particular, the solutions of

$$y'(t) = -ty(t - e^{-3t})$$

and

$$y'(t) = ty(t - r(t)), t^2 r(t) \in L_1([0,\infty)),$$

satisfy respectively,

$$y = e^{-t^2/2}[c + o(1)], \qquad c \text{ constant}$$

and

$$y = e^t[c + o(1)], \qquad c \text{ constant.}$$

Now, an explicite nonlinear scalar example is shown. Let $g(t,x) = -e^t x^3$ in equation (1.3):

$$x'(t) = -e^t x^3(t)$$

This ordinary system has the solutions

$$x(t, t_0, x_0) = \frac{|x_0|}{(1 + 2x_0^2(e^t - e^{t_0}))^{1/2}}$$

whence it is an h-system with $h(t) = e^{-t/2}$. Then, Theorem 1 implies that the solutions $y = y(t, t_0, y_{t_0})$ of the scalar equation

$$y'(t) = -e^t y^3(t - e^{-\alpha t}), \alpha > 2,$$

satisfy

$$y(t) = x(t) + e^{-t/2} \cdot \tilde{o}(1),$$

for t large enough.

Corollary 4 *If A is a stable matrix, then any solution of*

$$y' = Ay(t - r(t)), \quad r \in L_1([0,\infty))$$

satisfies

$$y = e^{tA}x_0 + e^{-\alpha t} \cdot \tilde{o}(1)$$

where x_0 is a constant vector, $0 > \alpha > maxRe\lambda$ for λ an eigenvalue of A and $\tilde{o}(1)$ is a convergent vector as $t \to \infty$.

When (1.3) is a linear and an h-system (see [6]) we have:

Corollary 5 *If system*

$$x' = A(t)x \tag{2.10}$$

is an h-system and $rh(t)^{-1}\|h^t\|_2\|A\|\|A_t\| \in L_1([0,\infty))$, then for any solution y of

$$y' = A(t)y(t - r(t))$$

satisfies

$$y = \Phi y_0 + h\tilde{o}(1) \quad as \quad t \to \infty$$

where y_0 is a constant vector and Φ is a fundamental matrix of (2.10).

3 An application: Asymptotic formulae for the solutions of (1.5)

Consider the functional differential equation

$$y'' + c(t)y(t - r(t)) = 0 \tag{3.1}$$

where $c : [0,\infty) \to \mathbf{R}$ and $r : [0,\infty) \to [0,q]$ are continuous functions.

As usually, a solution of eq. (3.1) is a function $y = y(t; t_0, \varphi, \psi)$ such that y satisfies the delay-differential equations (3.1) and

$$y_{t_0} = \varphi, \quad y'_{t_0} = \psi,$$

where $\varphi, \psi \in C([-q,0], \mathbf{R})$.

For $r = r(t)$ small, in some sense which will be precised, we hope that the solutions y of Eq (3.1) behave asymptotically as the solutions z of the ordinary differential equation

$$z'' + c(t)z(t) = 0. \tag{3.2}$$

We will prove that any solution y of Eq (3.1) are defined on all of $I = [0, \infty)$ and it satisfies as $t \to \infty$:

$$\begin{aligned} y &= (\delta_1 + o(1))z_1 + (\delta_2 + o(1))z_2 \\ y' &= (\delta_1 + o(1))z_1' + (\delta_2 + o(1))z_2' \end{aligned} \tag{3.3}$$

where $\{z_1, z_2\}$ is a fundamental system of solutions of Eq (3.2) and $\{\delta_1, \delta_2\}$ are constants. Let

$$y(t) = A(t)z_1(t) + B(t)z_2(t) \tag{3.4}$$

under the condition

$$A'z_1 + B'z_2 = 0 \tag{3.5}$$

Then, we have $y' = Az_1' + Bz_2'$ and $y'' = A'z_1' + B'z_2' + Az_1'' + Bz_2''$. Thus $y'' = A'z_1' + B'z_2' - c(Az_1 + Bz_2)$. Therefore

$$A'z_1' + B'z_2' = c(t)[y(t) - y(t - r(t))]. \tag{3.6}$$

Solving Eqs. (3.5) and (3.6), we get

$$\begin{aligned} A' &= -w^{-1}z_2 \cdot c(t)[y(t) - y(t - r(t))] \\ B' &= w^{-1}z_1 \cdot c(t)[y(t) - y(t - r(t))] \end{aligned} \tag{3.7}$$

where w is the Wronskian of system $\{z_1, z_2\}$. Now, we have

$$\begin{aligned} |y(t) - y(t - r(t))| &= |\textstyle\int_{t-r(t))}^{t} y'(s)ds| = |\int_{-r(t)}^{0} y'(t+s)ds| \\ &= |\textstyle\int_{-r(t)}^{0} y_t'(s)ds| = |\int_{-r(t)}^{0}(Az_1' + Bz_2')_t(s)ds|. \end{aligned}$$

Thus

$$|y(t) - y(t - r(t))| \le r(t) \max_{i=1,2} \|z_{it}'\| \cdot (\|A_t\| + \|B_t\|).$$

Then, by system (3.7), the vector $x = (A, B)$ satisfies a system of functional differential equations of the type

$$x' = F(t, x_t) \tag{3.8}$$

satisfying the conditions (i) $F : I \times C_0 \to \mathbf{R}$ is a continuous function (ii) $|F(t, \varphi)| \le \lambda(t)\|\varphi\|$, $(t, \varphi) \in I \times C_0$.

In this point, we need the following Theorem concerning the asymptotic behavior of system (3.8).

Theorem 2 *Assume the above conditions (i) and (ii), where* $\lambda \in C(I, \mathbf{R})$ *satisfies* $\lambda(t) \in L_1(I)$. *Then the solutions with continuous initial conditions of Eq (3.8) are defined on all of* I *and they converge as* $t \to \infty$.

The proof of this theorem is omitted because it is similar to that of Theorem 1.

Thus, we get:

Theorem 3 *Assume that* $r(t)|c(t)| \cdot |z_i(t)| \cdot \|z'_{it}\| \in L_1(I) \quad i = 1, 2$. *Then any solution* $y = y(t; t_0, y_{t_0}, y'_{t_0})$ *satisfies formulae (3.3)* .

Proof. The application of Theorem 2 implies that A and B converge as $t \to \infty$. The formulae (3.3) follow from (3.4) and $y' = Az'_1 + Bz'_2$.

So, we have

Corollary 6 *If* $r \in L_1(I)$, *then any solution* y *of the functional differential equation*

$$y'' + ay(t - r(t)) = 0, \qquad a > 0 \quad constant$$

satisfies for $t \to \infty$,

$$y = (\delta_1 + o(1))sinat + (\delta_2 + o(1))cosat$$

$$y' = a(\delta_1 + o(1))cosat - a(\delta_2 + o(1))sinat$$

More generally, using Green-Liouville formulae ([7]) for the solutions of (3.2) we get:

Corollary 7 *If* $c(t) \in C^2(I)$, $c > 0$ *and* $c^{-3/2}c''$, $r(t) \cdot |c^{3/4}(t)| \|c_t^{1/4}\| \in L_1(I)$ *then any solution* y *of the functional differential equation*

$$y'' + c(t)y(t - r(t)) = 0$$

satisfies for $t \to \infty$

$$y = c(t)^{-1/4}[(\delta_1 + o(1))exp(i \textstyle\int^t c^{1/2}(s)ds) + (\delta_2 + o(1))exp(-i \int^t c^{1/2}(s)ds)]$$

$$y' = c(t)^{1/4}[i(\delta_1 + o(1))exp(i \textstyle\int^t c^{1/2}(s)ds) + i(\delta_2 + o(1))exp(-i \int^t c^{1/2}(s)ds)]$$

For more related results, see [9].

ACKNOWLEDGEMENT. This work was partially supported by Fondecyt 1930839.

REFERENCES

1. BELLMAN, R. Research problem: Functional Differential Equation, Bull. Amer. Math. Soc., 71 (1965), 495.

2. COOKE, K. Functional differential equations with asymptotically vanishing lag, Rend. Circ. Mat. Palermo Ser. II 16 (1967), 36-59.

3. COOKE, K. Linear functional differential equations of asymptotically autonomous type, J. Diff. Eq. 7 (1970), 154-174.

4. COOKE, K. Asymptotic theory for the delay differential equation $u'(t) = -au(t - r(u))$, J. Math. Anal. Appl. 19 (1967), 160-173.

5. GROOSMAN, S. & JORKE, J. A. Asymptotic behavior and exponential stability criteria for differential delay equations. J. Diff. Eqs. 12 (2), 1972, 236-255.

6. PINTO, M. Asymptotic integration of systems resulting from the perturbations of an h-system, J. Math. Anal. Appl. 131 (1988), 194-216.

7. PINTO, M. Asymptotic integration of second order linear differential equations, J. Math. Anal. Appl. 111 (1985), 194-216.

8. PINTO, M. Continuation, nonoscillation and asymptotic formulae of solutions of second order nonlinear differential equations, Nonlinear Analysis T.M.A. Vol. 26, 11 (1991), 981-993.

9. PINTO, M. Green-Liouville formulae for second order delay differential equations, To appear.

10. PINTO, M. Asymptotic integration of the functional differential equation $y'(t) = a(t)y(t - r(t, y))$, J. Math. Anal. Appl. 175 N^O 1 (1993), 46-52

11. BURTON, T. A. & HADDOCK, J. R. On the delay-differential equations $x'(t) + a(t)f(x(t - r(t))) = 0$ and $x''(t) + a(t)f(x(t - r(t))) = 0$, J. Math. Anal. Appl. 54 (1976), 37-48.

12. HADDOCK, J. R. On the asymptotic behavior of solutions of $x'(t) = -a(t)f(x(t - r(t)))$, Siam J. Math. Anal. 5 (1974), 569-573.

13. TRENCH, W. Functional perturbations of second order linear differential equations, Siam J. Math. Anal. 16 (4) (1985), 741-756.

14. YONEYAMA, T. On the stability for the delay-differential equation $x'(t) = -a(t)f(x(t - r(t)))$, J. Math. Anal. Appl. 120 (1986), 271-275.

15. GONZALEZ, P. & PINTO, M. Stability properties of the solutions of the nonlinear functional differential systems, J. Math. Anal. Appl. 181 N^0 2 (1994), 562-573.