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COINCIDENCE THEOREMS FOR NONLINEAR HYBRID CONTRACTIONS

Y.J. CHO
Deptartment of Mathematics
Gyeongsang National University
Chinju 660-701, Korea

B. FISHER
Department of Mathematics
and Computer Science
University of Leicester
Leicester LE1 7RH, England

G.S. GENGA
Deptartment of Mathematics
Sunchon National University
Sunchon 540-742, Korea

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ABSTRACT. In this paper, we give some common fixed point theorems for single-valued mappings and multi-valued mappings satisfying a rational inequality Our theorems generalize some results of B Fisher, M L. Diviccaro et al. and V Popa

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1. INTRODUCTION

Let (X, d) be a metric space and let f and g be mappings from X into itself. In [1], S Sessa defined f and g to be weakly commuting if

$$d(gfx, fgx) \leq d(gx, fx)$$

for all x in X It can be seen that two commuting mappings are weakly commuting, but the converse is false as shown in the Example of [2]

Recently, G. Jungck [3] extended the concept of weak commutativity in the following way

DEFINITION 1.1. Let f and g be mappings from a metric space (X, d) into itself. The mappings f and g are said to be compatible if

$$\lim_{n\to\infty} (fgx_n, gfx_n) = 0$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$ for some z in X

It is obvious that two weakly commuting mappings are compatible, but the converse is not true Some examples for this fact can be found in [3].

Recently, H. Kaneko [4] and S. L. Singh et al. [5] extended the concepts of weak commutativity and compatibility [6] for single-valued mappings to the setting of single-valued and multi-valued mappings, respectively

Let (X, d) be a metric space and let $CB(X)$ denote the family of all nonempty closed and bounded subsets of X. Let H be the Hausdorff metric on $CB(X)$ induced by the metric d, i e ,

$$H(A, B) = \max\left\{\sup_{x\in A} d(x, B),\ \sup_{y\in B} d(y, A)\right\}$$

for A, $B \in CB(X)$, where $d(x, A) = \inf_{y \in A} d(x, y)$.

It is well-known that $(CB(X), H)$ is a metric space, and if a metric space (X, d) is complete, then $(CH(X), H)$ is also complete.

Let $\delta(A, B) = \sup\{d(x, y) : x \in A \text{ and } y \in B\}$ for all A, $B \in CB(X)$. If A consists of a single point a, then we write $\delta(A, B) = \delta(a, B)$ If $\delta(A, B) = 0$, then $A = B = \{a\}$ [7]

LEMMA 1.1 [8]. Let A, $B \in CB(X)$ and $k > 1$. Then for each $a \in A$, there exists a point $b \in B$ such that $d(a, b) \leq kH(A, B)$.

Let (X, d) be a metric space and let $f : X \to X$ and $S : X \to CB(X)$ be single-valued and multi-valued mappings, respectively.

DEFINITION 1.2. The mappings f and S are said to be weakly commuting if for all $x \in X$, $fSx \in CB(X)$ and

$$H(Sfx, fSx) \leq d(fx, Sx),$$

where H is the Hausdorff metric defined on $CB(X)$

DEFINITION 1.3. The mappings f and S are said to be compatible if

$$\lim_{n \to \infty} d(fy_n, Sfx_n) = 0$$

whenever $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $\lim_{n \to \infty} fx_n = \lim_{n \to \infty} y_n = z$ for some $z \in X$, where $y_n \in Sx_n$ for $n = 1, 2, \ldots$.

REMARK 1.1. (1) Definition 1 3 is slightly different from the Kaneko's definition [6]

(2) If S is a single-valued mapping on X in Definitions 1.2 and 1.3, then Definitions 1 2 and 1.3 become the definitions of weak commutativity and compatibility for single-valued mappings

(3) If the mappings f and S are weakly commuting, then they are compatible, but the converse is not true

In fact, suppose that f and S are weakly commuting and let $\{x_n\}$ and $\{y_n\}$ be two sequences in X such that $y_n \in Sx_n$ for $n = 1, 2, \cdots$ and $\lim_{n \to \infty} fx_n = \lim_{n \to \infty} y_n = z$ for some $z \in X$ From $d(fx_n, Sx_n) \leq d(fx_n, y_n)$, it follows that $\lim_{n \to \infty} d(fx_n, Sx_n) = 0$. Thus, since f and S are weakly commuting, we have

$$\lim_{n \to \infty} H(Sfx_n, fSx_n) = 0.$$

On the other hand, since $d(fy_n, Sfx_n) \leq H(fSx_n, Sfx_n)$, we have

$$\lim_{n \to \infty} d(fy_n, Sfx_n) = 0,$$

which means that f and S are compatible

EXAMPLE 1.1. Let $X = [1, \infty)$ be a set with the Euclidean metric d and define $fx = 2x^4 - 1$ and $Sx = [1, x^2]$ for all $x \geq 1$ Note that f and S are continuous and $S(X) = f(X) = X$ Let $\{x_n\}$ and $\{y_n\}$ be sequences in X defined by $x_n = y_n = 1$ for $n = 1, 2, \cdots$ Thus we have

$$\lim_{n \to \infty} fx_n = \lim_{n \to \infty} y_n = 1 \in X, \quad y_n \in Sx_n.$$

On the other hand, we can show that $H(fSx_n, Sfx_n) = 2\left(x_n^4 - 1\right)^2 \to 0$ if and only if $x_n \to 1$ as $n \to \infty$ and so, since $d(fy_n, Sfx_n) \leq H(fSx_n, Sfx_n)$, we have

$$\lim_{n \to \infty} d(fy_n, Sfx_n) = 0.$$

Therefore, f and T are compatible, but f and T are not weakly commuting at $x = 2$

We need the following lemmas for our main theorems, which is due to G Jungck [2]

LEMMA 1.2. Let f and g be mappings from a metric space (X, d) into itself If f and g are compatible and $fz = gz$ for some $z \in X$, then

$$fgz = ggz = gfz = ffz.$$

LEMMA 1.3. Let f and g be mappings from a metric space (X, d) into itself If f and g are compatible and fx_n, $gx_n \to z$ for some $z \in X$, then we have the following

(1) $\lim_{n\to\infty} gfx_n = fz$ if f is continuous at z,

(2) $fgz = gfz$ and $fz = gz$ if f and g are continuous at z

2. COINCIDENCE THEOREMS FOR NONLINEAR HYBRID CONTRACTIONS

In this section, we give some coincidence point theorems for nonlinear hybrid contractions, i e., contractive conditions involving single-valued and multi-valued mappings In the following Theorem 2 1, $S(X)$ and $T(X)$ mean $S(X) = \cup_{x\in X} Sx$ and $T(X) = \cup_{x\in X} Tx$, respectively

THEOREM 2.1. Let (X, d) be a complete metric space. Let $f, g : X \to X$ be continuous mappings and $S, T : X \to CB(X)$ be H-continuous multi-valued mappings such that

$$T(X) \subset f(X) \quad \text{and} \quad S(X) \subset g(X), \tag{2.1}$$

the pairs f, S and g, T are compatible mappings, (2 2)

$$H_p(Sx, Ty) \leq \frac{cd(fx, Sx)d^p(gy, Ty) + bd(fx, Ty)d^p(gy, Sx)}{\delta(fx, Sx) + \delta(gy, Ty)} \tag{2 3}$$

for all x, $y \in X$ for which $\delta(fx, Sx) + \delta(gy, Ty) \neq 0$, where $p \geq 1$, $b \geq 0$ and $1 < c < 2$ Then there exists a point $z \in X$ such that $fx \in Sz$ and $gz \in Tz$, i.e , z is a coincidence point of f, S and of g, T

PROOF. Choose a real number k such that $1 < k < \left(\frac{2}{c}\right)^{\frac{1}{p}}$ and let x_0 be an arbitrary point in X Since $Sx_0 \subset g(X)$, there exists a point $x_1 \in X$ such that $gx_1 \in Sx_0$ and so there exists a point $y \in Tx_1$ such that

$$d(gx_1, y) \leq kH(Sx_0, Tx_1),$$

which is possibly by Lemma 1 1 Since $Tx_1 \subset f(X)$, there exists a point $x_2 \in X$ such that $y = fx_2$ and so we have

$$d(gx_1, fx_2) \leq kH(Sx_0, Tx_1).$$

Similarly, there exists a point $x_3 \in X$ such that $gx_3 \in Sx_2$ and

$$d(gx_3, fx_2) \leq kH(Sx_2, Tx_1).$$

Inductively, we can obtain a sequence $\{x_n\}$ in X such that

$$fx_{2n} \in Tx_{2n-1}, \quad n \in N,$$

$$gx_{2n+1} \in Sx_{2n}, \quad n \in N_0 = N \cup \{0\},$$

$$d(gx_{2n+1}, fx_{2n}) \leq kH(Sx_{2n}, Tx_{2n-1}), \quad n \in N,$$

$$d(gx_{2n+1}, fx_{2n+2}) \leq kH(Sx_{2n}, Tx_{2n+1}), \quad n \in N_0,$$

where N denotes the set of positive integers.

First, suppose that for some $n \in N$

$$\delta(fx_{2n}, Sx_{2n}) + \delta(gx_{2n+1}, Tx_{2n+1}) = 0.$$

Then $fx_{2n} \in Sx_{2n}$ and $gx_{2n+1} \in Tx_{2n+1}$ and so x_{2n} is a coincidence point of f and S and x_{2n+1} is a coincidence point of g and T.

Similarly, $\delta(fx_{2n+2}, Sx_{2n+2}) + \delta(gx_{2n+1}, Tx_{2n+1}) = 0$ for some $n \in N$ implies that x_{2n+1} is a coincidence point of g and T and x_{2n+2} is a coincidence point of f and S.

Now, suppose that $\delta(fx_{2n}, Sx_{2n}) + \delta(gx_{2n+1}, Tx_{2n+1}) \neq 0$ for $n \in N_0$ Then, by (2 3), we have

$$
\begin{aligned}
&d^p(gx_{2n+1}, fx_{2n+2}) \\
&\quad \leq k^p H^p(Sx_{2n}, Tx_{2n+1}) \\
&\quad \leq k^p \frac{cd(fx_{2n}, Sx_{2n})d^p(gx_{2n+1}, Tx_{2n+1}) + bd(fx_{2n}, Tx_{2n+1})d^p(gx_{2n+1}, Sx_{2n})}{\delta(fx_{2n}, Sx_{2n}) + \delta(gx_{2n+1}, Tx_{2n+1})} \\
&\quad \leq k^p \frac{cd(fx_{2n}, gx_{2n+1})d^p(gx_{2n+1}, fx_{2n+2}) + bd(fx_{2n}, fx_{2n+2})d^p(gx_{2n+1}, gx_{2n+1})}{\delta(fx_{2n}, Sx_{2n}) + \delta(gx_{2n+1}, Tx_{2n+1})} \\
&\quad \leq k^p \frac{cd(fx_{2n}, gx_{2n+1})d^p(gx_{2n+1}, fx_{2n+2})}{d(fx_{2n}, gx_{2n+1}) + d(gx_{2n+1}, fx_{2n+2})}. \qquad (2.4)
\end{aligned}
$$

If $d(gx_{2n+1}, fx_{2n+2}) = 0$ and $d(fx_{2n}, gx_{2n+1}) \neq 0$ in (2 4), then $gx_{2n+1} = fx_{2n+2} \in Tx_{2n+1}$ and so x_{2n+1} is a coincidence point of g and T. But the case of $d(fx_{2n}, gx_{2n+1}) = 0$ and $d(gx_{2n+1}, fx_{2n+2}) \neq 0$ in (2.4) cannot occur.

In fact, if $d(fx_{2n}, gx_{2n+1}) = 0$ and $d(gx_{2n+1}, fx_{2n+2}) \neq 0$ in (2.4), then we have $d(gx_{2n+1}, fx_{2n+2}) = 0$, which is impossible From (2.4), we have

$$
\begin{aligned}
d^p(gx_{2n+1}, fx_{2n+2})&[d(fx_{2n}, gx_{2n+1}) + d(gx_{2n+1}, fx_{2n+2})] \\
&\leq k^p cd(fx_{2n}, gx_{2n+1})d^p(gx_{2n+1}, fx_{2n+2}),
\end{aligned}
$$

which implies that

$$d(gx_{2n+1}, fx_{2n+1}) \leq (k^p c - 1)d(fx_{2n}, gx_{2n+1}).$$

On the other hand, from (2.3), we have

$$
\begin{aligned}
&d^p(gx_{2n+3}, fx_{2n+2}) \\
&\quad \leq k^p H^p(Sx_{2n+2}, Tx_{2n+1}) \\
&\quad \leq k^p \frac{cd(fx_{2n+2}, Sx_{2n+2})d^p(gx_{2n+1}, Tx_{2n+1}) + bd(fx_{2n+2}, Tx_{2n+1})d^p(gx_{2n+1}, Sx_{2n+2})}{\delta(fx_{2n+2}, Sx_{2n+2}) + \delta(gx_{2n+1}, Tx_{2n+1})} \\
&\quad \leq k^p \frac{cd(fx_{2n+2}, gx_{2n+3})d^p(gx_{2n+1}, fx_{2n+2})}{d(fx_{2n+2}, gx_{2n+3}) + d(gx_{2n+1}, fx_{2n+2})},
\end{aligned}
$$

which implies that, if $\alpha = d(x_{2n+3}, fx_{2n+2})/d(fx_{2n+2}, gx_{2n+1})$, then $\alpha^p + \alpha^{p-1} \leq k^p c$ Thus $\alpha < 1$ and we have

$$d(gx_{2n+3}, fx_{2n+2}) \leq d(fx_{2n+2}, gx_{2n+1}).$$

Repeating the above argument, since $0 \leq k^p c - 1 < 1$, it follows that $\{gx_1, fx_2, gx_3, fx_4, \cdots, gx_{2n-1}, gx_{2n}, gx_{2n+1}, \cdots\}$ is a Cauchy sequence in X. Since (X, d) is a complete metric space, let $\lim_{n\to\infty} gx_{2n+1} = \lim_{n\to\infty} fx_{2n} = z$.

Now, we will prove that $fz \in Sz$, that is, z is a coincidence point of f and S. For every $n \in N$, we have

$$d(fgx_{2n+1}, Sz) \leq d(fgx_{2n+1}, Sfx_{2n}) + H(Sfx_{2n}, Sz). \qquad (2\ 5)$$

It follows from the H-continuity of S that

$$\lim_{n\to\infty} H(Sfx_{2n}, Sz) = 0 \qquad (2.6)$$

since $fx_{2n} \to z$ as $n \to \infty$. Since f and S are compatible mappings and $\lim_{n\to\infty} fz_n = \lim_{n\to\infty} y_n = z$, where $y_n = gx_{2n+1} \in Sx_{2n}$ and $z_n = x_{2n}$, we have

$$\lim_{n\to\infty} d(fy_n, Sfz_n) = \lim_{n\to\infty} d(fgx_{2n+1}, Sfx_{2n}) = 0. \tag{2 7}$$

Thus, from (2 5), (2.6) and (2.7), we have $\lim_{n\to\infty} d(fgx_{2n+1}, Sz) = 0$ and so, from

$$d(fz, Sz) \leq d(fz, fgx_{2n+1}) + d(fgx_{2n+1}, Sz)$$

and the continuity of f, it follows that $d(fz, Sz) = 0$, which implies that $fz \in Sz$ since Sz is a closed subset of X Similarly, we can prove that $gz \in Tz$, that is, z is a coincidence point of g and T This completes the proof

If we put $f = g = i_X$ (the identity mapping on X) in Theorem 2 1, we have the following

COROLLARY 2.2 [1] Let (X, d) be a complete metric space and let $S, T : X \to CB(X)$ be H-continuous multi-valued mappings such that

$$H^p(Sx, Ty) \leq \frac{cd(x, Sx)d^p(y, Ty) + bd(x, Ty)d^p(y, Sx)}{\delta(x, Sx) + \delta(y, Ty)} \tag{2.8}$$

for all x, $y \in X$ for which $\delta(x, Sx) + \delta(y, Ty) \neq 0$, where $p \geq 1$, $b \geq 0$ and $1 < c < 2$. Then S and T have a common fixed point in X, that is, $z \in Sz$ and $z \in Tz$

Assuming that $f = g$ and $S = T$ on X in Theorem 2.1, we have the following

COROLLARY 2.3. Let (X, d) be a complete metric space and let $f : X \to X$ be a continuous single-valued mapping and $S : X \to CB(X)$ be an H-continuous multi-valued mapping such that

$S(X) \subset f(X)$, (2 9)

f and S are continuous mappings, (2.10)

$$H^p(Sx, Ty) \leq \frac{cd(fx, Sx)d^p(fy, Sy) + bd(fx, Sy)d^p(fy, Sx)}{\delta(fx, Sx) + \delta(fy, Sy)} \tag{2 11}$$

for all x, $y \in X$ for which $\delta(fx, Sx) + \delta(fy, Sy) \neq 0$, where $p \geq 1$, $b \geq 0$ and $1 < c < 2$ Then there exists a point $z \in X$ such that $fz \in Sz$, i.e., z is a coincidence point of f and S.

REMARK 2.1. If we put $p = 1$ in Theorem 2.1, Corollaries 2.2 and 2.3, we can obtain further corollaries.

3. FIXED POINT THEOREMS FOR SINGLE-VALUED MAPPINGS

In this section, using Theorem 2.1, we can obtain some fixed point theorems for single-valued mappings in a metric space

If S and T are single-valued mappings from a metric space (X, d) into itself in Theorem 2 1, we have the following

THEOREM 3.1. Let (X, d) be a complete metric space Let f, g, S and T be continuous mappings from X into itself such that

$S(X) \subset g(X)$ and $T(X) \subset f(X)$, (3 1)

the pairs f, S and g, T are compatible mappings, (3 2)

either (i) $$d^p(Sx, Ty) \leq \frac{cd(fx, Sx)d^p(gy, Ty) + bd(fx, Ty)d^p(gy, Sx)}{d(fx, Sx) + d(gy, Ty)} \tag{3 3}$$

if $d(fx, Sx) + d(gy, Ty) \neq 0$ for all x, $y \in X$, where $p \geq 1$, $b \geq 0$ and $1 < c < 2$, or

(ii) $d(Sx, Ty) = 0$ if $d(fx, Sx) + d(gy, Ty) = 0$.

Then f, g, S and T have a unique common fixed point z in X Further, z is the unique common fixed point of f, S and of g, T

PROOF. The existence of the point w with $fw = Sw$ and $gw = Tw$ follows from Theorem 2 1 From (ii) of (3.3), since $d(fw, Sw) + d(gw, Tw) = 0$, it follows that $d(Sw, Tw) = 0$ and so $Sw = fw = gw = Tw$. By Lemma 1 2, since f and S are compatible mappings and $fw = Sw$, we have

$$Sfw = SSw = fSw = ffw, \tag{3 4}$$

which implies that $d(fSw, SSw) + d(gw, Tw) = 0$ and, using the condition (ii) of (3.3), we have

$$Sfw = SSw = Tw = gw = fw \tag{3 5}$$

and so $fw = z$ is a fixed point of S. Further, (3 4) and (3 5) implies that

$$Sz = fSw = SSw = fz = z.$$

Similarly, since g and T are compatible mappings, we have $Tz = gz = z$. Using (ii) of (3 3), since $d(fz, Sz) + d(gz, Tz) = 0$, it follows that $d(Sz, Tz) = 0$ and so $Sz = Tz$ Therefore, the point z is a common fixed point of f, g, S and T.

Next, we will show the uniqueness of the common fixed point z. Let z' be another common fixed point of f and S. Using the condition (ii) of (3.3), since $d(fz', Sz') + d(gz, Tz) = 0$, it follows that $d(z, z') = d(Tz, Sz') = 0$ and so $z = z'$. This completes the proof.

Now, we give an example of Theorem 3.1 with $p = 1$ and $f = g$

EXAMPLE 3.1. Let $X = \{1, 2, 3, 4\}$ be a finite set with the metric d defined by

$$d(1,3) = d(1,4) = d(2,3) = d(2,4) = 1,$$
$$d(1,2) = d(3,4) = 2.$$

Define mappings f, S, $T : X \to X$ by

$$f(1) = 1,\ f(2) = 2,\ f(3) = 4,\ f(4) = 3,$$
$$S(1) = S(2) = S(4) = 2,\ S(3) = 3,$$
$$T(1) = T(2) = T(3) = T(4) = 2.$$

From

$$Sf(1) = S(1) = 2 = f(2) = fS(1),$$
$$Sf(2) = S(2) = 2 = f(2) = fS(2),$$
$$d(Sf(3), fS(3)) = d(S(4), f(3)) = d(2,4) < 1 < 2 = d(3,4) = d(S(3), f(3))$$

and

$$d(Sf(4), fS(4)) = d(S(3), f(2)) = d(3,2) = 1 = d(2,3) = d(S(4), f(4)),$$

it follows that f and S are weakly commuting mappings and so they are compatible. Clearly, f, S and T are continuous and

$$S(x) = \{2, 3\} \subset X = f(X),\quad T(X) = \{2\} \subset X = f(X).$$

Further, we can show that the inequality (i) of (3.3) holds with $c = \frac{3}{2}$ and $b = 2$ and the condition (ii) of (3.3) holds only for the point 2. Therefore, all the conditions of Theorem 3 1 are satisfied and the point 2 is a unique common fixed point of f, S and T.

REMARK 3.1. Theorem 3.1 assures that f, g, S and T have a unique common fixed point in X However, either f or g or S or T can have other fixed points. Indeed, in Example 3 1, f and S have two fixed points.

REMARK 3.2. From the proof of Theorem 3 1, it follows that if the condition (ii) of (3 3) is omitted in the hypothesis of Theorem 3.1, then f, g, S and T have a coincidence point w, i e, $fw = gw = Sw = Tw$

If we put $f = g = i_X$ in Theorem 3 1, we have the following.

COROLLARY 3.2. Let (X, d) be a complete metric space and let $S, T : X \to X$ be continuous mappings such that

either (i) $$d^p(Sx, Ty) \leq \frac{cd(x, Sx)d^p(y, Ty) + bd(x, Ty)d^p(y, Sx)}{d(x, Sx) + d(y, Ty)} \qquad (3\ 5)$$

for all x, $y \in X$ if $d(x, Sx) + d(y, Ty) \neq 0$, where $p \geq 1$, $b \geq 0$ and $1 < c < 2$, or

(ii) $$d(Sx, Ty) = 0 \quad \text{if} \quad d(x, Sx) + d(y, Ty) = 0.$$

Then S and T have a unique common fixed point z in X

Assuming that $f = g$ and $S = T$ on X in Theorem 3.1, we have the following.

COROLLARY 3.3. Let (X, d) be a complete metric space and let $f, S : X \to X$ be continuous mappings such that

$S(X) \subset f(X)$, (3 6)

f and S are compatible mappings, (3 7)

either (i) $$d^p(Sx, Ty) \leq \frac{cd(x, Sx)d^p(fy, Sy) + bd(fx, Sy)d^p(fy, Sx)}{d(fx, Sx) + d(y, Sy)} \qquad (3\ 8)$$

for all x, $y \in X$ if $d(fx, Sx) + d(fy, Sy) \neq 0$, where $p \geq 1$, $b \geq 0$ and $1 < c < 2$, or

(ii) $$d(Sx, Sy) = 0 \quad \text{if} \quad d(fx, Sy) + d(fy, Sy) = 0.$$

Then f and S have a unique common fixed point z in X

REMARK 3.3. (1) If $p = 1$ in Corollary 3.2, we obtain the result of B Fisher [9].

(2) Theorem 3.1 is an extension of the results of M L. Diviccaro, S Sessa and B. Fisher [10]

REMARK 3.4. Conditions (3 6) and (3.7) are necessary in Corollary 3.3 (and so Theorem 3.1) [3]

EXAMPLE 3.1. Let $X = [0, 1]$ with the Euclidean metric $d(x, y) = |x - y|$ and define two mappings $f, S : X \to X$ by

$$Sx = \frac{1}{4} \quad \text{and} \quad fx = \frac{1}{2}x$$

for all $x \in X$. Note that f and S are continuous and $S(X) = \left\{\frac{1}{4}\right\} \subset \left[0, \frac{1}{2}\right] = f(X)$

Since $d(Sx, Sy) = 0$ for all x, $y \in X$, all the conditions of Corollary 3 3 are satisfied except the compatibility of f and S. In fact, let $\{x_n\}$ be a sequence in X defined by $x_n = \frac{1}{2}$ for $n = 1, 2, \cdots$. Then we have

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} \frac{1}{2}x_n = \frac{1}{4}, \quad \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} \frac{1}{4} = \frac{1}{4}$$

but

$$\lim_{n\to\infty} d(Sfx_n, fSx_n) = \lim_{n\to\infty} \left|\frac{1}{4} - \frac{1}{8}\right| = \frac{1}{8}.$$

Thus f and S are not compatible mappings But f and S have no common fixed points in X

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REFERENCES

[1] POPA, V., Theorems on multifunctions satisfying a rational inequality, *Comment. Math. Univ. Carolinae*, **24** (4) (1983), 673-680.

[2] SESSA, S., On a weak commutativity conditions of mappings in fixed point considerations, *Publ. Inst. Math.* (Beograd), **32** (46) (1982), 146-153

[3] JUNGCK, G , Compatible mappings and common fixed point, *Internat. J. Math. Math. Sci.* **9** (4) (1986), 771-779

[4] KANEKO, H , A Common fixed point of weakly commuting multi-valued mappings, *Math. Japon.*, **33** (5) (1988), 741-744

[5] SINGH, S.L., HA, K.S and CHO, Y J., Coincidence and fixed points of nonlinear hybrid contractions, *Internat. J. Math. Math. Sci.*, **12** (2) (1989), 247-256.

[6] KANEKO, H. and SESSA, S., Fixed point theorems for compatible multi-valued and single-valued mappings, *Internat. J. Math. Math. Sci.*, **12** (1989), 257-262

[7] RUS, I.A., Fixed point theorems for multi-valued mappings in a complete metric space, *Math. Japon.*, **20** (1975), 21-24

[8] NADLER, S.B , Multi-valued contraction mappings, *Pacific J. Math.*, **20** (2) (1969), 475-488.

[9] FISHER, B., Theorems on mappings satisfying a rational inequality, *Comment. Math. Univ. Carolinae*, **19** (1978), 37-44.

[10] DIVICCARO, M.L., SESSA, S. and FISHER, B., Common fixed point theorems with a rational inequality, *Bull. of Institute of Math.*, Academia Sinica, **14** (3) (1986), 277-285