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CONVERGENCE OF COMPOSITION SEQUENCES OF BILINEAR AND OTHER TRANSFORMATIONS $\{f_n\}, f_n \rightarrow z$

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ABSTRACT. This paper investigates convergence behavior of composition sequences $f_1 \circ f_2 \circ \ldots \circ f_n(z)$ and $f_n \circ f_{n-1} \circ \ldots \circ f_1(z)$ where the f_n's are bilinear transformations and $f_n \rightarrow z$ Additional results are provided for the case when the f_n's are more general functions

KEY WORDS AND PHRASES: Bilinear transformations, compositions, fixed points, continued fractions.

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1. INTRODUCTION

Starting with a sequence $\{f_n\}$ of complex valued functions, including bilinear transformations

$$f_n(z) := (a_n z + b_n)/(c_n z + d_n), \quad a_n d_n - b_n c_n \neq 0, \tag{1 1}$$

two composition sequences can be formed

$$F_n(z) := f_1 \circ f_2 \circ \ldots \circ f_n(z), \quad \text{(inner composition), and} \tag{1 2}$$

$$G_n(z) := f_n \circ f_{n-1} \circ \ldots \circ f_1(z), \quad \text{(outer composition).} \tag{1 3}$$

The sequence $\{F_n(z)\}$ (using (1.1)) arises in connection with normal continued fractions [1], and $\{G_n(z)\}$ occurs (using (1 1)) in the study of reverse continued fractions [2] and (more generally) in the computation of fixed points of functions written as infinite expansions [3]. Both sequences give perturbed orbits of $f(z)$ if $f_n \approx f$, and are thus interesting from a dynamical systems perspective

The investigation of the convergence behavior of sequences of the form (1.2) involving bilinear transformations (1.1) goes back at least to Paydon & Wall [4] (1942) and Schwerdtfeger [5] (1946), and was continued by Piranian & Thron [6] (1957), and DePree & Thron [7] (1962). Later, Magnus & Mandell [8] (1970) gave explicit results when $f_n \rightarrow f$, where f is categorized as "hyperbolic", "loxodromic", or "elliptic" (see below). The author pursued the elliptic case further and explored the remaining "parabolic" case [9] (1973) More recently the author investigated these cases with regard to outer compositional structures $\{G_n(z)\}$ in [10] (1991), [2] (1993). Barrlund, Karlsson & Wallin [11] (1993) have studied random inner and outer compositions of bilinear transformations—without the requirement that $f_n \rightarrow f$.

In addition, the author has explored structures (1.2) and (1.3) for sequences $\{f_n\}$ of more general complex functions in [12] (1988), [13] (1990), [14] (1992) Lorentzen has written a number of papers concerned with convergence properties of sequences of bilinear transformations and a definitive paper on inner compositions of more general functions [15] (1990)

This paper focuses on a case not falling into the four categories (hyperbolic, loxodromic, parabolic, or elliptic) mentioned above: the convergence behavior of (1.2) and (1.3) when $f_n \to z$, the identity function, and $\{f_n\}$ is described by (1.1) A minor additional case is considered also $f_n \to \alpha$, a certain constant. Further theory is developed for the setting in which the f_n's are not necessarily bilinear transformations but $f_n \to z$.

The earliest results for $f_n \to z$ are due probably to DePree & Thron [7]

THEOREM 1.1. If $\{F_n(z)\}$ converges to a bilinear transformation, where the f_n's are bilinear transformations, then $f_n \to z$.

and its partial converse.

THEOREM 1.2. If, for $f_n(z) := (a_n z + b_n)/(c_n z + d_n)$, $a_n d_n - b_n c_n = 1$, Πa_n, Πd_n, Σb_n, and Σc_n all converge absolutely, then $\lim_{n\to\infty} F_n(z) = (Pz+Q)/(Rz+S)$, where $PS - RQ \neq 0$.

The hypotheses of Theorem 1.2 are written in terms of the coefficients of f_n as it is described in (1.1). This seems reasonable, particularly in light of applications. However, there is a more "natural" approach to the study of compositions of the form (1.2) and (1.3), one that demonstrates the similarity of convergence behavior between $\{F_n(z)\}$, $\{G_n(z)\}$, and simple iteration $\{f^n(z)\}$ when $f_n \to f$ It is this approach that is taken in the current paper.

We first observe that any bilinear transformation f (or f_n) having two finite and distinct fixed points α and β (or α_n or β_n) can be written in "multiplier" form (Ford [16]):

$$\frac{f(z)-\alpha}{f(z)-\beta} = K\frac{z-\alpha}{z-\beta}, \quad |K| \le 1, \quad \text{and} \quad \frac{f_n(z)-\alpha_n}{f_n(z)-\beta_n} = K_n\frac{z-\alpha_n}{z-\beta_n}, \quad |K_n| \le 1. \tag{1.4}$$

Here K is called the "multiplier" of the transformation. Its value determines the character of f Equation (1.4) coupled with the strongly geometrical nature of bilinear transformations leads to clear geometrical convergence patterns for the iteration $\{f^n(z)\}$ (see Ford [16], e.g.). Using (1.4) judiciously also allows the formulation of hypotheses written in terms of α_n, β_n and K_n that lead to conclusions on the convergence behavior of $\{F_n(z)\}$ and $\{G_n(z)\}$.

When f is *hyperbolic* or *loxodromic* ($|K| < 1$), (1.4) shows that $f^n(z) \to \alpha$, the *attracting* fixed point of f, for each $z \neq \beta$, the *repelling* fixed point of f. Under rather mild restrictions on α_n, β_n and K_n the behavior of $\{F_n(z)\}$ and $\{G_n(z)\}$ when $f_n \to f$ with $|K| < 1$ is analogous to that of $\{f^n(z)\}$, [8], [10]. In the *parabolic* case (single attracting fixed point α), and the *elliptic* case ($|K| = 1$, $K \neq 1$), roughly parallel behavior has been shown to exist between $\{f^n(z)\}$ and $\{F_n(z)\}$ and $\{G_n(z)\}$ as well [2], [9].

However, in the present case ($f(z) := z$) one finds $\{f^n(z)\}$ exhibiting no dynamical behavior whatsoever, since $f^n(z) \equiv z$ Clearly f has an infinite number of *neutral* fixed points, no one of which exerts more dynamical influence than the others. Nevertheless, it will be shown that when $f_n \to z$ "slowly", with each $|K_n| < 1$ in (1.4), $G_n(z) \to \alpha = \lim \alpha_n$ for all z with one possible exception and $F_n(z) \to \Gamma$, a constant, for all values of z except $z = \beta$. Thus the "perturbed iterations" $\{F_n(z)\}$ and $\{G_n(z)\}$ possess *virtual* attracting fixed points even though $z = f(z) = \lim f_n$ does not.

To demonstrate the importance of the sequence $\{K_n\}$ in determining "slow" versus "fast" convergence, we have the following simple example.

EXAMPLE. $\frac{f_n(z)-\alpha}{f_n(z)-\beta} = K_n\frac{z-\alpha}{z-\beta}$, with $\alpha \neq \beta$ and (a) $K_n = 1 - 1/n$, and (b) $K_n = 1 - 1/n^2$, $n > 1$. Then (a) $\frac{G_n(z)-\alpha}{G_n(z)-\beta} = \prod_{j=2}^{n}\left(1-\frac{1}{j}\right)\frac{z-\alpha}{z-\beta} \to 0$ shows that $G_n(z) \to G(z) \equiv \alpha$ for all $z \neq \beta$, and (b) $\frac{G(z)-\alpha}{G(z)-\beta} = \prod_{n=2}^{\infty}\left(1-\frac{1}{n^2}\right)\frac{z-\alpha}{z-\beta} = \frac{1}{2}\frac{z-\alpha}{z-\beta}$ shows that $G(z) = \lim G_n(z) = a$ bilinear transformation. The same results clearly hold for $F(z) = \lim F_n(z)$

2. OUTER COMPOSITION

The basic theorem of this section is the following:

THEOREM 2.1. Suppose

(i) $0 < |K_n| < 1$, $K_n \to 1\,(f_n \to z)$ or $K_n \to 0\,(f_n \to \alpha)$,

(ii) $\alpha_n \to \alpha$, $\beta_n \to \beta$, with $\alpha \neq \beta$, and

(iii) $\Sigma|\alpha_n - \alpha_{n-1}| < \infty$, and $\Sigma|\beta_n - \beta_{n-1}| < \infty$

Then

1 If $\Pi\, K_n = 0$, $\{G_n(z)\}$converges to α for all $z \in \mathbb{C}$, with one possible exception

2 If $\Pi\, K_n = \Gamma \neq 0$, $\{G_n(z)\}$ converges to a bilinear transformation

PROOF OF 1. The proof of Theorem 2.1 involves the same techniques used by Magnus & Mandel [8] and the author [9] in the elliptic and parabolic cases for inner compositions.

We begin by solving (1 4) for $f_n(z)$, generating the following relationships.

$$a_n = (\alpha_n - K_n\beta_n)/(K_n\alpha_n - \beta_n), \quad b_n = \alpha_n\beta_n(K_n - 1)/(K_n\alpha_n - \beta_n),$$
$$c_n = (1 - K_n)/(K_n\alpha_n - \beta_n), \quad d_n = 1, \quad \text{provided} \quad K_n\alpha_n - \beta_n \neq 0. \tag{2 1}$$

The inequality in (1.1) is equivalent to $K_n(\alpha_n - \beta_n)^2 \neq 0$. If $K_n \to 1$, $\alpha_n \to \alpha$, $\beta_n \to \beta$, with $\alpha \neq \beta$, then $f_n \to z$. If $K \to 0$, then $f_n \to \alpha$

Next, set

$$\lambda_n(z) := (z - \alpha_n)/(z - \beta_n), \quad \text{and} \quad K_n(z) := K_n z, \quad \text{giving}$$
$$\lambda_n^{-1}(z) := (\beta_n z - \alpha_n)/(z - 1). \tag{2.2}$$

Thus

$$f_n(z) = \lambda_n^{-1} \circ K_n \circ \lambda_n(z).$$

Hence

$$\begin{aligned} G_n(z) &= f_n \circ f_{n-1} \circ \dots \circ f_1(z) \\ &= \lambda_n^{-1} \circ w_{n-1} \circ \dots \circ w_{h+1} \circ w_h \circ \dots \circ w_1(S(z)) \\ &= \lambda_n^{-1} \circ W_{n-1}^h \circ W_h(S_1(z)), \end{aligned} \tag{2 3}$$

where

$$\begin{aligned} w_j(z) &:= K_{j+1} \circ \lambda_{j+1} \circ \lambda_j^{-1}(z), \\ W_j(z) &:= w_j \circ w_{j-1} \circ \dots \circ w_1(z), \\ W_{n-1}^h(z) &:= w_{n-1} \circ w_{n-2} \circ \dots \circ w_{h+1}(z), \text{ and} \\ S(z) &:= K_1 \circ \lambda_1(z). \end{aligned}$$

We find that

$$w_j(z) = (p_j z + q_j)/(r_j z + 1), \tag{2.4}$$

where

$$\begin{aligned} p_j &= K_j(\beta_{j-1} - \alpha_j)/(\beta_j - \alpha_{j-1}), \\ q_j &= K_j(\alpha_j - \alpha_{j-1})/(\beta_j - \alpha_{j-1}), \quad \text{and} \\ r_j &= (\beta_{j-1} - \beta_j)/\beta_j - \alpha_{j-1}). \end{aligned}$$

The hypotheses of Theorem 2.1 imply $\Pi\, p_j = 0$, $\Sigma|q_n| < \infty$, and $\Sigma|r_n| < \infty$. It is of value to introduce the following additional notation

$$W_{h+j}^h(z) = w_{h+j} \circ w_{h+j-1} \circ \dots \circ w_{h+1}(z) = (A_{h+j}^h z + B_{h+j}^h)/(C_{h+j}^h z + D_{h+j}^h),$$
$$\text{with} \quad A_{h+1}^h = p_{h+1}, \quad B_{h+1}^h = q_{h+1}, \quad C_{h+1}^h = r_{h+1}, \quad \text{and} \quad D_{h+1}^h = 1.$$

Then

$$A_n^h = p_n A_{n-1}^h + q_n C_{n-1}^h \tag{2.5}$$
$$C_n^h = r_n A_{n-1}^h + C_{n-1}^h \tag{2.6}$$
$$B_n^h = p_n B_{n-1}^h + q_n D_{n-1}^h \tag{2 7}$$
$$D_n^h = r_n B_{n-1}^h + D_{n-1}^h \tag{2.8}$$

(2 5) and (2.6) give

$$A_n^h = p_n A_{n-1}^h + q_n \left(r_{n+1} + \sum_{j=h+2}^{n-1} r_j A_{j-1} \right). \tag{2 9}$$

It easily follows that A_n^h has 2^{n-h-1} terms, and, by writing out the first few terms and applying (2.9) inductively, one gets

LEMMA 2.1. $A_n^h = \prod_{j=h+1}^{n} p_j + \sum^{*} r_{k_1} q_{k_2} + \sum^{*} r_{k_1} q_{k_2} r_{k_3} q_{k_4} + \ldots + \sum^{*} r_{k_1} q_{k_2} \ldots r_{k_{2j-1}} q_{k_{2j}}$, where the * indicates a suppression of $\Pi\, p_j$ products and $h+1 \le k_1 < k_2 < \ldots k_{2j} \le n$

Given $\epsilon > 0$, for h sufficiently large one has $|p_{h+j}| < 1$ and

$$\left| A_n^h - \prod_{j=h+1}^{n} p_j \right| \le \sum |r_{k_1} q_{k_2}| + \sum |r_{k_1} q_{k_2} r_{k_3} q_{k_4}| + \ldots$$
$$\le \left(\sum_{h+1}^{\infty} |r_j| \sum_{h+1}^{\infty} |q_j| \right) + \left(\sum_{h+1}^{\infty} |r_j| \sum_{h+1}^{\infty} |q_j| \right)^2 + \ldots \le \epsilon .$$

Thus, for sufficiently large h and $n > h$, A_n^h is bounded by 1.

Next, from (2.6), one can get $C_n^h = r_{h+1} + \sum_{j=h+2}^{n-h} r_j A_{j-1}^h$, so that $C_n^h \approx 0$ for large h and $n > h$. Also, from (2.6), $|C_n^h - C_{n-1}^h| \le |r_n|\,|A_{n-1}^h|$, so that $|C_n^h - C_m^h| \le \sum_{j=m+1}^{n} |r_j|\,|A_{j-1}^h|$ for $n > m$ The Cauchy condition is met, and we see that

$$\lim_{n\to\infty} C_n^h = L(C,h) \approx 0 . \tag{2.10}$$

The following formula can be obtained by induction on (2.5)·

$$A_n^h = \prod_{j=h+1}^{n} p_j + \sum_{m=h+1}^{n-2} \left(\prod_{j=m+2}^{n} p_j \right) q_{m+1} C_m^h + q_n C_{n-1}^h . \tag{2 11}$$

Which implies, using $\Pi\, p_j = 0$,

$$\lim_{n\to\infty} A_n^h = 0. \tag{2.12}$$

In analogy with (2.9), (2.7) and (2.8) combine to give

$$B_n^h = p_n B_{n-1}^h + q_n \left(1 + \sum_{j=h+2}^{n-1} r_j B_{j-1}^h \right). \tag{2.13}$$

For $\epsilon > 0$ one can use a lemma analogous to Lemma 2.1 to show that, if h is sufficiently large, $\left| B_n^h - \left(\prod_{j=h+2}^{n} p_j \right) q_{h+1} \right| < \epsilon$ for $n > h$, so that $\{B_n^h\}$ is bounded by a small positive number. Treating D_n^h in (12) as we did C_n^h in (2.6) gives $D_n^h \approx 1$ for large h The Cauchy Condition then gives

$$\lim_{n\to\infty} D_n^h = L(D,h) \approx 1. \tag{2 14}$$

From (2.7) and (2.8) one obtains

$$B_n^h = \left(\prod_{j=h+2}^{n} p_j\right) q_{h+1} + \sum_{m=h+1}^{n-2} \left(\prod_{j=m+2}^{n} p_j\right) q_{m+1} D_m^h + q_n D_{n-1}^h .$$

Using $\Pi\, p_j = 0$, it follows that

$$\lim_{n\to\infty} B_n^h = 0. \tag{2 15}$$

Returning to (2.3), we choose and fix a value of h large enough to satisfy the various requirements already described. Then, from (2 10), (2.12), (2.14), and (2.15),

$$W_{n-1}^h(V_h) = \frac{A_{n-1}^h V_h + B_{n-1}^h}{C_{n-1}^h V_h + D_{n-1}^h} \to \frac{OV_h + O}{L(C,h)V_h + L(D,h)}, \quad \text{as} \quad n \to \infty \quad (L(D,h) \approx 1)$$

Setting $V_h := V_h(z) := W_{h+1}(S(z))$, one gets

$$\lim_{n\to\infty} G_n(z) = \lim_{n\to\infty} \lambda_n^{-1}(W_{n-1}^h(V_h)) = \alpha,$$

provided

$$V_h(z) \neq \infty \quad \text{if} \quad L(C,h) = 0 \quad \text{and} \quad V_h(z) \neq -L(D,h)/L(C,h) \quad \text{if} \quad L(C,h) \neq 0.$$

PROOF OF 2. Using the same set of formulae derived in the proof of part 1, one can show the following: For large values of h, $A_n^h \approx 1$, and

$$\lim_{n\to\infty} C_n^h = L(C,h) \approx 0 \tag{2 10'}$$
$$\lim_{n\to\infty} A_n^h = L(A,h) \approx 1 \tag{2 12'}$$
$$\lim_{n\to\infty} D_n^h = L(D,h) \approx 1 \tag{2.14'}$$
$$\lim_{n\to\infty} B_n^h = L(B,h) \approx 0. \tag{2 15'}$$

Therefore

$$W_{n-1}^h(V_h) = \frac{A_{n-1}^h V_h + B_{n-1}^h}{C_{n-1}^h V_h + D_{n-1}^h} \to \frac{(1+\epsilon_1)V_h(z) + \epsilon_2}{\epsilon_3 V_h(z) + (1+\epsilon_4)} = \frac{Az+B}{Cz+D} := \phi(z).$$

Hence $\lim_{n\to\infty} G_n(z) = \lim_{n\to\infty} \lambda_n^{-1}(W_{n-1}^h(V_h)) = \frac{\beta\phi(z)-\alpha}{\phi(z)-1}$, a bilinear transformation. □

Next, we look at a more general class of functions $f_n \to z$.

THEOREM 2.2. Given a sequence of functions $\{f_n\}$ where $f_n \to z$. Suppose

(i) there exists a convex set S where $S \supset f_n(S)$,

(ii) there exist $\{\alpha_n\}$ where $f_n(\alpha_n) = \alpha_n \in S$, $\alpha_n \to \alpha \in S$, and $\Sigma|\alpha_n - \alpha_{n-1}| < \infty$,

(iii) $|f_n'(z)| \leq K_n \leq 1$ for all $z \in S$, and $K_n \to 1$ with $\Pi\, K_n = 0$.

Then $G_n(z) \to \alpha$ for all z in S.

PROOF. First, we see that $|f_n(z) - f_n(w)| \leq \int_w^z |f_n'(s)|ds \leq K_n|z-w|$ implies $|f_n(z) - f_n(\alpha_n)| \leq K_n|z - \alpha_n|$ for all z in S. Then backward recursion using $|G_n(z) - \alpha| - |\alpha - \alpha_n| \leq |G_n(z) - \alpha_n| \leq K_n|G_{n-1}(z) - \alpha_{n-1}| + K_n|\alpha_n - \alpha_{n-1}|$ gives

$$|G_n(z) - \alpha| \leq |\alpha - \alpha_n| + \left(\prod_1^n K_j\right)|z - \alpha_1| + \sum_{j=2}^{n}\left(\prod_{i=j}^{n} K_i\right)|\alpha_j - \alpha_{j-1}| .$$

From this it is easily shown that $G_n(z) \to \alpha$. □

COROLLARY 2.1. Let $f_n(z) := K_n g_n(z)(z-\alpha_n)+\alpha_n$, for $|z| \le 1$. If (i) $0 < K_n \to 1^-$ with $\Pi K_n = 0$, $\alpha_n \to 0$, and (ii) $|f_n(z)| \le 1$, $|g_n(z)| \le 1$, $g_n(z) \to 1$, for $|z| \le 1$, then $G_n(z) \to 0$ for $|z| \le 1$

The proof follows immediately from Theorem 2.2.

EXAMPLES. $g_n(z) := 1 - 1/n + z/n$ produces quadratic functions $f_n(z) = a_n z^2 + b_n z + c_n$, and $g_n(z) := (2-\nu_n)/(2+\nu_n z^2)$ produces non-bilinear rational functions $f_n(z) = \frac{a_n z + b_n}{c_n z^2 + d_n} + e_n$ In both instances $K_n := 1 - 1/n$, $\alpha_n := 1/n^2$ are sufficient to satisfy conditions in the hypothesis of the theorem.

3. INNER COMPOSITION

We turn now to the functional sequence $\{F_n(z)\}$ described in (1.2) and useful in studying traditional continued fractions. This form of composition has the longest history. One of the earliest results for the case $f(z) := z$ is the following (DePree & Thron [7]):

THEOREM 3.1. Let $F_n(z) := f_1 \circ f_2 \circ \ldots \circ f_n(z) = (P_n z + Q_n)/(R_n z + S_n)$, and $f_n(z) = (a_n z + b_n)/(c_n z + d_n)$, with $a_n d_n - b_n c_n \equiv 1$. Suppose that

(i) $\Sigma|b_n|$ and $\Sigma|c_n|$ both converge

(ii) $|a_n| = 1 + \epsilon_n$, $\epsilon_n \ge 0$, $\Sigma\epsilon_n$ diverges

Then $\lim_{n\to\infty} F_n(z) = \Gamma$, a constant, for all $z \neq 0$

In dynamical terms we have

COROLLARY 3.1. Suppose that the following conditions hold:

(i) $\Sigma|\alpha_n|$ and $\Sigma|\beta_n|^{-1}$ both converge

(ii) $K_n \to 1$

(iii) $\left|\dfrac{K_n\beta_n - \alpha_n}{\sqrt{K_n}\,(\beta_n - \alpha_n)}\right| \quad 1 \quad$ and $\quad \sum\left(\left|\dfrac{K_n\beta_n - \alpha_n}{\sqrt{K_n}\,(\beta_n - \alpha_n)}\right| - 1\right)$ diverges

Then $\lim_{n\to\infty} F_n(z) = \Gamma$, a constant, for all $z \neq 0$.

PROOF. From (2.1) and $a_n d_n - b_n c_n = 1$, one gets

$$a_n = (\alpha_n - K_n\beta_n)/\triangle_n, \quad b_n = \alpha_n\beta_n(K_n - 1)/\triangle_n, \quad c_n = (1 - K_n)/\triangle_n, \tag{3.1}$$

$d_n = (K_n\alpha_n - \beta_n)/\triangle_n$, where $\triangle_n := \sqrt{K_n}\,(\alpha_n - \beta_n)$. Conditions (i), (ii), (iii) of the corollary then imply the hypotheses of Theorem 3.1

A different set of hypotheses on the fixed points leads to a similar conclusion: the techniques of proof of Theorem 2.1 can be used to prove an analogue of that theorem for inner composition The steps are nearly identically, so only an extended outline of the proof is given.

THEOREM 3.2. Suppose

(i) $0 < |K_n| < 1$, $K_n \to 1\,(f_n \to z)$ <u>or</u> $K_n \to 0\,(f_n \to \alpha)$

(ii) $\alpha_n \to \alpha$, $\beta_n \to \beta$, with $\alpha \neq \beta$,

(iii) $\Sigma\,|\alpha_n - \alpha_{n-1}| < \infty$, and $\Sigma|\beta_n - \beta_{n-1}| < \infty$, and

(iv) $\Pi K_n = 0$.

Then $\{F_n(z)\}$ converges to Γ, a constant, for all $z \in \mathbb{C}$ except $z = \beta$

PROOF. We write

$$\begin{aligned} F_n(z) &= \lambda_1^{-1} \circ w_1 \circ \ldots \circ w_h \circ w_{h+1} \circ \ldots \circ w_{n-1}(S_n(z)) \\ &= \lambda_1^{-1} \circ W_h \circ W_{n-1}^h(S_n(z)), \end{aligned} \tag{3.2}$$

where

$$\begin{aligned} w_j(z) &:= K_j \circ \lambda_j \circ \lambda_{j+1}^{-1}(z) \\ W_j(z) &:= w_1 \circ w_2 \circ \ldots \circ w_j(z) \\ W_{n-1}^h(z) &:= w_{h+1} \circ w_{h+2} \circ \ldots \circ w_{n-1}(z) \quad \text{and} \\ S_n(z) &:= K_n \circ \lambda_n(z). \end{aligned}$$

And find that

$$\begin{aligned} w_j(z) = (p_j z + q_j)/(r_j z + 1), \qquad & p_j = K_j(\beta_{j+1} - \alpha_j)/(\beta_j - \alpha_{j+1}), \\ q_j = K_j(\alpha_j - \alpha_{j+1})/(\beta_j - \alpha_{j+1}), \quad & r_j = (\beta_{j+1} - \beta_j)/(\beta_j - \alpha_{j+1}). \end{aligned} \tag{3 3}$$

Now write

$$\begin{aligned} & W_{h+j}^h = w_{h+1} \circ w_{h+2} \circ \ldots \circ w_{h+j}(z) = (A_{h+j}^h z + B_{h+j}^h)/(C_{h+j}^h z + D_{h+j}^h), \\ & \text{with} \quad A_{h+1}^h = p_{h+1}, \quad B_{h+1}^h = q_{h+1}, \quad C_{h+1}^h = r_{h+1}, \quad \text{and} \quad D_{h+1}^h = 1. \end{aligned}$$

As before

$$A_n^h = p_n A_{n-1}^h + r_n B_{n-1}^h \tag{3 4}$$
$$C_n^h = p_{hn} C_{n-1}^h + r_n D_{n-1}^h \tag{3 5}$$
$$B_n^h = q_n A_{n-1}^h + B_{n-1}^h \tag{3 6}$$
$$D_n^h = q_n C_{n-1}^h + D_{n-1}^h \tag{3 7}$$

By hypotheses, $\Sigma |q_j|$ and $\Sigma |r_j|$ both converge, and $p_j = K_j(1 + s_j)$, where $s_j = [(\alpha_{j+1} - \alpha_j) + (\beta_{j+1} - \beta_j)]/(\beta_j - \alpha_{j+1})$. Thus $\Sigma |s_j|$ converges, and this implies $\Pi\, p_j = (\Pi\, K_j)$ $\Pi(1 + s_j) = 0$ if $\Pi\, K_j = 0$. Clearly $\left| \prod_{j=h+1}^{n} p_j \right| < 1$ for h sufficiently large and $n > h$.

As before, we find that A_n^h is uniformly bounded with regard to n if h is large enough, and is in fact close to 0 Therefore (3.6) gives $\lim_{n\to\infty} B_n^h = L(B, h) \approx 0$. This, coupled with the recursive formula

$$A_n^h = \prod_{j=h+1}^{n} p_j + \sum_{m=h+2}^{n-1} \left(\prod_{j=m+1}^{n} p_j \right) r_m B_{m-1}^h + r_n B_{n-1}^h \tag{3 8}$$

gives $\lim_{n\to\infty} A_n^h = 0$. Similarly, C_n^h is bounded uniformly (and close to 0) for sufficiently large h and all $n > h$. Thus $\lim_{n\to\infty} D_n^h = L(D, h) \approx 1$.

Therefore

$$C_n^h = \left(\prod_{j=h+2}^{n} p_j \right) r_{h+1} + \sum_{m=h+2}^{n-1} \left(\prod_{j=m+1}^{n} p_j \right) r_m D_{m-1}^h + r_n D_{n-1}^h \tag{3 9}$$

shows that $\lim_{n\to\infty} C_n^h = 0$.

Now $F_n(z) = \lambda_1^{-1} \circ W_h \circ W_{n-1}^h(S_n(z))$, where $\lim_{n\to\infty} S_n(z) = (z - \alpha)/(z - \beta)$ and $\lim_{n\to\infty} W_{n-1}^h(S_n(z)) = \lim_{n\to\infty} \frac{A_{n-1}^h S_n(z) + B_{n-1}^h}{C_{n-1}^h S_n(z) + D_{n-1}^h} = L(B, h)/L(D, h) \approx 0$ for $z \neq \beta$. Hence $\lim_{n\to\infty} F_n(z) = \lambda_1^{-1} \circ W_h(L(B, h)/L(D, h))$, for $z \neq \beta$. □

Theorem 1.2, like Theorem 3 1, is an earlier result for the case $f_n(z) \to z$. In terms of fixed points we have

COROLLARY 3.2. If $0 < |K_n| < 1$, $K_n \to 1$, $\alpha_n \to \alpha$, $\beta_n \to \beta$, $\alpha \neq \beta$, and $\Pi\, K_n$ converges absolutely, then $\lim_{n\to\infty} F_n(z) = (Pz + Q)/(Rz + S)$, where $PS - RQ \neq 0$.

PROOF. From (3.1) it is not difficult to verify that the hypotheses of the corollary imply those of Theorem 1.2 For the products involving a_n and d_n use $K_n = 1 + (K_n - 1)$ and observe that the convergence of $\Sigma\, |1 - K_n|$ implies that of $\Sigma\, |1 - \sqrt{K_n}\,|$.

Finally we present a simple result for sequences of more general analytic functions Actually, Theorem 3.3 is a corollary to Theorem 2.2 [12], but its proof is so brief it is given here

THEOREM 3.3. Suppose $\{f_n(z)\}$ is a sequence of functions analytic on a convex and compact set S and such that $f_n \to z$ or $f_n \to \alpha$ on S, with (i) $S \supset f_n(S)$ for each n, (ii) $|f_n'(z)| \leq K_n < 1$, and (iii) $\Pi K_n \to 0$. Then $\lim_{n\to\infty} F_n(z) = \Gamma \in S$, uniformly for all z in S

PROOF. As in the proof of Theorem 2.2, condition (ii) implies $|f_n(z_1) - f_n(z_2)| \leq K_n|z_1 - z_2|$ Applying the Cauchy Condition,

$$|F_{n+p}(z) - F_n(z)| \leq K_1|f_2 \circ f_3 \circ \ldots \circ f_{n+p}(z) - f_2 \circ f_3 \circ \ldots \circ f_n(z)|$$

$$\vdots$$

$$\leq K_1K_2 \ldots K_n|f_{n+1} \circ f_{n+2} \circ \ldots \circ f_{n+p}(z) - z| \leq \left(\prod_1^n K_j\right)M \to 0$$

where $M := \text{diam}(S)$. Similarly $|F_n(z_1) - F_n(z_2)| \leq \left(\prod_1^n K_j\right)M \to 0$ for all $z_1, z_2 \in S$ □

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