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ON MAPS: CONTINUOUS, CLOSED, PERFECT, AND WITH CLOSED GRAPH

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ABSTRACT. This paper gives relationships between continuous maps, closed maps, perfect maps, and maps with closed graph in certain classes of topological spaces.

KEY WORDS AND PHRASES. Continuous, closed, perfect, closed graph, B-W compact, Frechet, fiber, Hausdorff, regular, compact, countably compact.
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1. INTRODUCTION.

Throughout, by a space we shall mean a topological space. No separation axioms are assumed and no map is assumed to be continuous or onto unless mentioned explicitly; cl(A) will denote the closure of the subset A in the space X. A space X is said to be T_1 at its subset A if each point of A is closed in X. X is said to be B-W Compact [1] if every infinite subset of X has at least one limit point. A point x in X is said to be a cluster point (w- limit point in the terminology of Thron [1]) of a subset A of X if every neighbourhood of x contains an infinite number of points of A. X is said to be a Frechet space if whenever $x \in cl(A)$, there is a sequence of points in A converging to x.A map $f:X \to Y$ is said to be perfect if it is continuous, closed, and has compact fibers $f^{-1}(y)$, $y \in Y$. For study of perfect maps, see [2] and its references.

The primary purpose of this paper is to give relationships between continuous maps,closed maps, perfect maps, and maps with closed graph. A generalization and an analogue of theorem 5 of Piotrowski and Szymanski [3] and analogues of theorem 1.1.17 and corollary 1.1.18 of Hamlett and Herrington [4] are also obtained.

NOTE. The definitions of subcontinuous and inversely subcontinuous maps can be found in Fuller [5].

2. MAIN RESULTS.

THEOREM 1 [4] .Let $f:X \to Y$ be continuous, where Y is Hausdorff. Then f has closed graph.

THEOREM 2. Let $f:X \to Y$ be closed with closed (compact) fibers,where X is regular (Hausdorff). Then f has closed graph.

PROOF. We prove only the parenthesis part; the other part, which can also be proved in a simple manner by using our proof of the parenthesis part, has been proved by Fuller [5,corollary 3.9]and by Hamlett and Herrington [4, theorem 1.1.17] by different techniques. Let $x \in X$, $y \in Y$, $y \neq f(x)$. Then $x \notin f^{-1}(y)$, which is compact. Since X is Hausdorff, there exist disjoint open sets U and V containing x and $f^{-1}(y)$ respectively. Then f is closed implies there exists an open set W containing y such that $f^{-1}(W)\subset V$ and therefore, $f(U)\cap W=\phi$. It follows that f has closed graph.

Combining theorems 1 and 2, we get the following

THEOREM 3. Let $f:X \to Y$ be perfect, where either X or Y is Hausdorff. Then f has closed graph.

The following theorem 4(theorem 5), part (b) of which is a generalization (analogue) of theorem 5 of Piotrowski and Szymanski [3], gives sufficient conditions under which the converse of theorem 1(theorem 2) holds.

THEOREM 4. Let $f:X \to Y$ have closed graph. Then f is continuous if any one of the following conditions is satisfied.

(a) Y is compact,

(b) X is Frechet and Y is B-W compact,

(c) f is subcontinuous.

PROOF. We give the proof of part (b) only; part (a) is well known (corollary 2(b) of Piotrowski and Szymanski [3],and theorem 1.1.10 of [4]), while part (c) is theorem 3.4 of Fuller [5]. Let F be a closed subset of Y and let $x\in clf^{-1}(F)-f^{-1}(F)$. Since X is a Frechet space, there exists a sequence $\{x_n\}$ of points in $f^{-1}(F)$ such that $x_n \to x$. Since f has closed graph, the set H of values of the sequence $\{f(x_n)\}$ is an infinite subset of the B-W compact set F and F is T_1 at H. Therefore, H has a cluster point $y\in F$, $y \neq f(x)$, and the set $U=X-f^{-1}(y)$ is an open set containing x. Then $x_n \to x$ implies there exists a positive integer n_0 such that $x_n\in U$ for all $n\geq n_0$. Again f has closed graph and the set $K=\{x_n:n\geq n_0\}\cup\{x\}$ is compact; it follows that f(K) is closed, which is a contradiction since it is easy to see that $y\in clf(K)-f(K)$. Hence f must be continuous.

THEOREM. 5. Let $f:X \to Y$ have closed graph. Then f is closed if any one of the following conditions is satisfied.

(a) X is compact,

(b) X is countably compact and Y is Frechet,

(c) f is inversely subcontinuous.

PROOF. We give the proof of part (b) only; part (a) is well known (corollary 2(a) of Piotrowski and Szymanski [3]), while part (c) is theorem 3.5 of Fuller [5].Let F be a closed subset of X and let $y\in clf(F)-f(F)$. Since Y is Frechet and T_1 at f(X), there exists a sequence $\{f(x_n)\}$of distinct points converging to y where $x_n\in F$.Now the set of values of the sequence $\{x_n\}$ is an infinite subset of the countably compact set F and therefore, it has a cluster point $x\in F$, $y \neq f(x)$. Since Y is T_1 at f(X), the set $V =Y-\{f(x)\}$ is an open set containing y. Then $f(x_n) \to y$ implies there exists a positive integer n_0 such that $f(x_n)\in V$ for all $n\geq n_0$. Since f has closed graph and the set $K =\{f(x_n):n\geq n_0\}\cup\{y\}$ is compact, it follows that $f^{-1}(K)$ is closed, which is a contradiction since it is easy to see that $x\in clf^{-1}(K)-f^{-1}(K)$. Hence f must be closed.

Combining theorems 1 and 5(theorems 2 and 4), we obtain the following theorem 6 (theorem 7), giving a relationship between continuous and closed maps. Theorem 6 includes theorem 16.19 of Thron [1], while theorem 7 includes and gives analogues of corollary 1.1.18 of Hamlett and Herrington [4].

THEOREM 6. Let $f:X \to Y$ be continuous, where Y is Hausdorff and one of the conditions (a), (b), (c) in theorem 5 is satisfied. Then f is closed.

The condition that X is countably compact in theorems 5(b)and 6(b) cannot be replaced by the weaker condition that X is B-W compact,as the following example shows.

EXAMPLE. Let X=N, the positive integers, with a base for a topology on X the family of all sets of the form $\{2n-1,2n\}$,$n\in N$, and $Y=\{0,1,1/2,....1/n,...\}$ as a subspace of the real line. The map $f:X \to Y$, defined by $f(2n-1)=1/n-1=f(2n)$ for $n\geq 2$ and $f(1)=0=f(2)$, is a continuous surjection which is not closed, although X is B-W compact and Y is Frechet, Hausdorff.

THEOREM 7. Let $f:X \to Y$ be closed with closed (compact) fibers, where X is regular (Hausdorff) and one of the conditions (a), (b), (c) in theorem 4 is satisfied. Then f is continuous(perfect).

Combining theorems 1 and 4, we obtain the following relationship between continuous maps and maps with closed graph.

THEOREM 8. Let $f:X \to Y$ be any map, where Y is Hausdorff and one of the conditions (a), (b), (c) of theorem 4 is satisfied. Then f is continuous if and only if it has closed graph.

Combining theorems 2 and 5, we obtain the following relationship between closed maps and maps with closed graph.

THEOREM 9. Let $f:X \to Y$ be any map with closed (compact) fibers, where X is regular (Hausdorff) and one of the conditions (a), (b), (c) of theorem 5 is satisfied. Then f is closed if and only if it has closed graph.

Combining theorems 3,4 and 5, we obtain the following relationship between perfect maps and maps with closed graph.

THEOREM 10.Let $f:X \to Y$ be any map with compact fibers, where either X is Hausdorff or Y is Hausdorff and one of conditions (a), (b), (c) of theorem 4 and one of the conditions (a), (b), (c) of theorem 5 are satisfied. Then f is perfect if and only if it has closed graph.

COROLLARY. Let $f:X \to Y$ be a bijection and one of the conditions (a), (b), (c) of theorem 4 and one of the conditions (a), (b), (c) of theorem 5 be satisfied.Then f has closed graph if and only if it is a homeomorphism and both X,Y are Hausdorff.

Combining theorems 8,9,and 10 we obtain the following

THEOREM 11. Let $f:X \to Y$ be any map with closed (compact) fibers,where X is regular (Hausdorff), Y is Hausdorff, and one of the conditions (a), (b), (c) of theorem 4 and one of the conditions (a), (b), (c) of theorem 5 are satisfied. Then the following conditions (i) to (iii) {(i) to (iv)} are equivalent.

(i) f is continuous.

(ii) f is closed.

(iii) f has closed graph.

(iv) f is perfect.

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