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ON PRIME AND SEMIPRIME NEAR-RINGS WITH DERIVATIONS

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ABSTRACT. Let N be a semiprime right near-ring, A a subset of N such that $0 \in A$ and $AN \subseteq A$, and d a derivation of N The purpose of this paper is to prove that if d acts as a homomorphism on A or as an anti-homomorphism on A, then $d(A) = \{0\}$

KEY WORDS AND PHRASES: Prime near-ring, semiprime near ring, ideal, derivation, homomorphism, anti-homomorphism
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1. INTRODUCTION

Throughout this paper N will be a right near-ring A derivation on N is defined to be an additive endomorphism satisfying the "product rule" $d(xy) = xd(y) + d(x)y$ for all $x, y \in N$ According to Bell and Mason [1], a near-ring N is said to be prime if $xNy = \{0\}$ for $x, y \in N$ implies $x = 0$ or $y = 0$, and semiprime if $xNx = \{0\}$ for $x \in N$ implies $x = 0$ Let S be a nonempty subset of N and d be a derivation of N If $d(xy) = d(x)d(y)$ or $d(xy) = d(y)d(x)$ for all $x, y \in S$, then d is said to act as a homomorphism or anti-homomorphism on S, respectively. As for terminologies used here without mention, we refer to Pilz [2].

Bell and Kappe [3] proved that if d is a derivation of a semiprime ring R which is either an endomorphism or anti-endomorphism, then $d = 0$ They also showed that if d is a derivation of a prime ring R which acts as a homomorphism or an anti-homomorphism on I, where I is a nonzero right ideal, then $d = 0$ on R

2. THE RESULTS

It is our aim in this paper to prove that the above conclusions hold for near-rings as follows

THEOREM Let N be a semiprime right near-ring, and d a derivation on N Let A be a subset of N such that $0 \in A$ and $AN \subseteq A$ If d acts as a homomorphism on A or as an anti-homomorphism on A, then $d(A) = \{0\}$.

In order to give the proof of the above theorem we need the following lemmas

LEMMA 1. If N is a right near-ring and d a derivation of N, then

$$c(yd(x) + d(y)x) = cyd(x) + cd(y)x \quad \text{for all} \quad x, y, c \in N.$$

A proof can be given by using a similar approach as in the proof of [1, Lemma 1]

LEMMA 2. Let N be a right near-ring, d a derivation of N, and A a multiplicative subsemigroup of N which contains 0 If d acts as an anti-homomorphism on A, then $a0 = 0$ for all $a \in A$

PROOF. Since $0a = 0$ for all $a \in A$ and d acts as an anti-homomorphism on A then we have $d(a)0 = 0$ for all $a \in A$ Taking $a0$ instead of a, one can obtain $a0 + d(a)0 = 0$ for all $a \in A$ Thus we get $a0 = 0$ for all $a \in A$

LEMMA 3. Let N be a right near-ring, and A a multiplicative subsemigroup of N

(a) If d acts as a homomorphism on A, then

$$d(y)xd(y) = yxd(y) = d(y)xy \quad \text{for all} \quad x, y \in A. \tag{2 1}$$

(b) If d acts as an anti-homomorphism on A, then

$$d(y)xd(y) = d(y)yx = xyd(y) \quad \text{for all} \quad x, y \in A. \tag{2 2}$$

PROOF. (a) Let d act as a homomorphism on A Then

$$d(xy) = xd(y) + d(x)y = d(x)d(y) \quad \text{for all} \quad x, y \in A. \tag{2 3}$$

Taking yx instead of x in (2.3) we get

$$yxd(y) + d(yx)y = d(yx)d(y) = d(y)d(xy) \quad \text{for all} \quad x, y \in A. \tag{2 4}$$

By Lemma 1, $d(y)d(xy) = d(y)xd(y) + d(y)d(x)y = d(y)xd(y) + d(yx)y$ Using this relation in (2 4), we obtain $yxd(y) = d(y)xd(y)$ for all $x, y \in A$ Similarly, taking yx instead of y in (2 3) one can prove the relation $d(y)xd(y) = d(y)xy$ for all $x, y \in A$.

(b) Since d acts as an anti-homomorphism on A, we have

$$d(xy) = xd(y) + d(x)y = d(y)d(x) \quad \text{for all} \quad x, y \in A. \tag{2 5}$$

Substituting xy for y in (2 5) leads to

$$\begin{aligned} xd(xy) + d(x)xy &= d(xy)d(x) \\ &= xd(y)d(x) + d(x)yd(x) \\ &= xd(xy) + d(x)yd(x) \qquad \text{for all} \qquad x, y \in A. \end{aligned}$$

From this relation we arrive at $d(x)xy = d(x)yd(x) = 0$ for all $x, y \in A$ Similarly taking xy instead of x in (2 5), one can prove the relation $d(y)xd(y) = xyd(y)$ for all $x, y \in A$.

PROOF OF THEOREM. (a) First suppose that d acts as a homomorphism on A. By Lemma 3 (a), we have

$$d(y)xd(y) = d(y)xy \quad \text{for all} \quad x, y \in A. \tag{2 6}$$

Right-multiplying (2.6) by $d(z)$, where $z \in A$, and using the hypothesis that d acts as a homomorphism on A together with Lemma 1, we obtain $d(y)xd(y)z = 0$ for all $x, y, z \in A$ Taking xr instead of x, where $r \in N$, we have $d(y)xrd(y)z = 0$ for all $x, y, z \in A$ and $r \in N$. Hence $d(y)xNd(y)x = \{0\}$ for all $x, y \in A$; and by semiprimeness

$$d(y)x = 0 \quad \text{for all} \quad x, y \in A. \tag{2 7}$$

Substituting yr for y in (2 7), where $r \in N$, leads to

$$yd(r)x + d(y)rx = 0 \quad \text{for all} \quad x, y \in A, r \in N. \tag{2.8}$$

Left-multiplying (2.8) by $d(z)$, where $z \in A$, we have that $d(z)yd(r)x + d(z)d(y)rx = 0$ According to (2 7) this relation reduces to $d(zy)rx = 0$. Hence we get $zd(y)rx = 0$ for all $x, y, z \in A$ and $r \in N$ By semiprimeness, we get

$$zd(y) = 0 = zrd(y) \quad \text{for all} \quad y, z \in A \quad \text{and} \quad r \in N. \tag{2.9}$$

Combining (2 7) and (2.9) shows that $d(yz) = 0$ for all $y, z \in A$ In particular, $d(xrx) = 0$ for all $x \in A$, $r \in N$, and since d acts as a homomorphism on A,

$$d(xr)d(x) = 0 = xd(r)d(x) + d(x)rd(x) \quad \text{for all} \quad x \in A, r \in N.$$

In view of (2.9), this gives $d(x)Nd(x) = \{0\}$ and hence $d(x) = 0$ for all $x \in A$

(b) Now assume that d acts as an anti-homomorphism on A Note that $a0 = 0$ for all $a \in A$ by Lemma 2. According to Lemma 3 (b),

$$d(y)xd(y) = xyd(y) \quad \text{for all} \quad x, y \in A, \tag{2 10}$$

$$d(y)xd(y) = d(y)yx \quad \text{for all} \quad x, y \in A. \tag{2 11}$$

Replacing x by $xd(y)$ in (2.10) and using Lemma 1 we get

$$d(y)xyd(y) + d(y)xd(y)y = xd(y)yd(y) \quad \text{for all} \quad x, y \in A. \tag{2 12}$$

Substituting xy for x in (2 10), we have

$$d(y)xyd(y) = xy^2d(y) \quad \text{for all} \quad x, y \in A. \tag{2 13}$$

Right-multiplying (2 10) by y we arrive at

$$d(y)xd(y)y = xyd(y)y \quad \text{for all} \quad x, y \in A. \tag{2 14}$$

Replacing x by y in (2.10), we have $d(y)yd(y) = y^2d(y)$, and left-multiplying this relation by x, we obtain

$$xd(y)yd(y) = xy^2d(y) \quad \text{for all} \quad x, y \in A. \tag{2 15}$$

Using (2 13), (2 14), and (2.15) in (2.12) one obtains $xyd(y)y = 0$ for all $x, y \in A$, hence $yryd(y)y = 0$ and $yd(y)yryd(y)y = yd(y)0 = 0$ for all $y \in A, r \in N$, and by semiprimeness

$$yd(y)y = 0 \quad \text{for all} \quad y \in A.$$

According to (2.14) we get $d(y)xd(y)y = 0$ for all $x, y \in A$ Using this relation in (2 11), we arrive at

$$d(y)yxy = 0 \quad \text{for all} \quad x, y \in A. \tag{2 16}$$

Replacing x by $xd(y)$ in (2.16), we have $d(y)yxd(y)y = 0 = d(y)yxrd(y)yx$ for all $x, y \in A, r \in N$, hence

$$d(y)yx = 0 \quad \text{for all} \quad x, y \in A. \tag{2 17}$$

Using (2 17) in (2 11), one obtains $d(y)xd(y) = 0 = d(y)xrd(y)x$ for all $x, y \in A, r \in N$, hence

$$d(y)x = 0 \quad \text{for all} \quad x, y \in A. \tag{2 18}$$

Therefore,

$$\begin{aligned} &xd(z)d(yn)x = 0 && \text{for all} \quad x, y, z \in A, \quad n \in N, \\ &xd(z)(yd(n) + d(y)n)x = 0 && \text{for all} \quad x, y, z \in A, \quad n \in N, \quad \text{and} \\ &xd(z)yd(n)x + xd(z)d(y)nx = 0 && \text{for all} \quad x, y, z \in A, \quad n \in N. \end{aligned}$$

In view of (2 18), this gives $xd(z)d(y)nx = 0 = xd(z)d(y)nxd(z)d(y)$, hence $xd(z)d(y) = 0$ for all $x, y, z \in A$ Since d acts as an anti-homomorphism on A, we have $xd(yz) = 0$ for all $x, y, z \in A$, so that $xyd(z) + xd(y)z = 0$ for all $x, y, z \in A$ By (2.18) we now get $xyd(z) = 0 = xd(z)ryd(z)$ for all $x, y, z \in A$ and $r \in N$; and taking x instead of y we get $xd(z) = 0$ for all $x, z \in A$ Recalling (2.18), we now have $d(xy) = 0$ for all $x, y \in A$, so $d(xxr) = 0$ for all $x \in A$ and $r \in N$. Thus $d(xr)d(x) = 0$, and we finish the proof as in case (a).

We now state some consequences of the theorem

COROLLARY 1. Let N be a semiprime right near-ring, and d a derivation of N If d acts as a homomorphism on N or as an anti-homomorphism on N, then $d = 0$

COROLLARY 2. Let N be a prime right near-ring, and d a derivation of N Let A be a nonzero subset of N such that $0 \in A$ and $AN \subseteq A$. If d acts as a homomorphism on A or as an anti-homomorphism on A then, $d = 0$

PROOF. By the theorem, we have $d(a) = 0$ for all $a \in A$ Then $d(ax) = ad(x) + d(a)x = ad(x) = 0 = ayd(x)$ for all $a \in A, x, y \in N$, and by primeness we get $a = 0$ or $d(x) = 0$ for all $a \in A, x \in N$ Since A is nonzero, we have $d(x) = 0$ for all $x \in N$

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