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# RESEARCH NOTES

## NOTES ON ($\alpha,\beta$)-DERIVATIONS

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**ABSTRACT.** Let $R$ be a prime ring of characteristic not 2, $U$ a nonzero ideal of $R$ and $0 \neq d$ a $(\alpha,\beta)$-derivation of $R$ where $\alpha$ and $\beta$ are automorphisms of $R$. i) $[d(U),a]=0$ then $a \in Z$ ii) For $a$, $b \in R$, the following conditions are equivalent (I) $\alpha(a)d(x) = d(x)\beta(b)$, for all $x \in U$ (II) Either $\alpha(a) = \beta(b) \in C_R(d(U))$ or $C_R(a) = C_R(b) = R'$ and $a[a,x] = [a,x]b$ (or $a[b,x] = [b,x]b$) for all $x \in U$ Let $R$ be a 2-torsion free semiprime ring and $U$ be a nonzero ideal of $R$ iii) Let $d$ be a $(\alpha,\beta)$-derivation of $R$ and $g$ be a $(\gamma,\delta)$-derivation of $R$. Suppose that $dg$ is a $(\alpha\gamma,\beta\delta)$-derivation and $g$ commutes both $\gamma$ and $\delta$ then $g(x)U\alpha^{-1}d(y) = 0$, for all $x, y \in U$. iv) Let $Ann(U) = 0$ and $d$ be an $(\alpha,\beta)$-derivation of $R$ and $g$ be a $(\gamma,\delta)$-derivation of $R$ such that $g$ commutes both $\gamma$ and $\delta$ If for all $x, y \in U$, $\beta^{-1}(d(x))Ug(y) = 0 = g(x)U\alpha^{-1}(d(y))$ then $dg$ is a $(\alpha\gamma,\beta\delta)$-derivation on $R$

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## 1. INTRODUCTION

Let $R$ be a ring and $X$ be a subset of $R$. Let $Ann_r(X) = \{a \in R \mid xa = 0 \text{ all } x \in X\}$ and $Ann_\ell(X) = \{a \in R \mid ax = 0 \text{ all } x \in X\}$ be the right and left annihilators, respectively, of the subset $X$ of $R$ If $R$ is a semiprime ring then the left and right and two-sided annihilators of an ideal $X$ coincide It will be denoted by $Ann(X)$. Let $U$ be an ideal of $R$ Note that if $\sigma$ is an automorphism of $R$ and $Ann(U) = 0$ then $Ann(\sigma(U)) = 0$. Let $R$ be a ring and $\alpha$, $\beta$ be two automorphisms of $R$ An additive mapping $d : R \to R$ is called an $(\alpha,\beta)$-derivation if $d(xy) = \alpha(x)d(y) + d(x)\beta(y)$ holds for all pairs $x$, $y \in R$

Throughout this note $R$ will represent an associative ring Let $R' = \{x \in R \mid d(x) = 0\}$ The centralizer of a subset $A$ of $R$ is $C_R(A) = \{y \in R \mid ay = ya,\ \forall a \in A\}$ $C_R(R) = Z$, the center of $R$

There are two motivations for this research Herstein [1] has proved Let $R$ be a prime ring of characteristic not 2, and $0 \neq d$ be a derivation of $R$ Then any element $a \in R$ satisfying $ad(x) = d(x)a$ for all $x \in R$, should be central In [2], Daif has proved the following theorem Let $R$ be a prime ring and $a$, $b \in R$ Then the following conditions are equivalent

(i) $ad(x) = d(x)b,\ \forall x \in R$

(ii) Either $a = b \in C_R(d(R))$ or $C_R(a) = C_R(b) = R'$ and $a[a,x] = [a,x]b$ (or $a[b,x] = [b,x]b$) for all $x \in R$ In the first part of this note we generalized these two theorems for an ideal $U$ and $(\alpha,\beta)$-derivation of $R$

In the second part, Bresar and Vukman [3] give some results concerning two derivations in semiprime rings We will generalize some of these results by taking an ideal of $R$ instead of $R$ and extend to more general mappings As a result of this, we will give a generalization of a well-known result of Posner which states that if $R$ is a prime ring of characteristic not 2 and $d$, $g$ are nonzero derivation of $R$ then $dg$ cannot be a derivation

## 2. RESULTS

**LEMMA 1.** Let $R$ be a prime ring of characteristics not 2, $(0) \neq U$ an ideal of $R$, $0 \neq d : R \rightarrow R$ a $(\alpha, \beta)$-derivation such that $\alpha d = d\alpha$, $d\beta = \beta d$ and $a \in R$. If $a \in C_R(d(U))$ then $a \in Z$

**PROOF.** Since $a \in C_R(d(U))$, $ad(x) = d(x)a$ for all $x \in U$ Replacing $x$ by $xy$, $y \in U$, we obtain $a\alpha(x)d(y) + ad(x)\beta(y) = \alpha(x)d(y)a + d(x)\beta(y)a$. Using hypothesis we have

$$d(x)[a, \beta(y)] = [\alpha(x), a]d(y).$$

Taking $yr$, $r \in R$, instead of $y$, we obtain

$$d(x)\beta(y)[a, \beta(r)] = [\alpha(x), a]\alpha(y)d(r) \quad \text{for all} \quad x, y \in U, r \in R.$$

If we replace $r$ by $\beta^{-1}(d(z)), z \in U$, we get $d(x)\beta(y)[a, d(z)] = [\alpha(x), a]\alpha(y)\beta^{-1}(d^2(z))$. Since $a \in C_R(d(U))$ we have $[\alpha(x), a]\alpha(y)\beta^{-1}(d^2(z)) = 0$ for all $x, y, z \in U$ Since $\alpha(U)$ is an ideal of $R$ and $R$ is prime we get $a \in Z$ or $d^2(U) = 0$. If $d^2(U) = 0$ then $0 = d^2(xy) = \alpha^2(x)d^2(y) + 2d(\alpha(x))d(\beta(y))$ and so $d(\alpha(x))d(\beta(y)) = 0$. By [4, Lemma 3] we have a contradiction Thus $a \in Z$.

**THEOREM 1.** Let $R$ be a prime ring of characteristic not 2, $0 \neq d : R \rightarrow R$ a $(\alpha, \beta)$-derivation, $(0) \neq U$ and ideal of $R$ and $a, b \in R$. Then the following conditions are equivalent

(I) $\alpha(a)d(x) = d(x)\beta(b)$, for all $x \in U$.

(II) Either $\beta(b) = \alpha(a) \in C_R(d(U))$ or $C_R(a) = C_R(b) = R'$ and $a[a,x] = [a,x]b$ (or $a[b,c] = [b,x]b$) for all $x \in U$.

**PROOF.** (I) $\Rightarrow$ (II) If $a \in C_R(d(U))$ then by Lemma 1 we get $\alpha(a) \in Z$. (I) gives $d(x)(\beta(b) - \alpha(a)) = 0$, for all $x \in U$. By [4, Lemma 3] it implies that $\beta(b) = \alpha(a)$. Similarly, if $\beta(b) \in C_R(d(U))$ then $\beta(b) = \alpha(a)$.

We assume henceforth that neither $\alpha(a)$ nor $\beta(b)$ in $C_R(d(U))$. Let in (I) $x$ be $rx$, where $r \in R$, and using (I), we have $\alpha(a)\alpha(r)d(x) + \alpha(a)d(r)\beta(x) = \alpha(r)d(x)\beta(b) + d(r)\beta(x)\beta(b)$ and so

$$\alpha([a, r])d(x) = d(r)\beta(xb) - \alpha(a)d(r)\beta(x). \tag{2.1}$$

Taking $y$ instead of $r$ where $y \in U$, in (2.1) and using (I) we obtain

$$\alpha([a, y])d(x) = d(y)\beta([x, b]), \quad \text{for all} \quad x, y \in U. \tag{2.2}$$

Now if $d(x) = 0$ then (2.2) gives us $d(y)\beta([x,b]) = 0$ for all $y \in U$ By [4, Lemma 3], we get $x \in C_R(b)$. Conversely, if $x \in C_R(b)$, then (2.2) gives us $\alpha([y, a])d(x) = 0$. Since by [4, Lemma 3] $a \notin Z$, we have $d(x) = 0$ Therefore $C_R(b) = R'$. Similarly, we can show that $C_R(a) = R'$. In particular, $d(a) = d(b) = 0$ and $ab = ba$.

Replace $r$ by $yb, y \in U$, in (2.1) we have $\alpha([a, y])\alpha(b)d(x) = d(y)\beta(b)(xb) - \alpha(a)d(y)\beta(bx) = \alpha(a)d(y)\beta(bx) = \alpha(a)d(y)\beta(xb) - \alpha(a)d(y)\beta(bx) = \alpha(a)d(y)\beta([x, b])$ and using (2.2) we get $\alpha([a, y])\alpha(b)d(x) = \alpha(a)\alpha([a, y])d(x)$ and so

$$\alpha([a, y]b - a[a, y])d(x) = 0 \quad \text{for all} \quad x, y \in U.$$

By [4, Lemma 3] we obtain

$$a[a, y] = [a, y]b \quad \text{for all} \quad y \in U.$$

Furthermore, replacing $x$ by $ax$ in (2.2) and using (2 2) and hypothesis we also have $a[b,x]=[b,x]b$

(II) $\Rightarrow$ (I) If $\alpha(a)=\beta(b)\in C_R(d(U))$ it is obviously $\alpha(a)d(x)=d(x)\beta(b)$ for all $x\in U$ Therefore it suffices to show that if $C_R(a)=C_R(b)=R'$ and $a[a,x]=[a,x]b$ for all $x\in U$ then $\alpha(a)d(x)=d(x)\beta(b)$ for all $x\in U$.

Since $d(a)=d(b)=0$, $ab=ba$, $[a,ax-xb]=a[a,x]-[a,x]b=0$ It gives $ax-xb\in R'$ and so $0=d(ax-xb)=\alpha(a)d(x)-d(x)\beta(b)$. This proves the theorem

For the second part we begin with

**LEMMA 2** [3, Lemma 1]. Let $R$ be a 2-torsion free semiprime ring and $a$, $b$ the elements of $R$ Then the following conditions are equivalent

(i) $axb=0$ for all $x\in R$

(ii) $bxa=0$ for all $x\in R$

(iii) $axb+bxa=0$ for all $x\in R$

If one of these conditions is fulfilled then $ab=ba=0$ too.

**LEMMA 3.** Let $R$ be a semiprime ring and $U$ a nonzero ideal of $R$ such that $Ann(U)=0$ Let $d$ be an $(\alpha,\beta)$-derivation of $R$ and $g$ be a $(\gamma,\delta)$-derivation of $R$. If $d(U)Ug(U)=0$ then $d(R)Ug(R)=0$.

**PROOF.** For all $x,y,z\in U$, $d(x)yg(z)=0$ Replace $x$ by $xs$, $s\in R$ we have $0=d(xs)yg(z)=\alpha(x)d(s)yg(z)+d(x)\beta(s)yg(z)$ Since $\beta(s)y\in U$, the last equation implies that $\alpha(x)d(s)yg(z)=0$, for all $x,y,z\in U$ and $s\in R$ Taking $tz$ instead of $z$, where $t\in R$, we have $0=\alpha(x)d(s)y\gamma(t)g(z)+\alpha(x)d(s)yg(t)\delta(z)$ Since $y\gamma(t)\in U$, it gives $\alpha(x)d(s)yg(t)\delta(z)=0$ for all $x,y,z\in U$ and $s,t\in R$ Therefore $d(s)yg(t)\delta(z)\in Ann(\alpha(U))=0$. Thus we get $d(s)yg(t)\delta(z)=0$ for all $y,z\in U$ and $s,t\in R$ Hence $d(s)yg(t)\in Ann(\delta(U))=0$. As a result of this, it implies that $d(R)Ug(R)=0$

**LEMMA 4.** Let $R$ be a semiprime ring and $U$ be a nonzero ideal of R such that $Ann(U)=0$. Let $a,b\in R$ be such that $aUb=0$ then $aRb=0$.

**PROOF.** For all $x\in U$ $0=axb$. Replace $x$ by $tbxrat$, where $t,r\in r$ we have $atbxratbx=0$ Since $R$ is semiprime ring, this implies that $atbU=0$ for all $t\in R$. Thus $atb\in Ann(U)=0$ we get $aRb=0$

**THEOREM 2.** Let $R$ be a 2-torsion free semiprime ring and $U$ be a nonzero ideal of $R$ with $Ann(U)=0$. Let $d$ be a $(\alpha,\beta)$-derivation of $R$ and $g$ be a $(\gamma,\delta)$-derivation of $R$. Suppose that $dg$ is a $(\alpha\gamma,\beta\delta)$-derivation and $g$ commutes both $\gamma$ and $\delta$. Then $g(x)U\alpha^{-1}d(y)=0$, for all $x,y\in U$.

**PROOF.** Since $g$ commutes both $\gamma$ and $\delta$, from the first par to the proof of [5, Lemma 1] there is no loss of generality in assuming $\beta=1$ and $\delta=1$ For all $x$, $y\in U$, $dg(xy)=d(\gamma(x)g(y)+g(x)y)=\alpha\gamma(x)dg(y)+d(\gamma(x))g(y)+\alpha(g(x))d(y)+dg(x)y$. On the other hand, since $dg$ is an $(\alpha\gamma,1)$-derivation we have $dg(xy)=\alpha\gamma(x)dg(y)+dg(x)y$. Comparing the two expressions so obtained for $dg(xy)$, we see that

$$d(\gamma(x))g(y)+\alpha(g(x))d(y)=0 \quad \text{for all} \quad x,y\in U. \tag{2 3}$$

Replacing $y$ by $yz$ where $z\in R$ in (2.3) we obtain $0=d(\gamma(x))g(yz)+\alpha(g(x))d(yz)=d(\gamma(x))\gamma(y)g(z)+d(\gamma(x))g(y)z+\alpha(g(x))\alpha(y)d(z)+\alpha(g(x))d(y)z=\{d(\gamma(x))g(y)+\alpha(g(x))d(y)\}z+d(\gamma(x))\gamma(y)g(z)+\alpha(g(x))\alpha(y)d(z)$. This relation reduces to

$$d(\gamma(x))\gamma(y)g(z)+\alpha(g(x))\alpha(y)d(z)=0 \quad \text{for all} \quad x,y\in U, z\in R. \tag{2.4}$$

Replace $y$ by $yg(t)$, $t\in U$ and take $z\in U$ we have $d(\gamma(x))\gamma(y)\gamma(g(t))g(z)+\alpha(g(x))\alpha(y)\alpha(g(t))d(z)=0$. Considering this relation (2.4) and (2.3) we obtain $d(\gamma(x))\gamma(y)\gamma(g(t))g(z)=-\alpha(g(x))\alpha(y)d(\gamma(t))g(z)=\alpha(g(x))\alpha(y)\alpha(g(t))d(z)$ for all $x,y,z\in U$. Comparing the last two relations we get $2\alpha(g(x))\alpha(y)\alpha(g(t))d(z)=0$. Since $R$ is 2-torsion free, it gives

$$g(x)yg(t)\alpha^{-1}d(z) = 0 \quad \text{for all} \quad x, y, z, t \in U.$$

Replacing $t$ by $tu, u \in U$ it follows $0 = g(x)y\gamma(t)g(u)\alpha^{-1}(d(z) + g(x)yg(t)u\alpha^{-1}(d(z))$ Since $y\gamma(t) \in U$ this relation reduces to $g(x)Ug(t)u\alpha^{-1}(d(z)) = 0$ for all $x, t, u, z \in U$ By Lemma 4 we have for all $x, t, u, z \in U, g(x)Rg(t)u\alpha^{-1}(d(z)) = 0$. In particular $g(x)u\alpha^{-1}(d(z))Rg(x)u\alpha^{-1}(d(z)) = 0$ for all $x, u, z \in U$. Since $R$ is semiprime we obtain $g(x)U\alpha^{-1}(d(z)) = 0$ for all $x, z \in U$

**COROLLARY.** Let $R$ be a prime ring of characteristic not 2, $d$ be an $(\alpha, \beta)$-derivation of $R$ and $g$ be a $(\gamma, \delta)$-derivation of $R$ such that $g$ commutes both $\gamma$ and $\delta$ If the composition $dg$ is a $(\alpha\gamma, \beta\delta)$-derivation then $d = 0$ or $g = 0$.

**THEOREM 3.** Let $R$ be a 2-torsion free semiprime ring and $U$ be a nonzero ideal of $R$ such that $Ann(U) = 0$. Let $d$ be a $(\alpha, \beta)$-derivation of $R$ and $g$ be a $(\gamma, \delta)$-derivation of $R$ such that $g$ commutes both $\gamma$ and $\delta$. If for all $x, y \in U, \beta^{-1}(d(x))Ug(y) = 0 = g(x)U\alpha^{-1}(d(y))$ then $dg$ is a $(\alpha\gamma, \beta\delta)$-derivation on $R$

**PROOF.** From Lemma 3 and Lemma 4, we get $\beta^{-1}(d(x))yg(z) = 0 = g(x)y\alpha^{-1}(d(z))$ for all $x, y, z \in R$ On the other hand, since $\beta^{-1}(d(x))yg(z) = 0$ for all $x, y, z \in R$ and since $\gamma$ is an automorphism of $R$ we obtain $d(\gamma(x))\beta(y)\beta(g(z)) = 0$ for all $x, y, z \in R$. Since $R$ is a semiprime ring, by Lemma 2 we get $d(\gamma(x))\beta(g(z)) = 0$ for all $x, z \in R$. Similarly from $g(x)U\alpha^{-1}d(y) = 0$, we get $\alpha(g(x))d(\delta(y)) = 0$ Therefore $dg$ is an $(\alpha\gamma, \beta\delta)$-derivation on $R$

## REFERENCES


[1] HERSTEIN, I.N., A note on derivations, *Canad. Math. Bull.* **22** (1979), 509-511.

[2] DAIF, M.N , A prime ring satisfying a condition for derivations, *Bull. Cal. Math. Soc.* **78** (1986), 303-304.

[3] BRESAR, M. and VUKMAN, J., Orthogonal derivation and extension of a theorem of Posner, *Radovi Matematicki* **5** (1989), 237-246

[4] AYDIN, N. and KAYA, K., Some generalization in prime ring with $(\sigma, \tau)$-derivation, *Doga-Tr. J. of Math.* **16** (1992), 169-176.

[5] BRESAR, M , On the compositions of $(\alpha, \beta)$-derivations of rings, and applications to von Neumann algebras, *Acta Sci. Math.* **56** (1992), 369-375.

[6] POSNER, E.C., Derivations in prime rings, *Proc. Amer. Math. Soc.* **8** (1957), 1093-1100