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ASYMPTOTIC EQUIVALENCE OF SEQUENCES AND SUMMABILITY

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ABSTRACT: For a sequence-to-sequence transformation A, let $R_mAx = \sum_{n\geq m}|(Ax)_n|$ and $\mu_mAx = \sup_{n\geq m}|(Ax)_n|$. The purpose of this paper is to study the relationship between the asymptotic equivalence of two sequences ($\lim_n x_n/y_n = 1$) and the variations of asymptotic equivalence based on the ratios R_mAx/R_mAy and μ_mAx/μ_mAy.

KEY WORDS: Asymptotically regular, Asymptotic equivalence.

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1. INTRODUCTION.

Let $x = (x)_n$ and $y = (y)_n$ be infinite sequences, and let A be a sequence-to-sequence transformation. We write $x \sim y$ if $\lim_n x_n/y_n = 1$. In order to compare rates of convergence of sequences, in [2] Pobyvanets introduced the concept of asymptotically regular matrices, which preserve the asymptotic equivalence of two nonnegative sequences, that is $x \sim y$ implies $Ax \sim Ay$. Furthermore, in [1] Fridy introduced new ways to compare rates by using the ratios R_mx/R_my, μ_mx/μ_my when they tend to zero. In [2] Marouf studied the relationship of these ratios when they have limit one. In the present study we investigate some further properties involved with the ratios such $\mu Ax/\mu Ay, RAx/RAy$ when they have limit one.

2. NOTATIONS AND BASIC THEOREMS.

For a summability transformation A, we use D_A to denote the domain of A:

$$D_A = \{x : \sum_{k=0}^{\infty} a_{nk}x_k \text{ converges for such } n \geq 0\}$$

and C_A to denote the summability field:

$$C_A = \{x : x \in D_A, \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} a_{n_k}x_k \text{ converges.}\}$$

Also

$$P_\delta = \{x : x_n \geq \delta > 0 \text{ for all } n\}$$

and

$$P = \{x : x_n > 0 \text{ for all } n.\}$$

For a sequence x in ℓ^1 or ℓ^∞, we also define $R_mx = \sum_{n\geq m}|x_n|$ and $\mu_mx = \sup_{n\geq m}|x_n|$ for $m \geq 0$.

We list the following results without proof.

THEOREM 1. (Pobyvanets [2]). A nonnegative matrix A is asymptotically regular if and only if for each fixed interger $m, \lim_{n\to\infty} a_{nm}/\sum_{k=0}^{\infty} a_{nk} 0$.

THEOREM 2. A matrix A is a $c_0 - c_0$ matrix (i.e. A preserves zero limits) if and only if

(a) $\lim_{n\to\infty} a_{nk} = 0$ for $k = 0, 1, 2, \ldots$.

(b) There exists a number $M > 0$ such that for each n $\sum_{k=0}^{\infty} |a_{nk}| < M$.

3. ASYMPTOTIC EQUIVALENCE PROPERTIES.

THEOREM 3. Let A be a nonnegative matrix. Suppose $x \sim y$, and $x, y \in P_\delta$ for some $\delta > 0$. Then $\mu Ax \sim \mu Ay$ if and only if for each $i = 0, 1, 2, \ldots$

$$\lim_{n\to\infty} a_{ni}/\sum_{i=0}^{\infty} a_{nj} = 0.$$

PROOF. If $\lim_{n\to\infty} a_{ni}/\sum_{j=0}^{\infty} a_{nj} = 0$, $i = 0, 1, 2, \ldots$, we want to prove that $\mu Ax \sim \mu Ay$.

Since $x \sim y$, there exists a null sequence ζ, such that

$$x_n = y_n(1 + \zeta_n) \quad n = 0, 1, 2, \ldots;$$

then

$$\begin{aligned}
\frac{(\mu Ax)_n}{(\mu Ay)_n} &= \frac{\sup_{k\geq n}(Ax)_k}{\sup_{k\geq n}(Ay)_k} \\
&= \frac{\sup_{k\geq n}\sum_{i=0}^{\infty} a_{ki}x_i}{\sup_{k\geq n}\sum_{i=0}^{\infty} a_{ki}y_i} \\
&= \frac{\sup_{k\geq n}\sum_{i=0}^{\infty} a_{ki}(y_i + y_i\zeta_i)}{\sup_{k\geq n}\sum_{i=0}^{\infty} a_{ki}y_i} \\
&\leq 1 + \frac{\sup_{k\geq n}\sum_{i=0}^{\infty} a_{ki}y_i|\zeta_i|}{\sup_{k\geq n}\sum_{i=0}^{\infty} a_{ki}y_i} \\
&\leq 1 + \frac{\sup_{k\geq n}\sum_{i=0}^{N} a_{ki}y_i|\zeta_i|}{\sup_{k\geq n}\sum_{i=0}^{\infty} a_{ki}y_i} + \frac{\sup_{k\geq n}\sum_{i=N+1}^{\infty} a_{ki}y_i|\zeta_i|}{\sup_{k\geq n}\sum_{i=0}^{\infty} a_{ki}y_i}
\end{aligned}$$

where N is a positive integer.

Since ζ is a null sequence, $\sup_j |\zeta_j| < \infty$, and for any $\epsilon > 0$ there is an $N \in \mathbf{N}$, such that if $i \geq N$, then $|\zeta_i| < \epsilon$. Hence

$$\begin{aligned}
\frac{(\mu Ax)_n}{(\mu Ay)_n} &\leq 1 + \sup_j |\zeta_j| \sum_{i=0}^{N} \frac{\sup_{k\geq n} a_{ki}y_i}{\sup_{k\geq n}\sum_{i=0}^{\infty} a_{ki}y_i} + \frac{\epsilon \sup_{k\geq N}\sum_{i=N+1}^{\infty} a_{ki}y_i}{\sup_{k\geq n}\sum_{i=0}^{\infty} a_{ki}y_i} \\
&\leq 1 + \sup_j |\zeta_j| \sum_{i=0}^{N} \frac{y_i \sup_{k\geq n} a_{ki}}{\delta \sup_{k\geq n}\sum_{i=0}^{\infty} a_{ki}} + \epsilon \\
&\leq 1 + \sup_j |\zeta_j| \sup_{0\leq j\leq N} y_j \sum_{i=0}^{N} \sup \frac{a_{ki}}{\sum_{i=0}^{\infty} a_{ki}} + \epsilon.
\end{aligned}$$

According to the hypothesis, there exists $N_1 \in \mathbf{N}$, such that if $k \geq N_1$, then $a_{ki}/\sum_{i=0}^{\infty} a_{ki} < \epsilon/N \sup_j \zeta_j \sup_{0\leq i\leq N} y_i$. So if $n \geq N$, we have

$$\frac{(\mu Ax)_n}{(\mu Ay)_n} \leq 1 + \epsilon + \epsilon.$$

This implies that $\lim_{n\to\infty} \frac{(\mu Ax)_n}{(\mu Ay)_n} \leq 1$. Similarly, we may prove $\lim_{n\to\infty} \frac{\sup_{k\geq n} \sum_{i=0}^{\infty} a_{ki}}{\sup_{k\geq n} \sum_{i=0}^{\infty} a_{ki}} \leq 1$ and the two inequalities yield $\lim_{n\to\infty} \frac{(\mu Ax)_n}{(\mu Ay)_n} = 1$.

Next, suppose $\mu Ax \sim \mu Ay$ for any $x \sim y$ such that $x, y \in P_\delta$ for some $\delta > 0$. We take $x = y = (1, 1, \ldots)$. Then $\mu Ax \sim \mu Ay$, i.e.,

$$\lim_{n\to\infty} \frac{\sup_{k\geq n} \sum_{i=0}^{\infty} a_{ki}}{\sup_{k\geq n} \sum_{i=0}^{\infty} a_{ki}} = 1.$$

Hence, there exists $M > 0$, such that $\{\sum_{i=0}^{\infty} a_{ki}\}_{k=0}^{\infty}$ is bounded by M.

If $\lim_{n\to\infty} a_{ni}/\sum_{j=0}^{\infty} a_{nj} \neq 0$ for some i. Then there exists $\lambda > 0$ and a sequence $n_1 < n_2 < \ldots$, such that $a_{u_i}/\sum_{j=0}^{\infty} a_{u_j} \geq \lambda$, $u = 1, 2, 3, \ldots$. Take $t > 0$, and define x and y by

$$y_n = 1, n = 0, 1, 2, \ldots$$

and

$$x_n = \begin{cases} 1 & \text{if } n \neq i \\ 1 + t & \text{if } n = i \end{cases}$$

It is clear that $x \sim y$ and $x, y \in P_1$. Consider the following limit:

$$\begin{aligned}
\lim_{u\to\infty} \quad & \frac{\sup_{k\geq u} \sum_{i=0}^{\infty} a_{n_k j} x_j}{\sup_{k\geq u} \sum_{i=0}^{\infty} a_{n_k j} y_j} \\
= \quad & \lim_{u\to\infty} \frac{\sup_{k\geq u}(\sum_{i=0}^{\infty} a_{n_k j} + t a_{n_k i})}{\sup_{k\geq u} \sum_{i=0}^{\infty} a_{n_k j}} \\
\geq \quad & \lim_{u\to\infty} \frac{\sup_{k\geq u}(\sum_{i=0}^{\infty} a_{n_k j} + t\lambda \sum_{j=0}^{\infty} a_{n_k i})}{\sup_{k\geq u} \sum_{i=0}^{\infty} a_{n_k j}} \\
= \quad & 1 + t\lambda.
\end{aligned}$$

We can choose $t = 1/\lambda$, which gives

$$\lim_{u\to\infty} \frac{(\mu Ax)_{n_u}}{(\mu Ay)_{n_u}} \geq 2.$$

This is a contradiction of $\mu Ax \sim \mu Ay$.

THEOREM 4. Suppose A is a nonnegative matrix; then $\mu x \sim \mu y$ implies $\mu Ax \sim \mu Ay$ for any bounded sequences $x, y \in P_\delta$, for some $\delta > 0$, if and only if A satisfies the following three conditions:

(i) $(\sum_{j=0}^{\infty} a_{kj})_{k=0}^{\infty}$ is a bounded sequence dominated by some B;

(ii) For any $j = 0, 1, 2, \ldots$

$$\lim_{n\to\infty} \frac{\sup_{k\geq n} a_{kj}}{\sup_{k\geq n} \sum_{i=0}^{\infty} a_{ki}} = 0;$$

(iii) For any infinite sequence $j_1 < j_2 < j_3 \ldots$

$$\lim_{n\to\infty} \frac{\sup_{k>n} \sum_{i=1}^{\infty} a_{kj_i}}{\sup_{k\geq n} \sum_{i=0}^{\infty} a_{kj}} = 1.$$

Before we prove this theorem, we shall give some examples of A which satisfy the above conditions (i), (ii), and (iii).

Example 1. $A = I$.

Example 2.

$$A = \begin{pmatrix} 1 & & & & & \\ \frac{1}{2^2} & 1 & & & & \\ \frac{1}{3^2} & \frac{1}{3^2} & 1 & & 0 & \\ \frac{1}{4^2} & \frac{1}{4^2} & \frac{1}{4^2} & 1 & & \\ \cdots & & & \ddots & & \\ \frac{1}{(n+1)^2} & \frac{1}{(n+1)^2} & \cdots & \frac{1}{(n+1)^2} & 1 & \ddots \end{pmatrix}$$

PROOF OF THEOREM 4. First, assume that for any bounded sequences $x, y \in P_\delta$, for some $\delta > 0$, $\mu x \sim \mu y$ implies $\mu Ax \sim \mu Ay$; we wish to prove that A satisfies the conditions (i), (ii) and (iii). Take $x = y = (1, 1, \ldots)$; then x, y are bounded, $x, y \in P_1$, and $\mu x \sim \mu y$; so $\mu Ax \sim \mu Ay$. But $(\mu Ax)_n = \sup_{k\geq n} \sum_{j=0}^{\infty} a_{kj}$. Hence, $(\sum_{j=0}^{\infty} a_{kj})_{k=0}^{\infty}$ should be bounded. This proves (i). To prove (ii) suppose there is a j such that

$$\overline{\lim_{n\to\infty}} \frac{\sup_{k>n} a_{kj}}{\sup_{k\geq n} \sum_{i=0}^{\infty} a_{ki}} = \lambda$$

for some $\lambda > 0$. As in the proof of Theorem 3, take $t > 0$ and define $y = (1, 1, \ldots)$ and

$$x_n = \begin{cases} 1 & \text{if } n \neq j, \\ 1 + t & \text{if } n = j. \end{cases}$$

Then $x, y \in P_1, x, y$ are bounded, and $\mu x \sim \mu y$; so we have $\mu Ax \sim \mu Ay$. But

$$\begin{aligned} & \overline{\lim_{n\to\infty}} \frac{\sup_{k>n} \sum_{i=0}^{\infty} a_{kj} x_i}{\sup_{k\geq n} \sum_{i=0}^{\infty} a_{ki} y_i} \\ = & \overline{\lim_{n\to\infty}} \frac{\sup_{k\geq n} (t a_{kj} + \sum_{i=0}^{\infty} a_{kj})}{\sup_{k\geq n} \sum_{i=0}^{\infty} a_{ki}} \\ \geq & \overline{\lim_{n\to\infty}} \frac{t \sup_{k>n} a_{kj}}{\sup_{k\geq n} \sum_{i=0}^{\infty} a_{ki}} - 1 \\ = & t\lambda - 1. \end{aligned}$$

By choosing $t = \frac{3}{\lambda}$, we get

$$\overline{\lim_{n\to\infty}} \frac{\sup_{k>n} \sum_{i=0}^{\infty} a_{kj} x_i}{\sup_{k\geq n} \sum_{i=0}^{\infty} a_{ki} x_i} \geq 3 - 1 = 2.$$

This is a contradiction $\mu Ax \sim \mu Ay$, so (ii) must hold.

Finally, we are going to prove (iii). For any given infinite sequence $j_1 < j_2 < \ldots$, we define x and y by

$$y_n = 2 \text{ for every } n,$$

and

$$x_n = \begin{cases} 2, & \text{if } n = j_u \text{ for } u = 1, 2, \ldots, \\ 1, & \text{otherwise.} \end{cases}$$

It is easy to see that x, y are bounded, $x, y \in P_1$ and $\mu x \sim \mu y$. This implies $\mu Ax \sim \mu Ay$. Hence we have

$$\begin{aligned} 1 &= \lim_{n\to\infty} \frac{\sup_{k\geq n} \sum_{j=0}^{\infty} a_{kj}x_j}{\sup_{k\geq n} \sum_{j=0}^{\infty} a_{kj}y_j} \\ &= \lim_{n\to\infty} \frac{\sup_{k\geq n} \sum_{j\in J} a_{kj}x_j + \sum_{j\notin J} a_{kj}x_j)}{2\sup_{k\geq n} \sum_{j=0}^{\infty} a_{kj}} \end{aligned}$$

where $J = \{j_1, j_2, j_3, \ldots\}$

$$\begin{aligned} &= \lim_{n\to\infty} \frac{\sup_{k\geq n}(2\sum_{j\in J} a_{kj} + \sum_{j\notin J} a_{kj})}{2\sup_{k\geq n} \sum_{j=0}^{\infty} a_{kj}} \\ &= \lim_{n\to\infty} \frac{(\sup_{k\geq n}(\sum_{j\in J} a_{kj} + \sum_{j=0}^{\infty} a_{kj})}{2\sup_{k\geq n} \sum_{j=0}^{\infty} a_{kj}} \\ &\leq \lim_{n\to\infty} \frac{\sup_{k\geq n} \sum_{j\in J} a_{kj} + \sup_{k\geq n} \sum_{j=0}^{\infty} a_{kj}}{2\sup_{k\geq n} \sum_{j=0}^{\infty} a_{kj}} \\ &= \lim_{n\to\infty} \frac{\sup_{k\geq n} \sum_{j\in J} a_{kj}}{2\sup_{k\geq n} \sum_{j=0}^{\infty} a_{kj}} + \frac{1}{2}. \end{aligned}$$

Hence

$$1 \leq \frac{1}{2} \lim_{n\to\infty} \frac{\sup_{k\geq n} \sum_{j\in J} a_{kj}}{\sup_{k\geq n} \sum_{j=0}^{\infty} a_{kj}} + \frac{1}{2}.$$

This implies

$$\lim_{n\to\infty} \frac{\sup_{k\geq n} \sum_{j\in J} a_{kj}}{\sup_{k\geq n} \sum_{j=0}^{\infty} a_{kj}} \geq 1.$$

On the other hand, it is clear that

$$\lim_{n\to\infty} \frac{\sup_{k\geq n} \sum_{j\in J} a_{kj}}{\sup_{k\geq n} \sum_{j=0}^{\infty} a_{kj}} \leq 1.$$

Combining the last two inequalities together, we get

$$\lim_{n\to\infty} \frac{\sup_{k\geq n} \sum_{j\in J} a_{kj}}{\sup_{k\geq n} \sum_{j=0}^{\infty} a_{kj}} = 1,$$

which proves (iii).

Conversely, assume A satisfies the conditions (i), (ii) and (iii), and suppose x, y are bounded by some $M > 0$, $x, y \in P_\delta$ for some $\delta > 0$, and $\mu x \sim \mu y$. For any $\epsilon > 0$, since x, y are bounded, there exists $N_1 \in \mathrm{N}$ such that if $j \geq N_1$, then

$$y_i \leq \lim_{k\to\infty} \sup_{i\geq k} y_i + \epsilon$$

and also there exists an infinite sequence $j_1 < j_2 < \ldots$, such that

$$x_{j_i} \geq \lim_{k\to\infty} \sup_{j\geq k} x_j - \epsilon$$

for $i = 1, 2, 3, \ldots$. Therefore

$$\lim_{n\to\infty} \frac{\sup_{k\geq n} \sum_{j=0}^{\infty} a_{kj}x_j}{\sup_{k\geq n} \sum_{j=0}^{\infty} a_{kj}y_j}$$

$$\geq \lim_{n\to\infty} \frac{\sup_{k>n}\sum_{i=0}^{\infty} a_{kj_i} x_{j_i}}{\sup_{k\geq n}\sum_{j=0}^{N_1} a_{kj} y_j + \sum_{j=N_1+1}^{\infty} a_{kj} y_j}$$

$$\geq \lim_{n\to\infty} \frac{\sup_{k>n}\sum_{i=0}^{\infty} a_{kj_i}(\lim_{\ell\to\infty}\sup_{i\geq \ell} x_i - \epsilon)}{M\sup_{k\geq n}\sum_{j=0}^{N_1} a_{kj} + \sup_{k\geq n}\sum_{j=N_1+1}^{\infty} a_{kj}(\lim_{\ell\to\infty}\sup_{i\geq \ell} y_j + \epsilon)}$$

$$\geq \lim_{n\to\infty} \frac{(\sup_{k>n}\sum_{i=0}^{\infty} a_{kj_i})\lim_{\ell\to\infty}\sup_{i\geq\ell} x_i - \epsilon\sup_{k>n}\sum_{i=1}^{\infty} a_{kj_i}}{M\sup_{k\geq n}\sum_{j=0}^{N_1} a_{kj} + \epsilon\sup_{k\geq n}\sum_{j=0}^{\infty} a_{kj} + \sup_{k\geq n}(\sum_{j=N_1+1}^{\infty} a_{kj})\lim_{\ell\to\infty}\sup_{i\geq\ell} y_j}$$

$$\geq \lim_{n\to\infty} \frac{(\sup_{k>n}\sum_{i=0}^{\infty} a_{kj_i})\lim_{\ell\to\infty}\sup_{i\geq\ell} x_i}{M\sup_{k\geq n}\sum_{j=0}^{N_1} a_{kj} + \epsilon\sup_{k\geq n}\sum_{j=0}^{\infty} a_{kj} + (\sup_{k\geq n}\sum_{j=N_1+1}^{\infty} a_{kj})\lim_{\ell\to\infty}\sup_{i\geq\ell} y_j}$$

$$- \lim_{n\to\infty} \frac{\epsilon\sup_{k>n}\sum_{i=0}^{\infty} a_{kj_i}}{(\sup_{k\geq n}\sum_{N_1}^{\infty} a_{kj})\lim_{\ell\to\infty}\sup_{i\geq\ell} y_j}$$

$$\geq \lim_{n\to\infty} \frac{(\sup_{k>n}\sum_{i=0}^{\infty} a_{kj_i})\lim_{\ell\to\infty}\sup_{i\geq\ell} x_i}{M\sup_{k\geq n}\sum_{j=0}^{N_1} a_{kj} + \epsilon\sup_{k\geq n}\sum_{j=0}^{\infty} a_{kj} + (\sup_{k\geq n}\sum_{j=N_1+1}^{\infty} a_{kj})\lim_{\ell\to\infty}\sup_{i\geq\ell} y_j}$$

$$- \frac{\epsilon}{\delta}$$

(here, we used (iii) to deduce that

$$\lim_{n\to\infty} \frac{\sup_{k>n}\sum_{j=0}^{\infty} a_{kji}}{\sup_{k\geq n}\sum_{j=N_1+1}^{\infty} a_{kj}} = \lim_{n\to\infty} \frac{\frac{\sup_{k>n}\sum_{i=1}^{\infty} a_{kj_i}}{\sup_{k\geq n}\sum_{i=1}^{\infty} a_{kj}}}{\frac{\sup_{k>n}\sum_{i=1}^{\infty} a_{kj_i}}{\sup_{k\geq n}\sum_{i=1}^{\infty} a_{kj}}} = \frac{1}{1} = 1)$$

$$\geq \lim_{n\to\infty} \frac{1}{B_1 + B_2 + B_3} - \frac{\varepsilon}{\delta}$$

$$\geq \lim_{n\to\infty} \frac{1}{\frac{M\sup_{k\geq n}\sum_{j=0}^{N_1} a_{kj}}{\delta\sup_{k\geq n}\sum_{i=1}^{\infty} a_{kj_i}} + \frac{\epsilon\sup_{k\geq n}\sum_{j=0}^{\infty} a_{kj}}{\delta\sup_{k\geq n}\sum_{i=1}^{\infty} a_{kj_i}} + \frac{\sup_{k\geq n}\sum_{j=N_1+1}^{\infty} a_{kj}}{\sup_{k\geq n}\sum_{i=1}^{\infty} a_{kj_i}}} - \frac{\varepsilon}{\delta},$$

where

$$B_1 = \frac{M\sup_{k\geq n}\sum_{i=0}^{N_1} a_{kj}}{(\sup_{k\geq n}\sum_{i=1}^{\infty} a_{kj_i})\lim_{\ell\to\infty}\sup_{i\geq\ell} x_i}, B_2 = \frac{\varepsilon\sup_{k\geq n}\sum_{j=0}^{\infty} a_{kj}}{(\sup_{k\geq n}\sum_{i=1}^{\infty} a_{kj_i})\lim_{\ell\to\infty}\sup_{i\geq 1} x_i},$$

$$B_3 = \frac{(\sup_{k\geq n}\sum_{j=N_1+1}^{\infty} a_{kj})\lim_{\ell\to\infty}\sup_{i\geq\ell} y_i}{(\sup_{k\geq n}\sum_{i=1}^{\infty} a_{kj_i})\lim_{\ell\to\infty}\sup_{i\geq\ell} x_i}.$$

For the fixed N_1, combining conditions (ii) and (iii), we can easily prove

$$\frac{\sup_{k>n}\sum_{j=0}^{N_1} a_{kj}}{\sup_{k\geq n}\sum_{i=1}^{\infty} a_{kj_i}} \to 0 \text{ as } n\to\infty.$$

Hence, for the given $\varepsilon > 0$, there is $N_2 \in \mathbf{N}$, such that if $n \geq N_2$, then

$$\frac{\sup_{k>n}\sum_{j=0}^{N_1} a_{kj}}{\sup_{k\geq n}\sum_{i=1}^{\infty} a_{kj_i}} < \varepsilon,$$

$$\frac{\sup_{k\geq n}\sum_{j=0}^{\infty}a_{kj}}{\sup_{k\geq n}\sum_{i=1}^{\infty}a_{kj_i}} < 1+\varepsilon \quad \text{(by (iii))},$$

and

$$\frac{\sup_{k\geq n}\sum_{j=0}^{N_1}a_{kj}}{\sup_{k\geq n}\sum_{i=1}^{\infty}a_{kj_i}} < 1+\varepsilon \quad \text{(by (iii))}.$$

These imply that if $n \geq N_2$

$$\frac{1}{\frac{M\sup_{k\geq n}\sum_{j=0}^{N_1}a_{kj}}{\delta\sup_{k\geq n}\sum_{i=1}^{\infty}a_{kj_i}}+\frac{\varepsilon\sup_{k\geq n}\sum_{j=0}^{\infty}a_{kj}}{\delta\sup_{k\geq n}\sum_{i=1}^{\infty}a_{kj_i}}+\frac{\sup_{k\geq n}\sum_{j=N_1+1}^{\infty}a_{kj}}{\sup_{k\geq n}\sum_{i=1}^{\infty}a_{kj_i}}} \geq \frac{1}{\frac{M\varepsilon}{\delta}+\frac{\varepsilon}{\delta}(1+\varepsilon)+1+\varepsilon}.$$

Hence

$$\lim_{n\to\infty}\frac{\sup_{k\geq n}\sum_{j=0}^{\infty}a_{kj}x_j}{\sup_{k\geq n}\sum_{j=0}^{\infty}a_{kj}y_j} \geq \frac{1}{\frac{M\varepsilon}{\delta}+\frac{\varepsilon}{\delta}(1+\varepsilon)+1+\varepsilon}-\frac{\varepsilon}{\delta}.$$

Since ε is arbitrary, we have

$$\lim_{n\to\infty}\frac{\sup_{k\geq n}\sum_{j=0}^{\infty}a_{kj}x_j}{\sup_{k\geq n}\sum_{j=0}^{\infty}a_{kj}y_j} \geq 1.$$

Similarly, we can prove

$$\lim_{n\to\infty}\frac{\sup_{k\geq n}\sum_{j=0}^{\infty}a_{kj}x_j}{\sup_{k\geq n}\sum_{j=0}^{\infty}a_{kj}y_j} \leq 1.$$

Thus, we have finished the proof.

REMARK.

Let A be a nonnegative matrix, $A = (a_{ij})$. If A satisfies the following two conditions, then A satisfies the conditions (i), (ii), (iii) of theorem 4:

a) There exists $\lambda > 0$, such that

$$\lim_{n\to\infty} a_{nn} = \lambda$$

b) $\lim_{n\to\infty}\sum_{j\neq n} a_{nj} = 0$

PROOF OF THE REMARK. If A satisfies the above conditions a and b, it is easy to see that A satisfies (i) in theorem 4. To prove (iii), let $j_1, j_2, \ldots$ be an infinity sequence: $j_1 < j_2 < \ldots$. Then

$$\lim_{n\to\infty}\frac{\sup_{k\geq n}\sum_{i=1}^{\infty}a_{kj_i}}{\sup_{k\geq n}\sum_{j=0}^{\infty}a_{kj}} \geq \lim_{n\to\infty}\frac{\sup_{j_i\geq n}a_{j_ij_i}}{\sup_{k\geq n}\sum_{j=0}^{\infty}a_{kj}}$$

$$= \frac{\lim_{n\to\infty}\sup_{j_i\geq n}a_{j_ij_i}}{\lim_{n\to\infty}\sup_{k\geq n}\sum_{j=0}^{\infty}a_{kj}} = \frac{\lambda}{\lambda+0} = 1$$

This inequality gives that

$$\lim_{n\to\infty}\frac{\sup_{k\geq n}\sum_{i=1}^{\infty}a_{kj_i}}{\sup_{k\geq n}\sum_{j=0}^{\infty}a_{kj}} = 1$$

Next, let's prove (ii) of theorem 4. In fact, for any fixed $j = 0, 1, 2, \ldots$

$$\lim_{n\to\infty}\frac{\sup_{k\geq n}a_{kj}}{\sup_{k\geq n}\sum_{i=0}^{\infty}a_{kj}}$$

$$\leq \lim_{n\to\infty}\frac{\sup_{k\geq n}\sum_{j<k}a_{kj}}{a_{nn}}$$

$$= \frac{\lim_{n\to\infty}\sup_{k\geq n}\sum_{j<k}a_{kj}}{\lim_{n\to\infty}a_{nn}}$$

$$\leq \frac{\lim_{n\to\infty}\sup_{k>n}\sum_{j\neq k} a_{kj}}{\lambda}$$

$$= \frac{\lim_{n\to\infty}\sum_{j\neq n} a_{nj}}{\lambda}$$

$$= 0$$

Next, we give some examples to show that the conditions of theorem 4 are necessary.

Example 3. Let A be defined as follows:

$$A = \begin{pmatrix} 2 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & \\ & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & \\ & & 2 & 0 & 2 & 0 & 0 & 0 & 0 & \\ & & & 1 & 2 & 0 & 0 & 0 & 0 & \\ & & & & 2 & 0 & 2 & 0 & 0 & \\ & & & & & 1 & 2 & 0 & 0 & \\ & & & & & & 2 & 0 & 2 & \\ & 0 & & & & & & 1 & 2 & \\ & & & & & & & & 2 & \\ & & & & & & & & & \ddots \end{pmatrix}$$

It is easy to see that A satisfies the conditions (i) and (ii), not (iii).

Take

$$x = (2,2,2,2,\ldots)$$
$$y = (2,1,1,1,2,1,1,1,1,2,1,1,1,1,2,\ldots)$$

x, y are bounded sequences and $x, y \in P_1$. For $m = 1,2,3,\ldots$ we have $\mu_m(x) = \mu_m(y) = 2$. Hence $\dfrac{\mu_m(x)}{\mu_m(y)} = 1$. But

$$Ax = (8,8,8,\ldots)$$
$$Ay = (6,3,\ldots) \quad y = (y_i) \; y_i \leq 6 \;\; i = 1,2,\ldots$$

This implies

$$\frac{\mu_m Ax}{\mu_m Ay} \to \frac{8}{6} = \frac{4}{3} \neq 1, \text{ as } n \to \infty.$$

Example 4.

Let

$$A = \begin{pmatrix} 1 & & & & \\ & \frac{1}{2} & & 0 & \\ & & \frac{1}{4} & & \\ & & & \frac{1}{8} & \\ & & & & \ddots \\ & & 0 & & \end{pmatrix}.$$

A satisfies (i) and (ii) not (iii).

Take

$$x = (2,2,2,\ldots)$$
$$y = (2,1,2,1,\ldots).$$

x and y are bounded and $x, y \in P_1$. We also have

$$\frac{\mu_m x}{\mu_m y} = 1, \quad m = 1, 2, \ldots$$

$$\begin{aligned} Ax &= (2, 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots) \\ Ay &= (2, \frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8}, \ldots). \end{aligned}$$

Then, if m is odd,

$$\frac{(\mu Ax)_m}{(\mu Ay)_m} = 2$$

if m is even

$$\frac{(\mu Ax)_m}{(\mu Ay)_m} = 1$$

$\Rightarrow \frac{(\mu Ax)_m}{(\mu Ay)_m}$ has no limit as $m \to \infty$.

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REFERENCES

[1] Fridy, J.A. Minimal Rates of Summability. Can. J. Math., Vol. XXX, 4 (1978), 808–816.

[2] Marouf, M. S. Asymptotic Equivalence And Summability. Internat. J. Math & Math. Sci. Vol. 16 No. 4 (1993) 755–762.

[3] Pobyvanets, I.P. Asymptotic Equivalence of Some Linear Transformation Defined by a Nonnegative Matrix and Reduced to Generalized Equivalences in the Sense of Cesáro and Abel, Mat. Fig. 28 (1980), 83–87, 123.

[4] Powel, R.E. and Shah, S.E. Summability Theory and its Applications, Van Nostrand Rienhold London, 1972.