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A CHANGE OF SCALE FORMULA FOR WIENER INTEGRALS OF CYLINDER FUNCTIONS ON ABSTRACT WIENER SPACE

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ABSTRACT The purpose of this paper is to establish the existence of analytic Wiener and Feynman integrals for a class of certain cylinder functions which is of the form :

$$F(x) = f((h_1, x)^{\sim}, \cdots, (h_n, x)^{\sim}), \quad x \in B,$$

on the abstract Wiener space, and to establish the relationship between the Wiener integral and the analytic Feynman integral for such cylinder functions on the abstract Wiener space. We then establish a change of scale formula for Wiener integrals of such cylinder functions on the abstract Wiener space.

KEY WORDS AND PHRASES Wiener measure, analytic Wiener integral, analytic Feynman integral, change of scale formula.
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1. INTRODUCTION

In [4], Cameron and Storvick expressed the analytic Wiener and Feynman integrals as the limits of Wiener integrals for certain Banach algebra $\mathcal{S}(L_2^{\nu}[a,b])$ of functionals. Using these results, they found a rather nice change of scale formula for Wiener integrals on a classical Wiener space [5]. In [13;14], Yoo, Yoon and Skoug extended these results to an Yeh-Wiener space and to an abstract Wiener space.

In [13], Skoug and Yoo expressed the analytic Wiener and Feynman integrals as the limits of Wiener integrals, and then they established a change of scale formula for Wiener integrals on the Fresnel class of the abstract Wiener space.

In this paper,we will show that the analytic Wiener and Feynman integrals of certain cylinder functions on the abstract Wiener space exist, and we will establish the relationship between the Wiener integral and the analytic Feynman integral for such cylinder functions on the abstract Wiener space. Using these results, we will establish a change of scale formula for Wiener integrals of such cylinder functions on the abstract Wiener space.

Note that the Fresnel class on the abstract Wiener space consists of bounded functions but not all cylinder functions are bounded in general.

2. DEFINITIONS AND PHRASES

Let H be a real separable infinite dimensional Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $|\cdot| = \sqrt{\langle \cdot, \cdot \rangle}$. Let $||\cdot||_0$ be a measurable norm on H with respect to the Gauss measure μ. Let B denote the completion of H with respect to $||\cdot||_0$. Let i denote the natural injection from H into

B. The adjoint operator i^* of i is one-to-one and maps B^* continuously onto a dense subset of H^*, where H^* and B^* are topological duals of H and B, respectively. By identifying H with H^* and B^* with i^*B^*, we have a triplet (B^*, H, B) such that $B^* \subset H^* \equiv H \subset B$ and $\langle h, x\rangle = (h, x)$ for all h in B^* and x in H, where $(\cdot,\cdot)$ denotes the natural dual pairing between B^* and B. By a well known result of Gross [8;12], $\mu \cdot i^{-1}$ has a unique countably additive extension ν to the Borel σ-algebra $\mathcal{B}(B)$ on B. The triplet (B, H, ν) is called an abstract Wiener space and ν is called a Wiener measure. We denote the Wiener integral of a functional F by $\int_B F(x)\nu(dx)$. For more details see [8;12].

Let $\{e_j\}_{j=1}^{\infty}$ denote a complete orthonormal system in H such that e_j's are in B^*. For each $h \in H$ and $x \in B$, we define a stochastic inner product $(\cdot,\cdot)^\sim$ between H and B as follows:

$$(h,x)^\sim = \begin{cases} \lim\limits_{n\to\infty} \sum_{j=1}^{n} \langle h, e_j\rangle (e_j, x), & \text{if the limit exists,} \\ 0, & \text{otherwise.} \end{cases} \tag{2.1}$$

It is well known [11] that for every $h \in H$, $(h,x)^\sim$ exists for ν-a.e. x in B and it has a Gaussian distribution with mean zero and variance $|h|^2$. Furthermore, it is easy to show that $(h,x)^\sim = (h,x)$ for ν-a.e. x in B if $h \in B^*$, $(h,x)^\sim$ is essentially independent of the complete orthonormal set used in its definition, and finally that if $\{h_1, \cdots, h_k\}$ is an orthonormal set of elements in H, then $(h_1,x)^\sim, \cdots, (h_k,x)^\sim$ are independent Gaussian functionals with mean zero and variance one. Note that if both h and x are in H, then $(h,x)^\sim = \langle h, x\rangle$.

Throughout this paper, let $\mathbb{R}^n$ denote the n-dimensional Euclidean space and let $\mathbb{C}, \mathbb{C}_+$, and $\mathbb{C}_+^\sim$ denote the complex numbers, the complex numbers with positive real part, and the non-zero complex numbers with nonnegative real part, respectively.

DEFINITION 2.1 Let (B, H, ν) be an abstract Wiener space. A function F is called a *cylinder function* on B if there exists a finite subset $\{g_1, \cdots, g_k\}$ of H such that

$$F(x) = \psi((g_1,x)^\sim, \cdots, (g_k,x)^\sim), \quad x \in B, \tag{2.2}$$

where ψ is a complex-valued Borel measurable function on $\mathbb{R}^k$. It is easy to show that there exists a linearly independent set $\{h_1, \cdots, h_n\}$ of H such that every cylinder function F of the form (2.2) is expressed as

$$F(x) = f((h_1,x)^\sim, \cdots, (h_n,x)^\sim), \quad x \in B, \tag{2.3}$$

where f is a complex-valued Borel measurable function on $\mathbb{R}^n$. Thus we lose no generality in assuming that every cylinder function on B is of the form (2.3).

DEFINITION 2.2 Let F be a complex-valued measurable function on B such that the integral

$$J(F;\lambda) = \int_B F(\lambda^{-\frac{1}{2}}x)\,\nu(dx) \tag{2.4}$$

exists for all real $\lambda > 0$. If there exists a function $J^*(F;z)$ analytic on $\mathbb{C}_+$ such that $J^*(F;\lambda) = J(F;\lambda)$ for all real $\lambda > 0$, then we define $J^*(F;z)$ to be the *analytic Wiener integral* of F over B with parameter z, and for each $z \in \mathbb{C}_+$, we write

$$I^{aw}(F;z) = J^*(F;z). \tag{2.5}$$

Let q be a non-zero real number and let F be a function on B whose analytic Wiener integral exists for each z in $\mathbb{C}_+$. If the following limit exists, then we call it the *analytic Feynman integral* of F over B with parameter q, and we write

$$I^{af}(F;q) = \lim_{z\to -iq} I^{aw}(F;z), \tag{2.6}$$

where z approaches $-iq$ through $\mathbb{C}_+$ and $i^2 = -1$.

DEFINITION 2.3 Let (B, H, ν) be an abstract Wiener space. Let n be a positive integer, and let $\{h_1, \cdots, h_n\}$ be an orthonormal set of elements in H. For $1 \leqslant p < \infty$, let $\mathcal{F}(n;p)$ denote the class of cylinder functions F with the form as follows:

$$F(x) = f((h_1, x)^\sim, \cdots, (h_n, x)^\sim), \quad x \in B, \tag{2.7}$$

where $f : \mathbb{R}^n \to \mathbb{C}$ is in $L_p(\mathbb{R}^n)$, the space of functions whose p-th powers are Lebesgue integrable on $\mathbb{R}^n$.

Let $\mathcal{F}(n;\infty)$ denote the class of cylinder functions F with the form as follows:

$$F(x) = f((h_1, x)^\sim, \cdots, (h_n, x)^\sim), \quad x \in B, \tag{2.8}$$

where $f : \mathbb{R}^n \to \mathbb{C}$ is in $C_0(\mathbb{R}^n)$, the space of continuous functions on $\mathbb{R}^n$ that vanish at infinity.

We will close this section by mentioning the following useful theorem which is called the Wiener Integration Formula.

THEOREM 2.4 Let (B, H, ν) be an abstract Wiener space and let $\{h_1, \cdots, h_n\}$ be an orthonormal set of elements in H. Let $F : B \to \mathbb{C}$ be a function defined by the formula

$$F(x) = f((h_1, x)^\sim, \cdots, (h_n, x)^\sim), \quad x \in B,$$

where $f : \mathbb{R}^n \to \mathbb{C}$ is a Lebesgue measurable function. Then

$$\begin{aligned} \int_B F(x)\, \nu(dx) &= \int_B f((h_1, x)^\sim, \cdots, (h_n, x)^\sim)\, \nu(dx) \\ &= \left(\frac{1}{2\pi}\right)^{\frac{n}{2}} \int_{\mathbb{R}^n} f(\vec{u}) \cdot \exp\{-\frac{1}{2}|\vec{u}|^2\}\, d\vec{u}, \end{aligned} \tag{2.9}$$

where $\vec{u} = (u_1, \cdots, u_n) \in \mathbb{R}^n$, $|\vec{u}|^2 = \sum_{j=1}^n u_j^2$, and $d\vec{u} = du_1 \cdots du_n$.

In the next section, we will use several times the following well-known integration formula:

$$\int_{\mathbb{R}} \exp\{-au^2 + ibu\}\, du = \sqrt{\frac{\pi}{a}} \exp\{-\frac{b^2}{4a}\}, \tag{2.10}$$

where a is a complex number with $Re\, a > 0$, b is a real number, and $i^2 = -1$.

3. THE MAIN RESULTS

We will begin this section by showing that the analytic Wiener integral of F exists for every $F \in \cup_{1 \leq p \leq \infty} \mathcal{F}(n;p)$ and that the analytic Feynman integral of F exists for every $F \in \mathcal{F}(n:1)$.

THEOREM 3.1 Let (B, H, ν) be an abstract Wiener space and let $F \in \mathcal{F}(n;p)$ be given by (2.7) or (2.8), where $1 \leqslant p \leqslant \infty$. Then :

(i) the analytic Wiener integrals of F exist, and for every $z \in \mathbb{C}_+$,

$$I^{aw}(F; z) = \left(\frac{z}{2\pi}\right)^{\frac{n}{2}} \int_{\mathbb{R}^n} f(\vec{u}) \exp\{-\frac{z}{2}|\vec{u}|^2\}\, d\vec{u}, \tag{3.1}$$

(ii) for every non-zero real number q, and for $F \in \mathcal{F}(n;1)$,the analytic Feynman integral of F exists and is given by

$$I^{af}(F; q) = \left(-\frac{iq}{2\pi}\right)^{\frac{n}{2}} \int_{\mathbb{R}^n} f(\vec{u}) \exp\{\frac{iq}{2}|\vec{u}|^2\}\, d\vec{u}, \tag{3.2}$$

where $\vec{u} = (u_1, \cdots, u_n) \in \mathbb{R}^n$, $|\vec{u}|^2 = \sum_{j=1}^n u_j^2$, and $d\vec{u} = du_1 \cdots du_n$.

PROOF. By Theorem 2.4, we have that for all real $\lambda > 0$

$$J(F;\lambda) = \int_B F(\lambda^{-\frac{1}{2}}x)\,\nu(dx)$$
$$= (\frac{\lambda}{2\pi})^{\frac{n}{2}} \int_{\mathbb{R}^n} f(\vec{u}) \exp\{-\frac{\lambda}{2}|\vec{u}|^2\}\, d\vec{u}.$$

Let $J^*(F;z) = (\frac{z}{2\pi})^{\frac{n}{2}} \int_{\mathbb{R}^n} f(\vec{u})\exp\{-\frac{z}{2}|\vec{u}|^2\}\,d\vec{u}$, $z \in \mathbb{C}_+$. Then $J^*(F;\lambda) = J(F;\lambda)$ for all real $\lambda > 0$.

We will use Morera's Theorem to show that $J^*(F;z)$ is an analytic function of z in $\mathbb{C}_+$. First of all, by the Dominated Convergence Theorem, we can show that $J^*(F;z)$ is continuous on $\mathbb{C}_+$; an appropriate dominating function is obtained almost exactly as in the following argument.

Now let Γ be any rectifiable simple closed curve lying in $\mathbb{C}_+$. We need only show that

$$\int_\Gamma J^*(F;z)\,dz = 0.$$

But this will clearly follow from the Cauchy Integral Theorem if we can justify moving the line integral along Γ inside the other integrals defining $J^*(F;z)$. Let $\alpha \equiv \sup\{|z| : z \in \Gamma\}$ and $\beta \equiv \inf\{Re\, z : z \in \Gamma\}$. If F belongs to $\mathcal{F}(n;1)$, then the function $(\frac{\alpha}{2\pi})^{\frac{n}{2}}|f(\vec{u})|$ dominates $(\frac{|z|}{2\pi})^{\frac{n}{2}}|f(\vec{u})|\exp\{-\frac{Re\,z}{2}|\vec{u}|^2\}$ and is integrable on $\mathbb{R}^n$. If F belongs to $\mathcal{F}(n;p)$ $(1 < p < \infty)$, then the function $(\frac{\alpha}{2\pi})^{\frac{n}{2}}|f(\vec{u})|\exp\{-\frac{\beta}{2}|\vec{u}|^2\}$ dominates $(\frac{|z|}{2\pi})^{\frac{n}{2}}|f(\vec{u})|\exp\{-\frac{Re\,z}{2}|\vec{u}|^2\}$ and is integrable on $\mathbb{R}^n$ by Hölder's Inequality. If F belongs to $\mathcal{F}(n;\infty)$, then the function $(\frac{\alpha}{2\pi})^{\frac{n}{2}} M\exp\{-\frac{\beta}{2}|\vec{u}|^2\}$ dominates $(\frac{|z|}{2\pi})^{\frac{n}{2}}|f(\vec{u})|\exp\{-\frac{Re\,z}{2}|\vec{u}|^2\}$ and is integrable on $\mathbb{R}^n$, where M is a bound with $|f(\vec{u})| \leqslant M$ for all $\vec{u} \in \mathbb{R}^n$.

Hence we can apply Fubini's Theorem to the integral $\int_\Gamma J^*(F;z)\,dz$ and then we have $\int_\Gamma J^*(F;z)\,dz = 0$, because the function $(\frac{z}{2\pi})^{\frac{n}{2}}\exp\{-\frac{z}{2}|\vec{u}|^2\}$ is analytic on $\mathbb{C}_+$. Then we have established (3.1). Finally the proof of (3.2) is immediate.

In order to obtain our main results, we need the following lemma:

LEMMMA 3.2 Let (B,H,ν) be an abstract Wiener space and let $\{h_1,\cdots,h_n\}$ be as in Definition 2.3. Let $F \in \mathcal{F}(n;p)$ be given by (2.7) or (2.8), where $1 \leqslant p \leqslant \infty$. Then for every $z \in \mathbb{C}_+$, the functional

$$\exp\{\frac{(1-z)}{2}\sum_{j=1}^n [(h_j,x)^\sim]^2\}\cdot F(x)$$

is Wiener integrable on B.

PROOF. By Theorem 2.4, we have that for every $z \in \mathbb{C}_+$,

$$\int_B \exp\{\frac{(1-z)}{2}\sum_{j=1}^n [(h_j,x)^\sim]^2\}\cdot F(x)\,\nu(dx)$$
$$= (\frac{1}{2\pi})^{\frac{n}{2}} \int_{\mathbb{R}^n} f(\vec{u})\exp\{-\frac{z}{2}|\vec{u}|^2\}\,d\vec{u}$$

where $\vec{u} = (u_1,\cdots,u_n) \in \mathbb{R}^n$, $|\vec{u}|^2 = \sum_{j=1}^n u_j^2$, and $d\vec{u} = du_1\cdots du_n$.

We can show that the last integral has a finite value by using the same argument as in the proof of Theorem 3.1. Thus the proof of this lemma is complete.

THEOREM 3.3 Let (B,H,ν) be an abstract Wiener space and let $\{h_1,\cdots,h_n\}$ be as in Definition 2.3. Let $F \in \mathcal{F}(n;p)$ be given by (2.7) or (2.8), where $1 \leqslant p \leqslant \infty$. Then for every $z \in \mathbb{C}_+$, the analytic Wiener integral $I^{aw}(F;z)$ of F is expressed as follows:

$$I^{aw}(F;z) = z^{\frac{n}{2}} \int_B \exp\{\frac{(1-z)}{2}\sum_{j=1}^n [(h_j,x)^\sim]^2\}\cdot F(x)\,\nu(dx). \tag{3.3}$$

PROOF. By Lemma 3.2, the right hand side of (3.3) has a finite value. Using Theorem 2.4, we obtain that

$$\begin{aligned}
&\int_B \exp\{\frac{(1-z)}{2}\sum_{j=1}^{n}[(h_j,x)^\sim]^2\}\cdot F(x)\,\nu(dx)\\
&=\int_B \exp\{\frac{(1-z)}{2}\sum_{j=1}^{n}[(h_j,x)^\sim]^2\}\cdot f((h_1,x)^\sim,\cdots,(h_n,x)^\sim)\,\nu(dx)\\
&=z^{-\frac{n}{2}}(\frac{z}{2\pi})^{\frac{n}{2}}\int_{\mathbf{R}^n} f(\vec{u})\cdot\exp\{-\frac{z}{2}|\vec{u}|^2\}\,d\vec{u},
\end{aligned}$$

where $\vec{u}=(u_1,\cdots,u_n)\in\mathbf{R}^n$, $|\vec{u}|^2=\sum_{j=1}^n u_j^2$, and $d\vec{u}=du_1\cdots du_n$. Therefore, we have established the equality (3.3) by (3.1) in Theorem 3.1

Now we shall express the analytic Feynman integral $I^{af}(F;q)$ of F in $\mathcal{F}(n;1)$ as the limit of a sequence of Wiener integrals on the abstract Wiener space.

THEOREM 3.4 Let (B,H,ν) be an abstract Wiener space and let $\{h_1,\cdots,h_n\}$ be as in Definition 2.3. Let $F\in\mathcal{F}(n;1)$ be given by (2.7). If $\{z_k\}_{k=1}^{\infty}$ is a sequence of complex numbers from $\mathbb{C}_+$ such that z_k approaches $-iq$ through $\mathbb{C}_+$, where q is a non-zero real number and $i^2=-1$, then the analytic Feynman integral $I^{af}(F;q)$ of F is expressed as follows:

$$I^{af}(F;q)=\lim_{k\to\infty}(z_k)^{\frac{n}{2}}\int_B \exp\{\frac{(1-z_k)}{2}\sum_{j=1}^{n}[(h_j,x)^\sim]^2\}\cdot F(x)\,\nu(dx). \tag{3.4}$$

PROOF. By Theorem 2.4, we can show that

$$\begin{aligned}
&(z_k)^{\frac{n}{2}}\int_B \exp\{\frac{(1-z_k)}{2}\sum_{j=1}^{n}[(h_j,x)^\sim]^2\}\cdot F(x)\,\nu(dx)\\
&=(z_k)^{\frac{n}{2}}\int_B \exp\{\frac{(1-z_k)}{2}\sum_{j=1}^{n}[(h_j,x)^\sim]^2\}\cdot f((h_1,x)^\sim,\cdots,(h_n,x)^\sim)\,\nu(dx)\\
&=(\frac{z_k}{2\pi})^{\frac{n}{2}}\int_{\mathbf{R}^n} f(\vec{u})\cdot\exp\{-\frac{z_k}{2}|\vec{u}|^2\}\,d\vec{u},
\end{aligned}$$

where $\vec{u}=(u_1,\cdots,u_n)\in\mathbf{R}^n$, $|\vec{u}|^2=\sum_{j=1}^n u_j^2$, and $d\vec{u}=du_1\cdots du_n$. Using the argument similar to that as in the proof of Theorem 3.1, we conclude that

$$\begin{aligned}
&\lim_{k\to\infty}(z_k)^{\frac{n}{2}}\int_B \exp\{\frac{(1-z_k)}{2}\sum_{j=1}^{n}[(h_j,x)^\sim]^2\}\cdot F(x)\,\nu(dx)\\
&=\lim_{k\to\infty}(\frac{z_k}{2\pi})^{\frac{n}{2}}\int_{\mathbf{R}^n} f(\vec{u})\cdot\exp\{-\frac{z_k}{2}|\vec{u}|^2\}\,d\vec{u}\\
&=(\frac{-iq}{2\pi})^{\frac{n}{2}}\int_{\mathbf{R}^n} f(\vec{u})\cdot\exp\{\frac{iq}{2}|\vec{u}|^2\}\,d\vec{u}\\
&=I^{af}(F;q),
\end{aligned}$$

where the last equality follows from (3.2) in Theorem 3.1 Thus the proof of this theorem is complete.

Now we can obtain a change of scale formula for Wiener integrals on $\mathcal{F}(n;p)$ which follows from Theorem 3.3 and Definition 2.2.

THEOREM 3.5 Let (B,H,ν) be an abstract Wiener space. Let $\rho>0$ be given and let $\{h_1,\cdots,h_n\}$ be as in Definition 2.3. Then for every $F\in\mathcal{F}(n;p)$,

$$\int_B F(\rho x)\,\nu(dx)=\rho^{-n}\int_B \exp\{\frac{(\rho^2-1)}{2\rho^2}\sum_{j=1}^{n}[(h_j,x)^\sim]^2\}\cdot F(x)\,\nu(dx), \tag{3.5}$$

where $1 \leqslant p \leqslant \infty$.

PROOF. First, we can show that for all real $z > 0$,

$$I^{aw}(F; z) = \int_B F(z^{-\frac{1}{2}}x)\, \nu(dx)$$

by Definition 2.2. Using Theorem 3.3 and taking $z = \rho^{-2}$ in the above equality, we have the desired result.

EXAMPLE Let (B, H, ν) be an abstract Wiener Space and n be a positive integer and let $\{h_1, \cdots, h_n\}$ be an orthonormal set of elements of H. Define $F : B \to \mathbb{C}$ by

$$\begin{aligned} F(x) &\equiv f((h_1, x)^\sim, \cdots, (h_n, x)^\sim) \\ &= \exp[-\alpha \sum_{j=1}^{n} ((h_j, x)^\sim)^2] \end{aligned} \tag{3.6}$$

where α is a complex number with $Re(\alpha) > 0$.

It is easy to see that $F \in \cap_{1 \leq p \leq \infty} \mathcal{F}(n : p)$ since $Re(\alpha) > 0$, and so F satisfies the hypothesis of all the theorems in this paper. But because of the special form of F(see(3.6)) we can easily evaluate the integrals on the right-hand side of equations (3.1) and (3.2). Thus, for non-zero real q and $z \in \mathbb{C}_+$, it follows that
$I^{aw}(F : z) = (\frac{z}{2\alpha + z})^{\frac{n}{2}}$ and that $I^{af}(F : q) = \lim_{z \to -iq} I^{aw}(F : z) = (\frac{-iq}{2\alpha - iq})^{\frac{n}{2}}$

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