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L_1 SPACES FAIL A CERTAIN APPROXIMATIVE PROPERTY

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ABSTRACT. In this paper the author studies some cases of Banach space that does not have the property P_1. He shows that if $X = \ell_1$ or $L_1(\mu)$ for some non-purely atomic measure μ, then X does not have the property P_1. He also shows that if $X = \ell_\infty$ or $C(Q)$ for some infinite compact Hausdorff space Q, then X^* does not have the property P_1.

KEY WORDS AND PHRASES: Property P_1, classical Banach spaces ℓ_1, $L_1(\mu)$ ℓ_1^n, compact width
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1. INTRODUCTION

The Banach space X is said to have the property P_1, if for each $\in\ > 0$ and each $r > 0$, there is $\delta > 0$, such that for each x and y in X, there is $z \in \overline{B(x, \in)}$ satisfying that for each θ with $0 < \theta < \delta$

$$B(x, r+\delta) \cap B(y, r+\theta) \subseteq B(z, r+\theta)$$

where $B(x,r)$ is the open ball of radius r and centered at x, and $\overline{B(x,r)}$ is its clouser u

The property P_1 plays an important role in approximation theory, and many authors used it This property appears in approximation by compact operators, simultaneous approximation and other areas (see for example Roversi [1], Lau [2], Mach [3] and Kamal [4]). Mach [3] showed that if X is uniformly convex then it has the property P_1 [3], and that if $X = C(Q)$, or $X = B(Q)$ then X has the property P_1 [4]. Mach [4, page 259] asked if the space $L_1(\mu)$ has the property P_1

In this paper the author studies some cases of normed linear space X, for which X does not have the property P_1 In section 2, it is shown that if $X = \ell_1$ then X does not have the property P_1, and in section 3, it is shown that if μ is a non-purely atomic measure, then $L_1(\mu)$ does not have the property P_1 These two results give a negative answer for the question of Mach [4] In section 3, it will be shown also that if $X = (\ell_\infty)^*$, or $X = (C(Q))^*$, where Q is an infinite compact Hausdorff space, then X does not have the property P_1

In this paper ℓ_1 is the Banach space of all real sequences $x = \{x_i\}$ satisfying that $\Sigma |x_i| < \infty$, together with the norm $\|x\| = \Sigma |x_i|$ Also ℓ_1^n is the Banach space of all real n-tuples $x = (x_1, x_2, \ldots, x_n)$ together with the norm $\|x\| = \sum_{i=1}^{n} |x_i|$.

2. l_1 DOES NOT HAVE THE PROPERTY P_1

The proof of the fact that ℓ_1 does not have the property P_1 depends on the behavior of the property P_1 in ℓ_1^n In Lemma 2.3, it will be shown that if $\epsilon > 0$ is fixed, and δ_n corresponds to ϵ for $X = \ell_1^n$ in Lemma 2 1, then $\delta_n \to 0$ when $n \to \infty$, so using the fact that ℓ_1^n is a norm-one-complemented subspace of ℓ_1, it will be shown in Theorem 2.4, that ℓ_1 does not have the property P_1.

LEMMA 2.1. If the Banach space X has the property P_1 then for each $\epsilon > 0$, there is $\delta > 0$ such that for each $y \in X$, there is $z \in \overline{B(0,\epsilon)}$ such that if $0 < \theta < \delta$ then

$$B(0, 1+\delta+\theta) \cap B(y, 1+\theta) \subseteq B(z, 1+\theta).$$

PROOF. Let $r = 1$ and let $\epsilon > 0$ be given By the definition of the property P_1 there is $\delta > 0$ such that for each x and y in X, there is $z \in B(x, \epsilon)$ satisfying the following; for each θ' such that $0 < \theta' < \delta'$

$$B(x, 1+\delta') \cap B(y, 1+\theta') \subseteq B(z, 1+\theta').$$

Let $x = 0$ and $\delta = 1/2\,\delta'$, then for all θ satisfying $0 < \theta < \delta$;

$$B(0, 1+\delta+\theta) \cap B(y, 1+\theta) \subseteq B(0, 1+\delta') \cap B(y, 1+\theta) \subseteq B(z, 1+\theta).$$

LEMMA 2.2. Let $n \geq 3$ be a positive integer, let $\delta > 0$ be given and let $(z_1, ..., z_n)$ be an n-tuple of real numbers

If $\sum_{i=1}^{n} z_i \geq \delta$, and for each $i \leq n-1$

$$z_1 + ... + z_{i-1} - z_i + z_{i+1} + ... + z_n \leq -\delta,$$

then $\sum_{i=1}^{n} |z_i| \geq (2n-3)\delta$.

PROOF. For each $i = 1, 2, ..., n$, let $y_i = z_1 + ... + z_{i-1} - z_i + z_{i+1} + ... + z_n$, then $\sum_{i=1}^{n} y_i = (n-2)\sum_{i=1}^{n} z_i$, thus

$$y_n = (n-2)\sum_{i=1}^{n} z_i - \sum_{i=1}^{n-1} y_i.$$

Therefore

$$\sum_{i=1}^{n} |z_i| \geq |y_n| \geq (n-2)\delta + (n-1)\delta = (2n-3)\delta.$$

LEMMA 2.3. Let $n \geq 3$ be a positive integer and let $\delta > 0$ be a given real number such that $(2n-5)\delta \leq 1$ Then the element $x_n = (\delta, \delta, ..., \delta, -(n-2)\delta)$ in ℓ_1^n satisfies the following conditions

(1) $\forall \theta$ such that $\theta > 0$

$$B(0, 1+\delta+\theta) \cap B(x_n, 1+\theta) \neq \phi.$$

(2) If $z \in \ell_1^n$ and for each θ with $0 < \theta < \delta$;

$$B(0, 1+\delta+\theta) \cap B(x_n, 1+\theta) \subseteq B(z, 1+\theta),$$

then $||z|| \geq (2n-3)\delta$

PROOF. Let $\{e_i'\}_{i=1}^{n}$ be the standard basis in ℓ_1^n, that is $e_i' = (x_1^i, x_2^i, ..., x_n^i)$, where $x_j^i = 1$ if $i = j$ and $x_j^i = 0$ if $i \neq j$ and let

$$x_n = \delta\sum_{i=1}^{n-1} e_i' - (n-2)\delta e_n' = (\delta, \delta, ..., \delta, -(n-2)\delta) \in \ell_1^n.$$

Then $||x_n|| = (n-1)\delta + (n-2)\delta = (2n-3)\delta \leq 1 + 2\delta \leq 2 + \delta$, therefore for each θ such that $0 < \theta < \delta$

$$B(0, 1+\delta+\theta) \cap B(x_n, 1+\theta) \neq \phi.$$

Assume that $z = (z_1, z_2, ..., z_n) \in \ell_1^n$ is such that for each θ with $0 < \theta < \delta$

$$B(0, 1+\delta+\theta) \cap B(x_n, 1+\theta) \subseteq B(z, 1+\theta).$$

It will be shown that

(1) $\sum_{i=1}^{n} z_i \geq \delta$, and

(2) for each $i \leq n-1$

$$z_1 + \ldots + z_{i-1} - z_i + z_{i+1} + \ldots + z_n \leq -\delta.$$

If these are true then by Lemma 2.2, $\|z\| = \sum_{i=1}^{n} |z_i| \geq (2n-3)\delta$

(1) Assume that $z_1 + \ldots\ldots + z_n < \delta$. Let

$$\begin{aligned} y &= \delta \sum_{i=1}^{n-2} e'_i + \frac{1+\delta}{2} e'_{n-1} + \left[\frac{1-(2n-5)}{2}\right] e'_n \\ &= \left(\delta, \ldots, \delta, \frac{1+\delta}{2}, \frac{1-(2n-5)}{2}\right) \in \ell_1^n. \end{aligned}$$

Then

$$\|y\| = (n-2)\delta + \frac{1+\delta}{2} + \frac{1-(2n-5)\delta}{2} = 1+\delta.$$

On the other hand

$$\begin{aligned} \|y - x_n\| &= \left|\frac{1+\delta}{2} - \delta\right| + \left|\frac{1-(2n-5)\delta}{2} + (n-2)\delta\right| \\ &= 1. \end{aligned}$$

Thus, for each θ such that $0 < \theta < \delta$,

$$y \in B(0, 1+\delta+\theta) \cap B(x_n, 1+\theta).$$

But

$$\begin{aligned} \|y - z\| = \sum_{i=1}^{n} |y_i - z_i| \geq \left|\sum_{i=1}^{n} y_i - \sum_{i=1}^{n} z_i\right| &= \left|1 + \delta - \sum_{i=1}^{n} z_i\right| \\ &= 1 + \left(\delta - \sum_{i=1}^{n} z_i\right) \\ &> 1, \end{aligned}$$

so for any $\theta < \left(\delta - \sum_{i=1}^{n} z_i\right)$, $y \notin B(z, 1+\theta)$.

(2) Assume that for a certain $i_0 \leq n-1$

$$z_1 + \ldots + z_{i_0-1} - z_{i_0} + z_{i_0+1} + \ldots + z_n > -\delta.$$

Let

$$y = \left(\frac{1+\delta}{2}\right) e'_{i_0} - \left(\frac{1+\delta}{2}\right) e'_n = \left(0, 0, \ldots, 0, \underbrace{\frac{1+\delta}{2}}_{i_0\text{-th term}}, 0, \ldots, 0, -\left(\frac{1+\delta}{2}\right)\right) \in \ell_1^n.$$

Then

$$\|y\| = 1+\delta,$$

and

$$\begin{aligned}\|y - x_n\| &= (n-2)\delta + \left|\frac{1+\delta}{2} - \delta\right| + \left|\frac{1+\delta}{2} + (n-2)\delta\right| \\ &= (n-2)\delta + \frac{1-\delta}{2} + \frac{1-(2n-5)\delta}{2} \\ &= 1.\end{aligned}$$

Thus, for each θ such that $0 < \theta < \delta$, $y \in B(0, 1+\delta+\theta) \cap B(x_n, 1+\theta)$. But

$$\begin{aligned}\|y - z\| &= \sum_{i=1}^{n} |y_i - z_i| \\ &\geq |(y_1 + \ldots + y_{i_0-1} - y_{i_0} + y_{i_0+1} + \ldots + y_n) - (z_1 + \ldots + z_{i_0-1} - z_{i_0} + z_{i_0+1} + \ldots + z_n)| \\ &= |-1 - \delta - (z_1 + \ldots + z_{i_0-1} - z_{i_0} + z_{i_0+1} + \ldots + z_n)| \\ &= |1 + [\delta + (z_1 + \ldots + z_{i_0-1} - z_{i_0} + z_{i_0+1} + \ldots + z_n)]| \\ &> 1.\end{aligned}$$

Thus, for some $\theta > 0$, $y \notin B(z, 1+\theta)$.

THEOREM 2.4. ℓ_1 does not have the property P_1

PROOF. It will be shown that for each $\delta > 0$, there is $x_\delta \in \ell_1$, such that if $z \in \ell_1$ and for all θ with $0 < \theta < \delta$ it is true that $B(0, 1+\delta+\theta) \cap B(x_\delta, 1+\theta) \subseteq B(z, 1+\theta)$, then $\|z\| > \frac{1}{2}$. Let $\{e_i\}_{i=1}^{\infty}$ be the standard basis in ℓ_1, and let $\delta > 0$ be given If $\delta > 1$ then for each $\theta > 0$

$$B(0, 1+1+\theta) \cap B(x_1, 1+\theta) \subseteq B(0, 1+\delta+\theta) \cap B(x_1, 1+\theta).$$

Thus one can take x_1 to be x_δ So without loss of generality one may assume that $\delta \leq 1$

Let $n \geq 3$ be a positive integer satisfying $(2n-5)\delta \leq 1$ and $(2n-3)\delta > \frac{1}{2}$, and let x_n be as in Lemma 2.3 Define

$$x_\delta = \delta \sum_{i=1}^{n-1} e_i - (n-2)\delta e_n = (\delta, \delta, \ldots, \delta, -(n-2)\delta, 0, 0, \ldots) \in \ell_1.$$

Then $\|x_\delta\| = \|x_n\| \leq 2 + \delta$, thus

$$B(0, 1+\delta+\theta) \cap B(x_\delta, 1+\theta) \neq \phi \quad \text{for} \quad 0 < \theta < \delta.$$

Let $P_n : \ell_1 \to \ell_1^n$ be the mapping defined by $P_n(\{x_i\}_{i=1}^{\infty}) = \{x_i\}_{i=1}^{n}$ By the construction of x_δ its image under P_n is the element x_n

Assume that for some $z \in \ell_1$

$$B(0, 1+\delta+\theta) \cap B(x_\delta, 1+\theta) \subseteq B(z, 1+\theta) \quad 0 < \theta < \delta,$$

then in ℓ_1^n

$$B(0, 1+\delta+\theta) \cap B(x_n, 1+\theta) \subseteq B(P_n(z), 1+\theta) \quad 0 < \theta < \delta,$$

Thus by Lemma 2.3 $\|P_n(z)\| \geq (2n-3)\delta > \frac{1}{2}$. Therefore

$$\|z\| \geq \|P_n(z)\| > \frac{1}{2}.$$

3. OTHER SPACES THAT DO NOT HAVE THE PROPERTY P_1

The subspace Y of X is called a norm-one-complemented subspace of X if there is a linear projection $P : X \to Y$ satisfying that $\|P\| = 1$. If A is a subset of X, and $x \in X$ then

$$d(x, A) = \inf\{\|x - y\|; y \in A\},$$

and if B is another subset of X, then the deviation of A from B is defined by

$$\delta(A, B) = \sup\{d(x, B); x \in A\}.$$

The compact width of A in X is defined by

$$a(A, X) = \inf\{\delta(A, K); K \text{ is a compact subset of } X\}.$$

The compact width is said to be attained if there is a compact subset K of X satisfying that $a(A, X) = \delta(A, K)$

In this section it will be shown that if $X = (C(Q))^*$, where Q is an infinite compact Hausdorff space, $X = (\ell_\infty)^*$, or $X = L_1(\mu)$ where μ is non-purely atomic measure, then X does not have the property P_1.

The proof of the following proposition is elementary

PROPOSITION 3.1. Let X be a Banach space that has the property P_1, and let Y be a closed subspace of X If Y is a norm-one-complemented subspace of X, then Y has the property P_1

COROLLARY 3.2. If μ is non-purely atomic measure then $L_1(\mu)$ does not have the property P_1

PROOF. By Feder [5, Theorem 2], $L_1[0,1]$ has a subset A for which the compact width $a(A, L_1[0,1])$ is not attained, thus by Kamal [6, Theorem 4.3] $L_1[0,1]$ does not have the property P_1, but by Lacy [7, sec 8], $L_1[0,1]$ is a norm-one-complemented subspace of $L_1(\mu)$, therefore by Proposition 3 1, $L_1(\mu)$ does not have the property P_1.

NOTE 3.3. Theorem 2.4 together with Corollary 3.2 give a negative answer to the question of Mach [4, page 259].

COROLLARY 3.4. If $X = \ell_\infty$ or $X = C(Q)$ for some compact infinite Hausdorff space Q Then X^* does not have the property P_1

PROOF. If $X = \ell_\infty$ then ℓ_1 is a norm-one-complemented subspace of X^*, and if $X = C(Q)$ then by Kamal [8, Lemma 3.2], ℓ_1 is a norm-one-complemented subspace of X^*, in both cases one concludes by Proposition 3.1 that X^* does not have the property P_1.

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