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FULLY IDEMPOTENT NEAR-RINGS AND SHEAF REPRESENTATIONS

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ABSTRACT. Fully idempotent near-rings are defined and characterized which yields information on the lattice of ideals of fully idempotent rings and near-rings. The space of prime ideals is topologized and a sheaf representation is given for a class of fully idempotent near-rings which includes strongly regular near-rings.

KEY WORDS AND PHRASES: Fully idempotent and strongly idempotent near-rings, strongly regular near-rings, prime and irreducible ideals, sheaves.

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1. INTRODUCTION AND PRELIMINARIES

For basic terminology and results on near-rings, see [10] and [11]. Throughout this paper R will denote a zero-symmetric right near-ring. For subsets S and T of R, $\langle S\rangle$ will denote the ideal generated by S and, as is customary, $ST = \{st | s \in S,\ t \in T\}$. If I is an ideal of R, I is called prime if whenever A, B are ideals with $AB \subseteq I$ then either $A \subseteq I$ or $B \subseteq I$; I is completely prime if $ab \in I \Rightarrow a \in I$ or $b \in I$; I is semiprime if $A^2 \subseteq I \Rightarrow A \subseteq I$; I is irreducible (resp. strongly irreducible) if $A \cap B = I \Rightarrow A = I$ or $B = I$ (resp. $A \cap B \subseteq I \Rightarrow A \subseteq I$ or $B \subseteq I$). Thus a prime ideal is semiprime and strongly irreducible and any strongly irreducible ideal is irreducible. Note that unlike the situation in rings a prime ideal I need not have the property $aRb \subseteq I \Rightarrow a \in I$ or $b \in I$. Finally R is defined to be strongly regular if R is von Neumann regular and reduced (has no nilpotent elements). Equivalently (see [9] and [12]) R is strongly regular iff for all $x \in R\ \exists a,\ b \in R$ such that $x = x^2a = bx^2$.

In section 2 we will define and characterize fully idempotent near-rings, and topologize the set of strongly irreducible ideals. In section 3 a sheaf representation of a class of fully idempotent near-rings will be given, a class which includes the strongly regular near-rings. Of course rings are zero-symmetric near-rings, and while some results of this paper generalize what is known in the ring-theoretic case, other results (e.g. Propositions 2.3, 2.5, 2.6) appear to be new for both near-rings and rings. Finally, we note that in [13] and [14] Szeto has given an alternate approach to sheaves for classes of unital near-rings (and see concluding remark).

2. FULLY IDEMPOTENT NEAR-RINGS

A ring R is fully idempotent if each ideal I of R is idempotent, i.e. if $I = I^2$. Several characterizations of these rings were given by Courter [7] and they play an important role in the study of (von Neumann) regular and V-rings [6], both of which are proper subclasses of fully idempotent rings. In this section we examine the near-ring analogue of fully idempotent rings. We begin with the following definition.

DEFINITION 2.1. A near-ring R is called fully idempotent if for each ideal I of R, $I = \langle I^2 \rangle$.

PROPOSITION 2.2. The following assertions for a near-ring R are equivalent:

1. R is fully idempotent.
2. For each pair of ideals I, J of R, $I \cap J = \langle IJ \rangle$.
3. The set of ideals of R (ordered by inclusion) forms a lattice $(\mathcal{L}_R, \vee, \wedge)$ with $I \vee J = I + J$ and $I \wedge J = \langle IJ \rangle$ for each pair of ideals I, J of R.

PROOF. (1) $\Rightarrow$ (2): For each pair of ideals I, J of R, we always have $IJ \subseteq I \cap J$. Hence $\langle IJ \rangle \subseteq I \cap J$. For the reverse inclusion, let $a \in I \cap J$ and let $\langle a \rangle$ be the (two-sided) ideal generated by a. Then $a \in \langle a \rangle = \langle \langle a \rangle \cdot \langle a \rangle \rangle \subseteq \langle IJ \rangle$. Thus $I \cap J \subseteq \langle IJ \rangle$. Hence $I \cap J = \langle IJ \rangle$.

(2) $\Rightarrow$ (3): The set of ideals of a near-ring ordered by inclusion forms a lattice under the sum and intersection of ideals ([11], Thm. 2.20). Thus for each pair of ideals I, J of R, $I \vee J = I + J$ and by the assumption, $I \wedge J = I \cap J = \langle IJ \rangle$.

(3) $\Rightarrow$ (2): For each pair of ideals I, J of R, $IJ \subseteq I \cap J$ always. Hence $\langle IJ \rangle \subseteq I \cap J$. On the other hand, since by the assumption $\langle IJ \rangle$ is the greatest lower bound of I and J, therefore $I \cap J \subseteq \langle IJ \rangle$. Hence $I \cap J = \langle IJ \rangle$.

(2) $\Rightarrow$ (1): Taking $I = J$ in the hypothesis, we have $I = \langle I^2 \rangle$ for each ideal I of R. Hence R is fully idempotent.

EXAMPLES. If a near-ring is biregular (in the sense of Betsch) then, among other defining properties, for every $r \in R$ there is a central idempotent e such that $\langle r \rangle = Re$ [11, p. 94]. R is biregular in the sense of Szeto if for all $r \in R$ there is a central idempotent e such that $\{\sum_1^n r_i r s_i\} = Re$. As observed in [9, Prop. 4] these are biregular in the first sense too. Moreover biregular and regular near-rings are fully idempotent. In fact, they satisfy the stronger condition that for all ideals I, J, $I \cap J = IJ$. For if R is regular and $x \in I \cap J$ then $x = xyx$ for some y so $x \in IJ$. Conversely $IJ \subset I \cap J$ is always true. Similarly if R is Betsch-biregular and $x \in I \cap J$ then $\langle x \rangle = Re$ so $x = re = ree \in IJ$. Note that in [14] Szeto defined a unital near-ring R to be strongly biregular if it is regular and all idempotents are central. These are, in fact, precisely the strongly regular near-rings as we have defined them ([9]).

It will be useful in the sequel to refer to near-rings with the property $I \cap J = IJ$ for all ideals I, J. We will call them strongly idempotent.

Recall that the set of all ideals of a near-ring under sum and intersection forms a complete modular lattice (cf. [9], Thm. 2.20). This lattice, however, need not be distributive. Below we show that the ideal lattice of a fully idempotent near-ring is a complete Brouwerian, and hence distributive, lattice. A lattice $\mathcal{L}$ is called Brouwerian if for any $a, b \in \mathcal{L}$, the set of all $x \in \mathcal{L}$ satisfying $a \wedge x \leq b$ contains a greatest element c, the pseudo-complement of a relative to b [4].

PROPOSITION 2.3. Let R be a fully idempotent near-ring. The set $\mathcal{L}_R$ of ideals of R (ordered by inclusion) is a complete Brouwerian lattice under the sum and intersection of ideals.

PROOF. As remarked above $\mathcal{L}_R$ is a complete lattice under the sum and intersection. We now show that $\mathcal{L}_R$ is a Brouwerian lattice. Let B and C be ideals of R. By Zorn's lemma there is an ideal M of R which is maximal in the family S of ideals I satisfying $I \cap B \subseteq C$. We wish to show M is the greatest element of S. If $M = R$ we are done. If $M \neq R$ and $I \in S$ then $IB \subseteq C$ so $(I + M) \cap B = \langle (I + M) \cdot B \rangle \subseteq \langle IB + MB \rangle \subseteq C$. By maximality of M, $I + M = M$, i.e. $I \subseteq M$ as required.

COROLLARY. $\mathcal{L}_R$ is distributive.

PROOF. Follows from ([4], 11.11).

The next proposition shows that the concepts of prime, irreducible and strongly irreducible coincide for fully idempotent near-rings. First we record the following existence result for irreducible ideals:

LEMMA 2.4. If I is an ideal of any near-ring R and if $a \notin I$, there exists an irreducible ideal K such that $I \subseteq K$ and $a \notin K$.

PROOF. By Zorn's lemma, $S = \{\text{ideals } L \mid I \subseteq L,\ a \notin L\}$ has maximal elements. Let K be one. If $K = B \cap C$ where B and C both properly contain K then they both contain I so by maximality of K they both contain a. But then $a \in B \cap C = K$, a contradiction.

COROLLARY. Every proper ideal is contained in a proper irreducible ideal.

PROPOSITION 2.5. Let R be a fully idempotent near-ring and let P be an ideal of R. Then the following assertions are equivalent:

1. P is irreducible.
2. P is strongly irreducible.
3. P is prime.

PROOF. As (3) $\Rightarrow$ (2) $\Rightarrow$ (1) is clear, it suffices to show that (1) $\Rightarrow$ (3). Suppose $IJ \subseteq P$ for ideals I, J of R. Since R is fully idempotent, $I \cap J = \langle IJ \rangle$ by Proposition 2.2. On the other hand, $IJ \subseteq P$ implies that $\langle IJ \rangle \subseteq P$. Hence $I \cap J \subseteq P$. This implies that $(I \cap J) + P = P$. Since R is fully idempotent then by the corollary to Proposition 2.3, the ideal lattice of R is distributive. Hence $P = (I \cap J) + P = (I + P) \cap (J + P)$. Since P is irreducible, $I + P = P$ or $J + P = P$. This implies that $I \subseteq P$ or $J \subseteq P$. Hence P is a prime ideal.

PROPOSITION 2.6. Let R be a fully idempotent near-ring and $\mathcal{L}_R$ be the ideal lattice of R under the sum and intersection of ideals. Then the set B_R of direct summands of R is a Boolean sublattice of $\mathcal{L}_R$.

PROOF. The proposition will follow if we show that the sum and intersection of summands of R are summands of R; that is, if A_1 and A_2 are direct summands of R, then $A_1 + A_2$ and $A_1 \cap A_2$ are direct summands of R. Let B_1 and B_2 be the cosummands of A_1 and A_2, respectively, that is, $A_i + B_i = R$ and $A_i \cap B_i = (0)$; for $i = 1, 2$. We show that $A_1 + A_2$ is a summand with $B_1 \cap B_2$ as its cosummand. Suppose $I = (B_1 \cap B_2) + A_1 + A_2 \neq R$ and let $x \in R$, $x \notin I$. By Lemma 2.4 there is an irreducible (hence by Proposition 2.5 a strongly irreducible) ideal P such that $I \subseteq P$, $x \notin P$. Then $B_1 \cap B_2 \subseteq P$ so B_1 (say) $\subseteq P$. Since

$A_1 + B_1 = R$, $A_1 \not\subseteq P$. But $A_1 \subseteq I \subseteq P$ which is a contradiction. Therefore $I = R$. Now let $x \in B_1 \cap B_2$ and $y_k \in A_k$ $(k = 1,2)$. Then $(y_1 + y_2)x = y_1x + y_2x \in A_1B_1 + A_2B_2 \subseteq (A_1 \cap B_1) + (A_2 \cap B_2) = (0)$. Hence, by Proposition 2.2, $(A_1 + A_2) \cap (B_1 \cap B_2) = (0)$. Therefore $A_1 + A_2$ is a direct summand of R with $(B_1 \cap B_2)$ as its cosummand. Using an identical argument, we can show that $A_1 \cap A_2$ is a summand with $B_1 + B_2$ as its cosummand.

We now prove the following characterization theorem for fully idempotent near-rings.

THEOREM 2.7. The following are equivalent:

1. R is fully idempotent.
2. Every proper ideal is the intersection of the prime ideals containing it.

PROOF. (1) $\Rightarrow$ (2). First note that if R is fully idempotent every ideal is contained in some prime ideal by the corollary to Lemma 2.4, and Proposition 2.5. Let $\{P_\alpha\}$ be the primes containing I so $I \subseteq \cap P_\alpha$. Conversely if $a \notin I$ there exists a prime ideal P with $I \subseteq P$ and $a \notin P$.
(2) $\Rightarrow$ (1). Let I be an ideal of R. If $\langle I^2 \rangle = R$ then $I = \langle I^2 \rangle$. If $\langle I^2 \rangle \neq R$ then $I^2 \subseteq \langle I^2 \rangle = \cap P_\alpha \subseteq P_\alpha$ so $I \subseteq P_\alpha$ for all α, i.e. $I \subseteq \cap P_\alpha = \langle I^2 \rangle$. Since $\langle I^2 \rangle \subseteq I$ we are done.

COROLLARY 1. R is fully idempotent iff each ideal is semiprime [11, Prop. 2.93].

COROLLARY 2. A fully idempotent near-ring is a subdirect product of prime near-rings [11, Thm. 2.95].

Let S_R denote the set of proper strongly irreducible ideals of R. Since every near-ring has minimal prime ideals [11, Cor. 2.77], S_R is not empty. For any ideal I, define $\Theta_I = \{J \in S_R | I \not\subseteq J\}$ and let $T(S_R) = \{\Theta_I | I \text{ is an ideal of } R\}$.

THEOREM 2.8. The set $T(S_R)$ constitutes a topology on the set S_R and if every irreducible ideal is strongly irreducible the assignment $I \mapsto \Theta_I$ is a lattice isomorphism between the lattice $\mathcal{L}_R$ of ideals of R and the lattice of open subsets of S_R.

PROOF. First we show that the set $T(S_R)$ forms a topology on the set S_R. Since (0) is contained in every ideal, therefore $\Theta_{(0)} = \{J \in S_R : (0) \not\subseteq J\} = \phi$. Furthermore, $\Theta_R = \{J \in S_R : R \not\subseteq J\} = S_R$ because elements of S_R are proper. Now let $\Theta_{I_1}, \Theta_{I_2} \in T(S_R)$ with I_1, I_2 in $\mathcal{L}_R$. Then $\Theta_{I_1} \cap \Theta_{I_2} = \{J \in S_R : I_1 \not\subseteq J \text{ and } I_2 \not\subseteq J\} = \{J \in S_R : I_1 \cap I_2 \not\subseteq J\}$. Now consider an arbitrary family $(I_\lambda)_{\lambda \in \Lambda}$ of ideals of R. Then $\bigcup \Theta_{I_\lambda} = \bigcup \{J \in S_R : I_\lambda \not\subseteq J\}$
$= \{J \in S_R : \exists \lambda \in \Lambda \text{ such that } I_\lambda \not\subseteq J\} = \{J \in S_R : \sum_\lambda I_\lambda \not\subseteq J\} = \Theta_{\sum I_\lambda}$. Since $\sum I_\lambda$ is an ideal of R ([11], Thm. 2.1), it follows that $\bigcup_\lambda \Theta_{I_\lambda} \in T(S_R)$. This shows that the set $T(S_R)$ of subsets Θ_I, with $I \in \mathcal{L}_R$, is a topology on the set S_R. Define $\phi : \mathcal{L}_R \to T(S_R)$ by setting $\phi(I) = \Theta_I$. It is easily verified that ϕ preserves finite intersections and arbitrary unions. Hence ϕ is a lattice homomorphism. Finally we show that ϕ is an isomorphism. For this purpose, we show that $I_1 = I_2 \Leftrightarrow \Theta_{I_1} = \Theta_{I_2}$ for I_1, I_2 in $\mathcal{L}_R$. Suppose $\Theta_{I_1} = \Theta_{I_2}$. If $I_1 \neq I_2$, then $\exists x \in I_1$ such that $x \notin I_2$. Then by Lemma 2.4, there exists an irreducible and hence a strongly irreducible ideal J of R such that $I_2 \subseteq J$ and $x \notin J$, i.e. $I_1 \not\subseteq J$. Hence $J \in \Theta_{I_1}$. But $\Theta_{I_1} = \Theta_{I_2}$, so $J \in \Theta_{I_2}$. This means that $I_2 \not\subseteq J$. But this is a contradiction. Hence $I_1 = I_2$.

Recall that a near-ring R is <u>indecomposable</u> ([11, 2.42]) if it is not the direct sum of non-trivial ideals.

PROPOSITION 2.9. If R is a near-ring in which every irreducible ideal is strongly irreducible then R is indecomposable iff $T(S_R)$ is a connected space.

Proof. Recall that a topological space is connected if and only if it has no nonempty proper open and closed subsets. If I is an ideal of R, then Θ_I is both open and closed if and only if there exists some ideal J of R such that $\Theta_I \cup \Theta_J = S_R$ and $\Theta_I \cap \Theta_J = \phi$. By the 1-1 correspondence of the preceding Theorem, this implies that $I + J = R$ and $I \cap J = (0)$. Hence Θ_I is both open and closed if and only if I is a direct summand of R. Thus it follows that $T(S_R)$ is a connected space if and only if R has no nontrivial direct summands, that is, R is directly indecomposable.

We end this section by discussing near-rings in which each irreducible ideal is strongly irreducible. By Proposition 2.5, fully idempotent near-rings are in this class (and in this case S_R is the prime spectrum and $T(S_R)$ the spectral space). More generally in any distributive lattice it is easy to show that meet-irreducible elements are strongly irreducible (for example, dualize [1, Lemma 1, p. 58]). According to [1, Ch. IX] rings with the stated property are precisely those for which the conclusion of the Chinese Remainder Theorem holds, and an examination of the proof for rings shows that it holds for zero-symmetric near-rings also. Examples of near-rings which are not rings and which have a distributive lattice of ideals are the unital near-rings for which no homomorphic image is a ring [11, Cor. 2.25].

3. STRONGLY IDEMPOTENT AND STRONGLY REGULAR NEAR-RINGS

We first formulate the definition of a sheaf of near-rings as follows:

DEFINITION 3.1. Let X be a topological space and let $T(X)$ be the category of open sets of X and inclusion maps. A presheaf P of near-rings on X is a contravariant functor from the category $T(X)$ to the category N of near-rings, that is, it consists of the data:

(a) For every open set $U \subseteq X$, there exists a near-ring $P(U)$, and

(b) for every inclusion $V \subseteq U$ of open sets, there exists a near-ring homomorphism $\rho^P_{UV} : P(U) \to P(V)$, subject to the following conditions:

1. $P(\phi) = (0)$, where ϕ is the empty set of X;
2. $\rho^P_{UU} : P(U) \to P(U)$ is the identity map, and
3. if $W \subseteq V \subseteq U$ are three open sets, then $\rho^P_{UW} = \rho^P_{VW} \circ \rho^P_{UV}$.

If P is a presheaf on X, then $P(U)$ is called a section of the presheaf P on the set U and the maps ρ^P_{UV} are called restriction maps, for which the notation $\alpha|_V$ is occasionally used instead of $\rho^P_{UV}(\alpha)$ where $\alpha \in P(U)$. The presheaf P is called a sheaf if the following additional conditions are satisfied:

4. if U is an open set and $(V_\lambda)_{\lambda\in\Lambda}$ is an open covering of U and if $\alpha|_{V_\lambda} = \beta|_{V_\lambda}$ for $\alpha, \beta \in P(U)$ and for all V_λ, then $\alpha = \beta$;
5. if U is an open set and $(V_\lambda)_{\lambda\in\Lambda}$ is an open covering of U and if there are elements $\alpha_\lambda \in P(V_\lambda)$ for each $\lambda \in \Lambda$ such that for each pair $\lambda, \mu \in \Lambda$,
$\alpha_{\lambda}|_{V_\lambda \cap V_\mu} = \alpha_{\mu}|_{V_\lambda \cap V_\mu}$, then $\exists \alpha \in P(U)$ s.t. $\alpha|_{V_\lambda} = \alpha_\lambda$ for each $\lambda \in \Lambda$.

If a presheaf satisfies condition (4) only, it is called separated [2].

Next we state some preliminary lemmas. We write $E(I)$ for the near-ring generated by all the R-endomorphisms of the left R-group ${}_RI$.

LEMMA 3.2. If R is a strongly idempotent near-ring then for every pair of ideals I, J with $J \subseteq I$, any R-homomorphism from J to I factors through J.

PROOF. Since $J \cap I = JI$ then when $J \subseteq I$ we have $J = JI$. Also $J = J^2$ so if $a \in J$ then $a = a_1b = a_2a_3b$ for $a_i \in J$, $b \in I$. Then $f(a) = a_2f(a_3b) \in JI = J$.

We now describe a sheaf of near-rings on the prime spectrum of some near-rings.

THEOREM 3.3. Let R be a strongly idempotent near-ring in which $(R, +)$ is abelian. The assignment $\Theta_I \to E(I) = F_R(I)$ defines a sheaf F_R of near-rings on the prime spectrum of R.

PROOF. First we prepare the data for the existence of a presheaf. $F_R(I) = E(I)$ is a near-ring (in fact a ring) for every ideal I of R. We now define a restriction map: $\rho^F_{IJ} : E(I) \to E(J)$, whenever $\Theta_J \subseteq \Theta_I$, that is, when $J \subseteq I$ for ideals I, J of R. Let $\bar{\alpha} = \sum_{i=1}^{n} \sigma_i e_i \in E(I)$. Define $\rho_{IJ}(\bar{\alpha}) = \bar{\alpha}|_J$. Note that $\bar{\alpha}|_J \in E(J)$ by Lemma 3.2. Clearly, ρ^F_{IJ} is a near-ring homomorphism. Thus F_R satisfies the conditions of a presheaf. We now show that F_R is separated. Let $I = \sum_{\lambda \in \Lambda} I_\lambda$ and suppose $f, g \in F_R(I)$ such that $f|_{I_\lambda} = g|_{I_\lambda}$ for all $\lambda \in \Lambda$. For each $x \in I$, we can write $x = x_1 + \ldots + x_n$ for $x_n \in I_\lambda$. Since $(R, +)$ is commutative, we can write $f(x) = f(x_1) + f(x_2) + \ldots + f(x_n) = g(x_1) + g(x_2) + \ldots + g(x_n) = g(x_1 + \ldots + x_n) = g(x)$. Hence $f = g$, and so F_R is separated. Finally, in order to show that F_R is a sheaf, we verify condition (5) in Def. 3.1. Let $I = \sum_{\lambda \in \Lambda} I_\lambda$ be an ideal of R, and suppose for each $\lambda, \mu \in \Lambda$, $f_\lambda \in E(I_\lambda)$ and $f_\mu \in E(I_\mu)$ such that $f_{\lambda | I_\lambda \cap I_\mu} = f_{\mu | I_\lambda \cap I_\mu}$. Define $f : I_\lambda + I_\mu \to I_\lambda + I_\mu$ as follows: For $x \in I_\lambda + I_\mu$, write $x = x_\lambda + x_\mu$ with $x_\lambda \in I_\lambda$ and $x_\mu \in I_\mu$. We now define $f(x) = f_\lambda(x_\lambda) + f_\mu(x_\mu)$. We show that f is well-defined. Suppose $x = x_\lambda + x_\mu = x'_\lambda + x'_\mu$. Then since $(R, +)$ is commutative, we have $x_\lambda - x'_\lambda = x'_\mu - x_\mu \in I_\lambda \cap I_\mu$. Hence $f_\lambda(x_\lambda - x'_\lambda) = f_\mu(x'_\mu - x_\mu)$. Therefore $f_\lambda(x_\lambda) + f_\mu(x_\mu) = f_\lambda(x'_\lambda) + f_\mu(x'_\mu)$. Now suppose $x \in I_\nu \cap (I_\lambda + I_\mu)$ where I_ν is an ideal of R for some $\nu \in \Lambda$. Since every strongly idempotent near-ring is fully idempotent, by the corollary to Proposition 2.3, we can write $I_\nu \cap (I_\lambda + I_\mu) = (I_\nu \cap I_\lambda) + (I_\nu \cap I_\mu)$. Thus $x \in I_\nu \cap (I_\lambda + I_\mu)$ means that $x \in (I_\nu \cap I_\lambda) + (I_\nu \cap I_\mu)$. Hence we can write $x = x_\lambda + x_\mu$ where $x_\lambda \in I_\nu \cap I_\lambda$ and $x_\mu \in I_\nu \cap I_\mu$. Thus $f(x) = f_\lambda(x_\lambda) + f_\mu(x_\mu) = f_\nu(x_\lambda) + f_\nu(x_\mu) = f_\nu(x_\lambda + x_\mu) = f_\nu(x)$. This implies that the family $\{I_\lambda : \lambda \in \Lambda\}$ is stable under finite sums. We can now define a map $f : \sum I_\lambda \to \sum I_\lambda$ which satisfies condition (5) in Def. 3.1. Suppose $x \in \sum_{\lambda \in \Lambda} I_\lambda$. Then $x = x_1 + \ldots + x_n$ where $x_\lambda \in I_\lambda$. Since x belongs to a finite sum of I_λ's, by the preceding arguments, we can suppose that $x \in I_\mu$ for some μ. Hence we can define $f(x) = f_\mu(x)$, which is well-defined since two different f_μ agree on x when $f_\mu(x)$ is correctly defined. Clearly $f \in E(I)$ which extends each f_λ for $\lambda \in \Lambda$. This proves that F_R is a sheaf of near-rings.

Strongly regular near-rings satisfy the hypotheses of Theorem 3.3 (see [11, 9.159a]). Further information about these near-rings is obtained by using the work of Cornish [5]. For any prime ideal P, let $O_P = \{x | xy = 0 \text{ for some } y \notin P\}$. As observed in [6] any reduced zero-symmetric near-ring is a reduced system in the sense of [5, p. 883] and so we can apply [5, Thm. 3.5] to obtain:

THEOREM 3.4. For any prime ideal P of a reduced near-ring, O_P is the intersection of the minimal prime ideals contained in P. Hence P is a minimal prime iff $P = O_P$.

The next result is known in the unital case (eg. [11, Thm. 9.163]).

PROPOSITION 3.5. If R is strongly regular, every prime ideal is a minimal prime ideal.

PROOF. First note that all prime ideals in a strongly regular near-ring are completely prime [3, examples following Prop. 5.4]. Let $x \in P$ and choose $r \notin P$. If $rx = 0$, then $xr = 0$ since R is reduced, so $x \in O_P$ and we are done. If $rx \neq 0$ then $rx = rxsrx$ for some $s \in R$. Hence $(r - rxsr)x = 0$. Let $y = r - rxsr$. Then $y \notin P$ or else $r \in P$ (since $x \in P$). Hence $x \in O_P$ as required.

THEOREM 3.6. If R is strongly regular, $T(P_R)$ is Hausdorff.

PROOF. Suppose P and Q are distinct prime ideals with $a \in Q$, $a \notin P$. Then $a \in O_Q$ so $ab = 0$ for some $b \notin Q$. Because prime ideals are completely prime $\Theta_{(a)}$ and $\Theta_{(b)}$ are disjoint open sets containing P and Q respectively.

CONCLUDING REMARKS. As mentioned in the introduction, a different approach to sheaves of unital zero-symmetric near-rings was taken by Szeto. He used the prime ideal spaces of the Boolean algebra of central idempotents. In [14] he further concluded that an approach based on the prime ideal space of R was "unlikely for a near-ring due to the fact that the sum of ideals is not necessarily an ideal and an ideal is not necessarily embedded in a prime ideal". However, the last statement is false for unital near-rings [11, 271 and 2.72] and the sum of ideals is generally defined to make the first statement incorrect [11, Theorem 2.1].

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