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LUCAS PARTITIONS

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ABSTRACT. The Lucas sequence is defined by: $L_0 = 2$, $L_1 = 1$, $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$. Let $V(n)$, $r(n)$ denote respectively the number of partitions of n into parts, distinct parts from $\{L_n\}$. We develop formulas that facilitate the computation of $V(n)$ and $r(n)$.

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1. INTRODUCTION

Let S denote a non-empty subset of N, the set of all natural numbers. Let $V(n)$, $r(n)$, $r_E(n)$, $r_O(n)$ denote respectively the number of partitions of n into parts, distinct parts, evenly many distinct parts, oddly many distinct parts from S. Define $V(0) = r(0) = r_E(0) = 1$, $r_O(0) = 0$. Let $V(n)$ have the generating function:

$$F(z) = \sum_{n=0}^{\infty} V(n)z^n. \tag{1.1}$$

Let

$$1/F(z) = \sum_{n=0}^{\infty} a(n)z^n. \tag{1.2}$$

It follows from (1.1) and (1.2) that

$$\sum_{k=0}^{n} a(n-k)V(k) = 0 \quad \text{for} \quad n \geq 1. \tag{1.3}$$

Furthermore,

$$a(n) = r_E(n) - r_O(n). \tag{1.4}$$

REMARK. Apostol [1], p.311 and Hardy [3], p.285 prove that (1.4) holds when $S = N$, but the same reasoning applies to the more general case. Since also

$$r(n) = r_E(n) + r_O(n) \tag{1.5}$$

it follows that

$$a(n) = 2r_E(n) - r(n) = r(n) - 2r_O(n). \tag{1.6}$$

In this note, we consider the case where S is the set of all Lucas numbers, L_n, where $n \geq 0$. (The Lucas numbers are defined by: $L_0 = 2$, $L_1 = 1$, $L_n = L_{n-1} + L_{n-2}$ if $n \geq 2$.) We will show how to compute the $r(n)$ and the $a(n)$: via explicit formulas if $n = L_k$ or $n = 1 + L_{2k+1}$ for some k, recursively otherwise. The $V(n)$ can then be computed recursively via (1.3).

2. PRELIMINARIES

Notation and Definitions

$F_k = k^{th}$ Fibonacci number ($F_0 = 0, F_1 = 1, F_k = F_{k-1} + F_{k-2}$ if $k \geq 2$)

$L_k = k^{th}$ Lucas number ($L_0 = 2, L_1 = 1, L_k = L_{k-1} + L_{k-2}$ if $k \geq 2$)

$E(n) = \sum_{k=1}^{n} |a(k)|$

$[x]$ denotes the integer part of the real number x

$[a, b]$ denotes the set of all integers, t, such that $a \leq t \leq b$.

In particular, if $k \geq 3$, then

$I_k = [L_{k+1}, L_{k+2} - 1]$

$I_{k,1} = [L_{k+1}, 2L_k - 1]$

$I_{k,2} = [2L_k, 5F_k - 1]$

$I_{k,3} = [5F_k, L_{k+2} - 1]$

Lucas Identities

(1) $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$, with $L_0 = 2$, $L_1 = 1$

(2) $L_j < L_k$ iff $j < k$, unless $j = k - 1 = 0$

(3) $\{L_n\}$ is strictly increasing if $n \geq 1$

(4) $L_n = \alpha^n + \beta^n$, where $\alpha = \frac{1}{2}\left(1 + 5^{\frac{1}{2}}\right)$, $\beta = \frac{1}{2}\left(1 - 5^{\frac{1}{2}}\right)$

(5) $L_{2n} = L_n^2 - 2(-1)^n$

(6) $L_n > 1.6^n$ if $n \geq 4$

(7) $\sum_{i=0}^{k+1} L_i = L_{k+3} - 1$

(8) $\sum_{i=0}^{[\frac{1}{2}j]} L_{j-2i} = L_{j+1} + t$, where $t = \begin{cases} 1 & \text{if } 2|j \\ -2 & \text{if } 2\not| j \end{cases}$

(9) $L_{n+1} + L_{n-1} = 5F_n$

(10) $L_{k+2} - 5F_k = 2L_k - L_{k+1} = L_{k-2}$

REMARKS. (1) is the definition of the Lucas sequence (2) and (3) follow from (1). (4) follows from (1), using induction and the fact that α, β are the roots of $u^2 - u - 1 = 0$. (5) and (6) follow from (4). (7) through (9) may be proved using induction on n. (10) follows from (1) and (9).

3. THE MAIN THEOREMS

Let n be a natural number. We first address the issue of the representability of n as a sum of distinct Lucas numbers. Such a representation will be called a Lucas representation of n. If in addition, the summands are non-consecutive Lucas numbers, we say that the Lucas representation of n is special. We will show that every natural number has a special Lucas representation. If the special Lucas representation of n is unique, which is usually the case, we call it the minimal Lucas representation of n. Otherwise, n has two special Lucas representations. In this case, we define the minimal Lucas representation of n as the special Lucas representation that does not include $L_0 = 2$ as a summand.

For example, 13 has the unique special (and hence minimal) Lucas representation $13 = 11 + 2 = L_5 + L_0$; 12 has two special Lucas representations: $12 = 11 + 1 = L_5 + L_1$, and $12 = 7 + 3 + 2 = L_4 + L_2 + L_0$. The former is the minimal Lucas representation of 12.

THEOREM 1. Every natural number, n, has a special Lucas representation:

$$n = L_{k_1} + L_{k_2} + \text{etc.} + L_{k_r} \tag{3.1}$$

where $k_i - k_{i+1} \geq 2$ for all i such that $1 \leq i \leq r - 1$, if $r \geq 2$.

PROOF. (Induction on n) It suffices to consider the case where $n \neq L_k$. Therefore there exists unique $k_1 \geq 3$ such that $L_{k_1} < n < L_{k_1+1}$. Let $n_1 = n - L_{k_1}$. Now (1) implies $0 < n_1 < L_{k_1-1}$. By induction hypothesis, we have $n_1 = L_{k_2} + L_{k_3} + \text{etc.} + L_{k_r}$, with $r \geq 2$ and $k_i - k_{i+1} \geq 2$ for all i such that $2 \le i \le r$, if $r \ge 3$. Thus $L_{k_2} \le n_1$, hence $L_{k_2} < L_{k_1-1}$. Since $k_1 \ge 3$, (2) implies $k_2 < k_1 - 1$, that is, $k_1 - k_2 \ge 2$. Since $n = L_{k_1} + n_1$, the conclusion now follows.

LEMMA 1. Let $n = L_{j_1} + L_{j_2} + \text{etc.} + L_{j_s}$, with $s \ge 2$ and $j_i - j_{i+1} \ge 2$ for all i such that $1 \le i \le s-1$. Let $j = j_1$. Then $n \le L_{j+1} + (-1)^j$. Furthermore, $n = L_{j+1} + (-1)^j$ iff $s = 1 + \left[\frac{1}{2}j\right]$, $j_s = 0$, and $j_i = j - 2(i-1)$ for all i such that $1 \le i \le s-1$.

PROOF. Using (2) and our hypothesis, we have $L_{j_i} \le r + L_{j-2(i-1)}$ where $r = \begin{cases} 1 \text{ if } j_i = 0 \text{ and } j = 2i-1 \\ 0 \text{ otherwise} \end{cases}$. Thus we have:

$$n = \sum_{i=1}^{s} L_{j_i} \le r + \sum_{i=1}^{s} L_{j-2(i-1)} = r + \sum_{i=0}^{s-1} L_{j-2i} \le r + \sum_{i=0}^{[\frac{1}{2}j]} L_{j-2i}. \tag{3.2}$$

Now (8) implies $n \le L_{j+1} + (-1)^j$. If $s = 1 + \left[\frac{1}{2}j\right]$, $j_s = 0$, and $j_i = j - 2(i-1)$ for all i such that $1 \le i \le s-1$, then the weak inequalities in (3.2) may be replaced with equalities, which yields $n = L_{j+1} + (-1)^j$. Conversely, if $n = L_{j+1} + (-1)^j$, then the weak inequalities in (3.2) become equalities. This implies $s = 1 + \left[\frac{1}{2}j\right]$, $L_{j_i} = L_{j-2(i-1)}$ (and hence $j_i = j - 2(i-1)$) for all i such that $1 \le i \le s-1$, and $j_s = 0$.

LEMMA 2. $L_i = L_j + 1$ iff $(i,j) = (0,1), (2,0)$, or $(3,2)$.

PROOF. Suppose $L_i = L_j + 1$. If $j = 0$, then $L_i = 3$, so $i = 2$. If $j = 1$ and $i < j$, then $i = 0$. Now suppose $i > j \ge 1$. Then (3) and (1) imply $1 = L_i - L_j \ge L_i - L_{i-1} = L_{i-2}$. Therefore $L_{i-2} = 1$, so $i = 3$ and $j = 2$. The converse follows by direct substitution.

LEMMA 3. If

$$n = L_k \tag{3.3}$$

then this special Lucas representation of n is unique.

PROOF. Let k be the least index such that the special representation (3.3) is not unique. By inspection, $k \ge 4$. Thus n has a second special Lucas representation:

$$n = L_{j_1} + L_{j_2} + \text{etc.} + L_{j_s} \tag{3.4}$$

with $j_i - j_{i+1} \ge 2$ for all i such that $1 \le i \le s-1$. In fact, (2) implies $s \ge 2$. Let $j = j_1$. Now (3.4) implies $L_j < n$, so $L_j < L_k$. If $2 \nmid j$, then Lemma 1 implies $L_k \le L_{j+1} - 1$, so $L_k < L_{j+1}$ But then $L_j < L_k < L_{j+1}$, an impossibility. If $2|j$, then Lemma 1 implies $L_k \le L_{j+1} + 1$. Since $k \ge 4$, Lemma 2 implies $L_k \ne L_{j+1} + 1$. Therefore $L_k < L_{j+1} + 1$, so that $L_k \le L_{j+1}$. Since $L_j < L_k \le L_{j+1}$, we must have $L_k = L_{j+1}$, hence $k = j+1$. Now (3.4) yields $L_k = L_{k-1} + L_{j_2} + \text{etc.} + L_{j_s}$, hence $L_{k-2} = L_{j_2} + \text{etc.} + L_{j_s}$. By definition of k, we must have $s = 2$, $j_2 = k-2$. But then $j_1 - j_2 = 1$, an impossibility.

THEOREM 2. Let n have two distinct special Lucas representations:

$$n = L_{k_1} + L_{k_2} + \text{etc.} + L_{k_r} \quad \text{with } k_i - k_{i+1} \ge 2 \text{ for all } i \text{ such that } 1 \le i \le r-1; \tag{3.5}$$

$$n = L_{j_1} + L_{j_2} + \text{etc.} + L_{j_s} \quad \text{with } j_i - j_{i+1} \ge 2 \text{ for all } i \text{ such that } 1 \le i \le s-1. \tag{3.6}$$

Assume also that $j = j_1 < k_1$. Then $k_1 = 2s-1$, $k_2 = k_r = 1$, and $j_i = 2(s-i)$ for all i with $1 \le i \le s$.

PROOF. Let $k_1 = k$. Note that Lemma 3 implies $\text{Min}\{r,s\} \ge 2$. Thus $n \ge 5$ and $j \ge 2$. Let $j = k - m$, where $m \ge 1$. Lemma 1 implies $n \le L_{j+1} + (-1)^j = L_{k-m+1} + (-1)^{k-m}$. By hypothesis, $L_k < n$, so that $L_k < L_{k-m+1} + (-1)^{k-m}$. If $m \ge 2$, then $L_{k-m+1} \le L_{k-1}$ by (2), since if $k = 2$, then $j = 1 = s$. Thus $L_k < L_{k-1} + (-1)^{k-m}$, which implies $L_{k-2} < 1$, an impossibility, since $L_n \ge 1$ for all n. Therefore $m = 1$, so $0 < (-1)^{k-1}$ implies k is odd. Since $L_k < n \le L_k + 1$, we must have $n = L_k + 1 = L_k + L_1$, so $k_2 = k_r = 1$. Now (3.6) and Lemma 1 imply $j_i = 2(s-i)$ for all i such that $1 \le i \le s$.

THEOREM 3. Let n have the special Lucas representation:

$$n = L_{k_1} + L_{k_2} + \text{etc.} + L_{k_r}. \tag{3.7}$$

This special Lucas representation is unique unless $k_r = 1$ and $k_{r-1} = 2h+1$ for some $h \ge 1$, in which case n has a second special Lucas representation:

$$n = L_{k_1} + L_{k_2} + \text{etc.} + L_{k_{r-2}} + L_{2h} + L_{2h-2} + \text{etc.} + L_2 + L_0. \tag{3.8}$$

PROOF. If $r = 1$, then the special Lucas representation (3.7) is unique by Lemma 3 If $r \ge 2$, suppose that n has a second special Lucas representation:

$$n = L_{j_1} + L_{j_2} + \text{etc.} + L_{j_s}. \tag{3.9}$$

Again, by Lemma 3, $s \ge 2$. If $j_1 < k_1$, then the conclusion follows from Theorem 2, with $r = 2$ and $h = s$. Now suppose that $j_i = k_i$ for all i such that $1 \le i \le u-1$ (for some $u \ge 2$), but $j_u < k_u$. Let

$$m = n - \sum_{i=1}^{u-1} L_{k_i} = L_{k_u} + L_{k_{u+1}} + \text{etc.} + L_{k_r};$$

also

$$m = n - \sum_{i=1}^{u-1} L_{j_i} = L_{j_u} + L_{j_{u+1}} + \text{etc.} + L_{j_s}.$$

Now Theorem 2 implies $j_{u+i} = 2(s-u-i)$ for all i such that $0 \le i \le s-u$, $u = r-1$, $k_u = k_{r-1} = 2(s-u) - 1 = 2(s-r)+1$, $k_r = 1$. The conclusion now follows from Theorem 2, with $h = s - r$.

Combining the results of Theorems 1, 2, and 3, we have:

THEOREM 4. Every natural number, n, has a unique minimal Lucas representation:

$$n = L_{k_1} + L_{k_2} + \ldots + L_{k_{r-1}} + L_{k_r} \tag{3.10}$$

where (i) $k_i - k_{i+1} \ge 2$ for all i such that $1 \le i \le r-1$, if $r \ge 2$; (ii) if $r \ge 2$ and $k_r = 0$, then $k_{r-1} \ge 3$.

LEMMA 4. Let n have the minimal Lucas representation given by (3.10) in Theorem 4 above. Then $L_{k_1} < n < L_{k_1+1}$, if $r \ge 2$.

PROOF. (Induction on r) Clearly, $L_{k_1} < n$, so it suffices to show that $n < L_{k_1+1}$. Let $r = 2$, so $n = L_{k_1} + L_{k_2}$. If $k_2 \ge 1$, then by hypothesis, $k_2 \le k_1 - 2$, so (2) implies $L_{k_2} \le L_{k_1-2} < L_{k_1-1}$. Thus $n \le L_{k_1} + L_{k_1-2} < L_{k_1} + L_{k_1-1} = L_{k_1+1}$. If $k_2 = 0$, then by (1) and (2), we have $n = L_{k_1} + L_{k_2} = L_{k_1} + L_0 = L_{k_1} + 2 < L_{k_1} + 3 = L_{k_1} + L_2 \le L_{k_1} + L_{k_1-1} = L_{k_1+1}$, so $n < L_{k_1+1}$. If $r \ge 3$, let $n_1 = n - L_{k_1} = L_{k_2} + \text{etc.} + L_{k_r}$. Clearly, this is a minimal Lucas representation of n_1, so by induction hypothesis, we have $n_1 < L_{k_2+1}$, hence $n < L_{k_1} + L_{k_2+1}$. Since $1 \le k_2 + 1 \le k_1 - 1$ by hypothesis, (2) implies $L_{k_2+1} \le L_{k_1-1}$. Therefore $n < L_{k_1} + L_{k_1-1} = L_{k_1+1}$.

The three following theorems permit the computation of $r(L_n)$ and $a(L_n)$.

THEOREM 5. $r(L_n) = \left[\frac{1}{2}(n+2)\right]$ if $n \ge 0$.

PROOF. (Induction on n) The statement is true by inspection if $n = 0$ or 1. If $n \ge 2$, and if L_n is partitioned into several distinct parts, then (7) implies that the largest part must be L_{n-1}. Therefore, by (1), we have $r(L_n) = 1 + r(L_{n-2}) = 1 + \left[\frac{1}{2}n\right]$ (by induction hypothesis) $= \left[\frac{1}{2}(n+2)\right]$. (The "1" in the last equation arises from the trivial partition: $L_n = L_n$.)

THEOREM 6. $r_E(L_n) = \left[\frac{1}{4}(n+2)\right]$ if $n \ge 0$.

PROOF. (Induction on n) The statement is true by inspection if $n = 0$ or 1. If $n \geq 2$, then reasoning as in the proof of Theorem 5, we have $r_E(L_n) = r_O(L_{n-2}) = r(L_{n-2}) - r_E(L_n - 2) = \left[\frac{1}{2}n\right] - \left[\frac{1}{4}n\right] = \left[\frac{1}{4}(n+2)\right]$ by Theorem 5 and induction hypothesis.

THEOREM 7. $a(L_n) = \begin{cases} 0 & \text{if } n \equiv 2, 3 \pmod 4 \\ -1 & \text{if } n \equiv 0, 1 \pmod 4 \end{cases}$.

PROOF. This follows from (1.6) and from Theorems 5 and 6.

Having settled the case where n is a Lucas number, we now consider the case where n is a sum of two or more distinct, non-consecutive Lucas numbers. Then, by Theorem 4, n has a unique minimal Lucas representation:

$$n = \sum_{k=1}^{r} L_{k_i} \tag{3.11}$$

where $r \geq 2$, $k_i - k_{i+1} \geq 2$ for all i such that $1 \leq i \leq r-1$, and if $k_r = 0$, then $k_{r-1} \geq 3$.

Alternatively, we could write:

$$n = \sum_{j=0}^{s} c_j L_j \tag{3.12}$$

where (i) $c_s = 1$; (ii) $c_j = 0$ or 1 for all j such that $0 \leq j \leq s-1$; (iii) $c_{j-1}c_j = 0$ for all j such that $1 \leq j \leq s$; (iv) if $c_0 = 1$, then $c_2 = 0$.

If we omit the conditions (iii) and (iv), then (3.12) corresponds to a Lucas representation of n. The c_j will be called the digits of the representation.

Referring again to (3.11), let $n_1 = n - L_{k_1} > 0$, $n_2 = n_1 - L_{k_2} \geq 0$. Given any Lucas representation of n, define the initial segment as the first $k_1 - k_2$ digits; define the terminal segment as the remaining digits. In the minimal Lucas representation of n, the initial segment consists of a 1 followed by $k_1 - k_2 - 1$ 0's, and corresponds to the minimal Lucas representation of $L_{k_1-k_2-1}$, while the terminal segment corresponds to the minimal Lucas representation of n_1. Lucas representations of n may be obtained as follows:

Type I. Arbitrary combinations of Lucas representations of the integers corresponding to the initial and terminal segments in the minimal Lucas representation of n, namely $L_{k_1-k_2-1}$ and n_1. Clearly, the number of Type I Lucas representations of n is $r(L_{k_1-k_2-1})r(n_1) = \left[\frac{1}{2}(k_1 - k_2 + 1)\right] r(n_1)$.

Type II. Suppose that in a non-minimal Lucas representation of n, the initial segment ends in 10, while the terminal segment starts with 0. If this block of digits, consisting of 100, is replaced by 011, then a new Lucas representation of n is obtained. A necessary condition for the existence of Type II Lucas representations is that $2|(k_1 - k_2)$.

Type III. In the minimal Lucas representation of n, if $k_r = 1$ and $k_{r-1} = 2h + 1$ for some $h \geq 1$, then by Theorem 3, a new Lucas representation of n is obtained by replacing $L_{2h+1} + L_1$ by $L_{2h} + L_{2h-2} + \text{etc.} + L_2 + L_0$.

The three following theorems enable us to compute $r(1 + L_{2k+1})$ and $a(1 + L_{2k+1})$.

THEOREM 8. If $k \geq 1$, then $r(L_{2k+1} + 1) = k + 1$.

PROOF. Let $n = L_{2k+1} + 1 = L_{2k+1} + L_1$. Here $n_1 = L_1$, so the number of Type I Lucas representations of n is $r(L_{2k-1})r(L_1) = \left[\frac{2k+1}{2}\right]\left[\frac{3}{2}\right] = k$. Since L_1 has no Lucas representation but the minimal one, there are no Type II Lucas representations of n. By hypothesis and Theorem 3, there is a unique Type III Lucas representation of n. Therefore $r(L_{2k+1} + 1) = k + 1$.

THEOREM 9. If $k \geq 1$, then $r_E(L_{4k+1} + 1) = k$; $r_E(L_{4k-1} + 1) = k + 1$.

PROOF. Let $n = L_{2j+1} + 1 = L_{2j+1} + L_1$. As in the proof of Theorem 8, n has no Type II Lucas representations. A Type I Lucas representation has an even number of terms iff its initial segment has an odd number of terms. Therefore the number of such Type I Lucas representations of n is $r_O(L_{2j-1}) = r(L_{2j-1}) - r_E(L_{2j-1}) = \left[\frac{1}{2}(2j+1)\right] - \left[\frac{1}{4}(2j+1)\right] = j - \left[\frac{1}{4}(2j+1)\right]$ by (1.5) and Theorems 5 and 6. Whether $j = 2k$ or $2k - 1$, the number of Type I Lucas representations of n with evenly many terms is k, since $2k - \left[\frac{1}{4}(4k+1)\right] = k = (2k-1) - \left[\frac{1}{4}(4k-1)\right]$. The unique Type III

Lucas representation of n has $j+1$ terms, and thus contributes to $r_E(L_{2j+1}+1)$ iff j is odd. The conclusion now follows.

THEOREM 10. If $k \geq 1$, then $a(L_{4k+1}+1) = -1$; $a(L_{4k-1}+1) = 2$.

PROOF. This follows from (1.6) and from Theorems 8 and 9.

In Theorems 11, 12, and 13 below, we develop formulas for $r(n), r_E(n)$, and $a(n)$ in the case where $n \neq L_k$, $n \neq L_{2k+1}+1$. In order to do so, we must be able to count the number of Type II Lucas representations of n. We therefore need to determine the number of Lucas representations of n that do not include the largest possible Lucas number as a part. This question is addressed by Lemma 5.

LEMMA 5. Let n have the minimal Lucas representation:

$$n = L_{k_1} + L_{k_2} + \text{etc.} + L_{k_r}.$$

Let $n_1 = n - L_{k_1} \geq 0$. Let $\bar{r}(n)$ denote the number of Lucas representations of n that do not include L_{k_1} as a part; let $r_E(n)$, $r_0(n)$ denote respectively the number of such representations consisting of evenly, oddly many parts. Then

$$\bar{r}(n) = r(n) - r(n_1); \tag{3.13}$$

$$\bar{r}_E(n) = r_E(n) - r_0(n_1); \tag{3.14}$$

$$\bar{r}_0(n) = r_0(n) - r_0(n) - r_E(n_1). \tag{3.15}$$

PROOF. It follows from the definitions of $\bar{r}(n)$, $\bar{r}_E(n)$, $\bar{r}_0(n)$ that $r(n) - \bar{r}(n)$ is the number of Lucas representations of n that do include L_{k_1} as a part; $r_E(n) - \bar{r}_E(n)$, $r_0(n) - \bar{r}_0(n)$ are respectively the number of such representations consisting of evenly, oddly many parts. If $n \neq L_k$ let

$$n = L_{k_1} + L_{j_2} + \text{etc.} + L_{j_s} \quad (\text{with } s \geq 2) \tag{3.16}$$

be a Lucas representation of n that includes L_{k_1} as a part. (It follows from Lemma 4 that L_{k_1} is the largest part.) Corresponding to (3.16), there is a Lucas representation of n_1:

$$n_1 = L_{j_2} + \text{etc.} + L_{j_s}. \tag{3.17}$$

This correspondence is clearly a bijection, so that $r(n) - \bar{r}(n) = r(n_1)$. Furthermore, the number of parts in (3.16) and (3.17) differ in parity. Therefore $r_E(n) - \bar{r}_E(n) = r_0(n_1)$ and $r_0(n) - \bar{r}_0(n) = r_E(n_1)$. The conclusions (3.13), (3.14), (3.15) now follow if $n \neq L_k$. If $n = L_k$, so that $n_1 = 0$, then clearly no other Lucas representation of n includes L_k as a part. Therefore $\bar{r}(n) = r(n) - 1 = r(n) - r(0) = r(n) - r(n_1)$. Furthermore, $\bar{r}_E(n) = r_E(n) = r_E(n) - 0 = r_E(n) - r_0(n_1)$; $\bar{r}_0(n) = r_0(n) - 1 = r_0(n) - r_E(n_1)$.

THEOREM 11. Let n have the minimal Lucas representation:

$$n = L_{k_1} + L_{k_2} + \text{etc.} + L_{k_r} \tag{3.18}$$

where (i) $r \geq 2$; (ii) if $k_r = 1$, then $2|k_{r-1}$. Let $n_1 = n - L_{k_1}$, $n_2 = n_1 - L_{k_2} \geq 0$. Then

$$r(n) = \begin{cases} \frac{1}{2}(k_1 - k_2 + 1)r(n_1) & \text{if } 2\nmid(k_1 - k_2) \\ \left(1 + \frac{1}{2}(k_1 - k_2)\right)r(n_1) - r(n_2) & \text{if } 2|(k_1 - k_2) \end{cases}.$$

PROOF. By hypothesis, there are no Type III Lucas representations of n. As mentioned earlier, the number of Type I Lucas representations of n is $\left[\frac{1}{2}(k_1 - k_2 + 1)\right]r(n_1)$. If $2\nmid(k_1 - k_2)$, then there are no Type II Lucas representations of n, so that $r(n) = \left[\frac{1}{2}(k_1 - k_2 + 1)\right]r(n_1) = \frac{1}{2}(k_1 - k_2 + 1)r(n_1)$. If $2|(k_1 - k_2)$, then the number of Type II Lucas representations of n is the number of Lucas representations of n_1 that do not include L_{k_2} as a part, namely $\bar{r}(n_1)$. By Lemma 5, we have $\bar{r}(n_1) = r(n_1) - r(n_2)$. Therefore

$$r(n) = \left[\frac{1}{2}(k_1 - k_2 + 1)\right]r(n_1) + r(n_1) - r(n_2) = \left(1 + \frac{1}{2}(k_1 - k_2)\right)r(n_1) - r(n_2).$$

COROLLARY 1. If $n \geq 2$, then $r(L_n - 3) = \left[\frac{1}{2}n\right]$.

PROOF. (Induction on n) By inspection, the conclusion is true if $2 \leq n \leq 5$. If $n \geq 6$, then $L_n - 3 = L_{n-1} + (L_{n-2} - 3) = L_{n-1} + L_{n-3} + (L_{n-4} - 3)$. Now Theorem 11 implies $r(L_n - 3) = 2r(L_{n-2} - 3) - r(L_{n-4} - 3)$. By induction hypothesis, we have $r(L_{n-2}-3) = \left[\frac{1}{2}(n-2)\right] = \left[\frac{1}{2}n\right] - 1$, and $r(L_{n-4} - 3) = \left[\frac{1}{2}(n-4)\right] = \left[\frac{1}{2}n\right] - 2$. Therefore $r(L_n - 3) = 2\left(\left[\frac{1}{2}n\right] - 1\right) - \left(\left[\frac{1}{2}n\right] - 2\right) = \left[\frac{1}{2}n\right]$.

COROLLARY 2. If $n \geq 1$, then $r(L_n^2 - 1) = n$.

PROOF. If $m \geq 1$, then via (5), we have $r(L_{2m}^2 - 1) = r(L_{4m} + 1) = r(L_{4m} + L_1)$. Now Theorem 11 implies $r(L_{2m}^2 - 1) = (2m)r(L_1) = (2m)1 = 2m$. again, via (5), we have $r(L_{2m-1}^2 - 1) = r(L_{4m-2} - 3)$. Now Corollary 1 implies $r(L_{2m-1}^2) = \left[\frac{1}{2}(4m-2)\right] = 2m - 1$.

REMARK. Corollaries 1 and 2 imply (independently) that the function $r(n)$ is a surjection from N to N.

THEOREM 12. Let $n \neq L_k$, $n \neq L_{2k+1} + 1$. Then

$$r_E(n) = \begin{cases} \frac{1}{4}(k_1 - k_2 + 1)r(n_1) & \text{if } k_1 - k_2 \equiv 3 \pmod 4 \\ \frac{1}{4}(k_1 - k_2 + 3)r(n_1) - r_E(n_1) & \text{if } k_1 - k_2 \equiv 1 \pmod 4 \\ \frac{1}{4}(k_1 - k_2 + 2)r(n_1) - r_0(n_2) & \text{if } k_1 - k_2 \equiv 2 \pmod 4 \\ \left(1 + \frac{1}{4}(k_1 - k_2)\right)r(n_1) - r_E(n_1) - r_E(n_2) & \text{if } k_1 - k_2 \equiv 0 \pmod 4 \end{cases}.$$

PROOF. By hypothesis and Theorem 3, any Type III Lucas representation of n must arise from a corresponding Type III Lucas representation of $n_1 = n - L_{k_1}$. Thus it suffices to count the Type I and II Lucas representations of n consisting of evenly many parts. A Type I Lucas representation of n with evenly many parts will occur whenever the initial and terminal segments agree in parity. Therefore the number of such representations is given by:

$$\begin{aligned} &r_E(L_{k_1-k_2-1})r_E(n_1) + r_0(L_{k_1-k_2-1})r_0(n_1) = \\ &\quad \left[\tfrac{1}{2}(k_1 - k_2 + 1)\right]r_E(n_1) + \left(\left[\tfrac{1}{2}(k_1 - k_2 + 1)\right] - \left[\tfrac{1}{4}(k_1 - k_2 + 1)\right]\right)(r(n_1) - r_E(n_1)) = \\ &\quad \left(\left[\tfrac{1}{2}(k_1 - k_2 + 1)\right] - \left[\tfrac{1}{4}(k_1 - k_2 + 1)\right]\right)r(n_1) + \left(2\left[\tfrac{1}{4}(k_1 - k_2 + 1)\right] - \left[\tfrac{1}{2}(k_1 - k_2 + 1)\right]\right)r_E(n_1). \end{aligned}$$

If $2 \nmid (k_1 - k_2)$, then no Type II Lucas representations of n can arise. In particular, if $k_1 - k_2 \equiv 3 \pmod 4$, by simplifying the last formula, we obtain $r_E(n) = \frac{1}{4}(k_1 - k_{72} + 1)r(n_1)$. Similarly, if $k_1 - k_2 \equiv 1 \pmod 4$, we obtain $r_E(n) = \frac{1}{4}(k_1 - k_2 + 3)r(n_1) - r_E(n_1)$. If $2 | (k_1 - k_2)$, we wish to count the number of Type II Lucas representations on n that have evenly many terms. Each such representation originates from a Lucas representation of n whose initial segment has $\frac{1}{2}(k_1 - k_2)$ terms, and whose terminal segment's number of terms therefore differs in parity from $\frac{1}{2}(k_1 - k_2)$. If $k_1 - k_2 \equiv 2 \pmod 4$, then the number of Type II Lucas representations of n is $\overline{r}_E(n_1) = r_E(n_1) - r_0(n_2)$, by Lemma 5. In this case, the number of Type I Lucas representations of n is $\frac{1}{4}(k_1 - k_2 + 2) - r_E(n_1)$. Thus we obtain: $r_E(n) = \frac{1}{4}(k_1 - k_2 + 2)r(n_1) - r_0(n_2)$. If $k_1 - k_2 \equiv 0 \pmod 4$, then the number of Type II Lucas representations of n is $\overline{r}_0(n_1) = r_0(n_1) - r_E(n_2)$, by Lemma 5. In this case, the number of Type I Lucas representations of n is $\frac{1}{4}(k_1 - k_2)r(n_1)$. Therefore we obtain:

$$r_E(n) = \frac{1}{4}(k_1 - k_2)r(n_1) + r_0(n_1) - r_E(n_2) = \left(1 + \frac{1}{4}(k_1 - k_2)\right)r(n_1) - r_E(n_1) - r_E(n_2).$$

THEOREM 13. Let $n \neq L_k$, $n \neq L_{2k+1} + 1$. Then

$$a(n) = \begin{cases} -a(n_1) - a(n_2) & \text{if } k_1 - k_2 \equiv 0 \pmod 4 \\ -a(n_1) & \text{if } k_1 - k_2 \equiv 1 \pmod 4 \\ a(n_2) & \text{if } k_1 - k_2 \equiv 2 \pmod 4 \\ 0 & \text{if } k_1 - k_2 \equiv 3 \pmod 4 \end{cases}.$$

PROOF. This follows from (1.6) and from Theorems 11 and 12.

LEMMA 6. If $n \geq 1$, then

$$a(L_n - 1) = \begin{cases} 1 & \text{if } n \equiv 1 \pmod 4 \\ -1 & \text{if } n \equiv 2 \pmod 4 \\ 0 & \text{if } n \equiv 0, 3 \pmod 4 \end{cases}$$

PROOF. (Induction on n) The conclusion holds by inspection if $1 \leq n \leq 4$ If $n \geq 5$, then $L_n - 1 = L_{n-1} + (L_{n-2} + 1) = L_{n-1} + L_{n-3} + (L_{n-4} - 1)$. Now Theorem 13 implies $a(L_n - 1) = a(L_{n-4} - 1)$, so the conclusion follows from the induction hypothesis.

LEMMA 7. If $n \geq 1$, then $a(2L_n - 1) = (-1)^n$.

PROOF. The conclusion holds by inspection if $1 \leq n \leq 4$. If $n \geq 5$, then $2L_n - 1 = L_{n+1} + (L_{n-2} - 1) = L_{n+1} + L_{n-3} + (L_{n-4} - 1)$. Now Theorem 13 implies $a(2L_n - 1) = -a(L_{n-2} - 1) - a(L_{n-4} - 1)$. The conclusion now follows from Lemma 6.

LEMMA 8. If $j \leq n \leq L_{k-3} - 1$, then $a(2L_k + n) = 0$.

PROOF. $a(2L_k + n) = a(L_{k+1} + L_{k-2} + n)$ by (1), so the conclusion follows from the hypothesis and Theorem 13.

LEMMA 9. If $0 \leq n \leq L_{k-2} - 1$, then $a(5F_k + n) = a(n)$.

PROOF. By (9), we have $a(5F_k + n) = a(L_{k+1} + L_{k-1} + n)$. The conclusion now follows from the hypothesis and Theorem 13.

THEOREM 14. If n belongs to I_k, where $k \geq 2$, and $m = L_{k+3} - 1 - n$, then (i) $r(m) = r(n)$ and (ii) $a(m) = (-1)^k a(n)$.

PROOF. It is easily seen that m belongs to I_k iff n does. Now (7) implies there is a bijection between the partitions of m, n respectively into distinct Lucas parts. Thus $r(m) = r(n)$. Furthermore, since the left side of (7) has $k + 2$ terms, it follows that under this bijection, corresponding partitions of m and n will have numbers of parts that agree or disagree in parity accordingly as k is even or odd. Therefore $a(m) = (-1)^k a(n)$.

COROLLARY 3. If $k \geq 1$, then $a(L_{4k-2} - 2) = -1$; $a(L_{4k} - 2) = 2$.

PROOF. This follows from Theorems 10 and 14.

THEOREM 15. If $k \geq 3$, then $E(L_{k+2} - 1) = E(L_{k+1} - 1) + 2E(L_{k-2} - 1) + 2$.

PROOF. If $1 \leq i \leq 3 \leq k$, let $x_{k,i} = \Sigma\{|a(n)| : n \in I_{k,i}\}$. Thus $x_{k,1} + x_{k,2} + x_{k,3} = E(L_{k+2} - 1) - E(L_{k+1} - 1)$. Now $x_{k,3} = \Sigma\{|a(n)| : 5F_k \leq n \leq L_{k+2} - 1\} = E(L_{k+2} - 1) - E(5F_k - 1)$. But Lemma 9 implies $a(5F_k) = a(0) = 1$, so $E(5F_k) = 1 + E(5F_k - 1)$. Thus $x_{k,3} = E(L_{k+2} - 1) - E(5F_k) + 1$. But (10) implies $L_{k+2} = 5F_k + L_{k-2}$, so $x_{k,3} = E(5F_k + L_{k-2} - 1) - E(5F_k) + 1$. Now Lemma 9 implies $x_{k,3} = E(L_{k-2} - 1) + 1$. Also, $x_{k,2} = \Sigma\{|a(n)| : 2L_k \leq n \leq 5F_k - 1\} = 0$ by Lemma 8. Now $x_{k,1} = \Sigma\{|a(n)| : L_{k+1} \leq n \leq 2L_k - 1\}$. Theorem 14 implies that $x_{k,1} = \Sigma\{|a(n)| : 5F_k \leq n \leq L_{k+2} - 1\} = x_{k,3}$. Thus we have: $E(L_{k+2} - 1) - E(L_{k+1} - 1) = 2(1 + E(L_{k-2} - 1))$, from which the conclusion follows.

THEOREM 16. If $k \geq 2$ and if $L_{k+1} \leq n \leq L_{k+2} - 1$, then

$$E(n) = \begin{cases} E(2L_k) - E(2L_k - 2 - n) - 1 & \text{if } L_{k+1} \leq n \leq 2L_k - 2 \\ E(2L_k) & \text{if } 2L_k - 1 \leq n \leq 5F_k - 1 \\ E(2L_k) + E(n - 5F_k) + 1 & \text{if } 5F_k \leq n \leq L_{k+2} - 1 \end{cases}.$$

PROOF. If $2L_k - 1 \leq n \leq 5F_k - 1$, then Lemma 8 implies $E(n) = E(2L_k)$. If $5F_k \leq n \leq L_{k+2} - 1$, then $E(n) - E(5F_k) = \sum_{j=1}^{n-5F_k} |a(5F_k + j)| = \sum_{j=1}^{n-5F_k} |a(j)| = E(n - 5F_k)$ by Lemma 9. Also, Lemmas 8 and 9 imply $E(5F_k) = 1 + E(2L_k)$, so $E(n) = 1 + E(2L_k) + E(n - 5F_k)$. Finally, if $L_{k+1} \leq n \leq 2L_k - 2$, let $m = 2L_k - n$, so that $2 \leq m \leq L_{k-2}$. We must show that $E(2L_k - m) = E(2L_k) - E(m - 2) - 1$. Now Lemmas 8 and 9 imply $a(2L_k) = 0$ and $|a(2L_k - 1)| = 1$. Therefore $E(2L_k) = E(2L_k - 1) = 1 + E(2L_k - 2)$. Thus it suffices to show that

$E(2L_k - m) = E(2L_k-2)-E(m-2)$ when $2 \le m \le L_{k-2}$. This is trivially true when $m = 2$. If $3 \le m \le L_{k-2}$, then $L_{k+1} + 3 \le 2L_k - m \le 2L_k - 3$, so that by (1), (9), Lemma 9 and Theorem 14, we have $|a(2L_k-m)| = |a(L_{k+3}-1-2L_k+m)| = |a(5F_k+m-1)| = |a(m-1)|$. Therefore $E(2L_k-2) - E(2L_k-m) = \sum_{j=2}^{m-1} |a(2L_k-j)| = \sum_{j=2}^{m-1} |a(j-1)| = \sum_{i=1}^{m-2} |a(i)| = E(m-2)$, so we are done.

THEOREM 17. $\lim_{n\to\infty} \frac{E(n)}{n} = 0$.

PROOF. If $k \ge 2$, let $t_k = \max\{E(n)/n : n \in I_k\}$. It suffices to show that $\lim_{k\to\infty} t_k = 0$. If $n \in I_k$, then by Theorem 16, we have: $E(n) \le E(2L_k) + E(n - 5F_k) + 1$. Since $E(n)$ is non-decreasing and $n - 5F_k \le L_{k-2}$, it follows that $E(n) \le E(2L_k) + E(L_{k-2}) + 1$. By Theorem 16, we have $E(L_{k+1}) = E(2L_k) - E(L_{k-2}-2) - 1$, so we obtain $E(n) \le E(L_{k+1}) + E(L_{k-2}) + E(L_{k-2}-2) + 2$, hence $E(n) \le E(L_{k+1}) + 2E(L_{k-2})$. Since $n \le L_{k+1}$, we get $\frac{E(n)}{n} \le \frac{E(L_{k+1})}{L_{k+1}} + 2\frac{E(L_{k-2})}{L_{k+1}}$, so that $t_k \le E(L_{k+1})/L_{k+1} + 2E(L_{k-2})/L_{k+1}$. Since $E(n)$ is non-decreasing and L_k tends to infinity with k, it suffices to show that $\lim_{k\to\infty} E(L_k)/L_k = 0$. In fact, since $E(L_k) \le 1 + E(L_k - 1)$, it suffices to show that $\lim_{k\to\infty} E(L_k - 1)/L_k = 0$. If $k \ge 1$, let $c_k = E(L_k - 1)$. Thus $c_1 = 0$, $c_2 = c_3 = 2$, $c_4 = 4$. By Theorem 15, we have: $c_{k+2} = c_{k+1} + 2c_{k-2} + 2$. Let the $\{c_k\}$ have the generating function: $F(z) = \sum_{k=1}^{\infty} c_k z^k$. Using the method of [2], p. 337-350, we obtain: $F(z) = (2z^2 - 2z^3 + 2z^4)/(1 - z^2)$ $(1 - 2z + 2z^2 - 2z^3)$. Therefore $c_k = b_1 t_1^k + b_2 t_2^k + b_3 t_3^k + b_4 + b_5(-1)^k$, where the b_i are constants, and the t_i are the roots of the equation: $x^3 - 2x^2 + 2x - 2 = 0$. Using Cardan's formula, if $u = \frac{1}{3}(17 + 3\sqrt{33})^{1/3}$, $v = \frac{1}{3}(17 - 3\sqrt{33})^{1/3}$, then $t_1 = \frac{2}{3} + u + v \doteq 1.544$, $t_2 = \frac{2}{3} - \frac{1}{2}(u+v) + \frac{i\sqrt{3}}{2}(u - v)$, $t_3 = \bar{t}_2$. Thus $|t_2| = |t_3| \doteq 1.138$, $b_2 = b_3$, and $|b_2 t_2^k + b_3 t_3^k| = |2b_2 Re(t_2^k)| \le 2|b_2||t_2^k|$. Now $0 < |c_k/L_k| \le (|b_1|1.544^k + 2|b_2|1.138^k + |b_4| + |b_5|)/L_k$. But (6) implies that the right side of the last inequality tends to 0 as k tends to infinity. Therefore $\lim_{k\to\infty} c_k/L_k = 0$, we are done.

THEOREM 18. $a(n)$ assumes each of the values $0, \pm 1, \pm 2$ infinitely often.

PROOF. Theorem 7 implies $a(n) = 0, -1$ infinitely often, while Theorem 10 implies $a(n) = 2$ infinitely often. By Theorems 13 and 7, $a(L_{4k+5} + L_{4k}) = -a(L_{4k}) = 1$. Therefore $a(n) = 1$ infinitely often. Finally, with $k \ge 2$, let $n = L_{4k} + 5 = L_{4k} + L_3 + L_1$. Now Theorem 13 implies $a(n) = -a(5) = -2$. Therefore $a(n) = -2$ infinitely often.

THEOREM 19. $|a(n)| \le 2$ for all n.

PROOF. If $|a(n)| \ge 3$ for some n, let n be the least such integer. By Theorems 7 and 10, $n \ne L_k$, $L_{2k+1} + 1$. Let n have the minimal Lucas representation:

$$n = n_0 = L_{k_1} + L_{k_2} + \text{etc.} + L_{k_r} \text{ where } r \ge 2; \text{ let } n_i = n_{i-1} - L_{k_i}$$

for $1 \le i \le r$, with $n_r = 0$. By hypothesis and Theorem 13, we must have: $a(n) = -(a(n_1) + a(n_2))$, with $k_1 - k_2 \equiv 0 \pmod 4$. By Theorem 13 implies

$$a(n_1) + a(n_2) = \begin{cases} -a(n_3) & \text{if } k_2 - k_3 \equiv 0 \pmod 4 \\ 0 & \text{if } k_2 - k_3 \equiv 1 \pmod 4 \\ a(n_2) + a(n_3) & \text{if } k_2 - k_3 \equiv 2 \pmod 4 \\ a(n_2) & \text{if } k_2 - k_3 \equiv 3 \pmod 4 \end{cases}.$$

Therefore $a(n) = -(a(n_2) + a(n_3))$, with $k_2 - k_3 \equiv 2 \pmod 4$. If we apply Theorem 13 repeatedly, we eventually get $a(n) = -(a(n_{r-1}) + a(n_r)) = -(a(L_{k_r}) + a(0))$. Now Theorem 7 implies $|a(n)| \le 1$, contrary to hypothesis.

ACKNOWLEDGMENT. Theorems 14 through 17 are Lucas analogues of results about Fibonacci partitions announced by Weinstein in [4]. For each integer n, such that $0 \le n \le 100$, Table 1 lists $r(n), r_E(n), a(n), E(n)$, and $V(n)$.

Table 1

n	$r(n)$	$r_E(n)$	$a(n)$	$E(n)$	$V(n)$	n	$r(n)$	$r_E(n)$	$a(n)$	$E(n)$	$V(n)$
0	1	1	1	0	1	51	6	3	0	30	6308
1	1	0	-1	1	1	52	6	2	-2	32	6877
2	1	0	-1	2	2	53	6	3	0	32	7491
3	2	1	0	2	3	54	8	4	0	32	8155
4	2	1	0	2	5	55	5	3	1	33	8862
5	2	2	2	4	6	56	5	3	1	34	9622
6	2	1	0	4	9	57	7	3	-1	35	10438
7	3	1	-1	5	12	58	6	3	0	35	11316
8	2	1	0	5	16	59	6	3	0	35	12247
9	2	1	0	5	20	60	6	3	0	35	13249
10	3	2	1	6	26	61	6	3	0	35	14319
11	3	1	-1	7	33	62	6	3	0	35	15464
12	3	1	-1	8	41	63	6	3	0	35	16678
13	3	2	1	9	50	64	6	3	0	35	17981
14	4	2	0	9	62	65	7	4	1	36	19369
15	3	2	1	10	75	66	5	2	-1	37	20845
16	3	1	-1	11	90	67	5	2	-1	38	22413
17	3	1	-1	12	107	68	8	4	0	38	24089
18	4	2	0	12	129	69	6	3	0	38	25868
19	3	2	1	13	151	70	6	4	2	40	27754
20	3	2	1	14	178	71	6	3	0	40	29759
21	5	2	-1	15	208	72	7	3	-1	41	31893
22	4	2	0	15	244	73	4	2	0	41	34149
23	4	2	0	15	281	74	4	2	0	41	36541
24	4	2	0	15	326	75	5	3	1	42	39078
25	5	3	1	16	375	76	7	3	-1	43	41771
26	3	1	-1	17	431	77	5	2	-1	44	44609
27	3	1	-1	18	491	78	5	3	1	45	47619
28	4	2	0	18	561	79	8	4	0	45	50802
29	4	2	0	18	638	80	7	4	1	46	54170
30	4	3	2	20	723	81	7	3	-1	47	57715
31	4	2	0	20	816	82	7	3	-1	48	61471
32	6	3	0	20	922	83	9	5	1	49	65434
33	5	2	-1	21	1037	84	6	3	0	49	69613
34	5	2	-1	22	1163	85	6	3	0	49	74013
35	5	3	1	23	1302	86	7	3	-1	50	78664
36	6	3	0	23	1458	87	8	4	0	50	83561
37	4	2	0	23	1624	88	8	5	2	52	88715
38	4	2	0	23	1808	89	8	4	0	52	94140
39	6	3	0	23	2009	90	6	3	0	52	99862
40	5	3	1	24	2231	91	7	3	-1	53	105871
41	5	2	-1	25	2467	92	7	3	-1	54	112190
42	5	2	-1	26	2729	93	7	4	1	55	118835
43	6	3	0	26	3012	94	8	4	0	55	125830
44	4	2	0	26	3321	95	6	3	0	55	133160
45	4	3	2	28	3651	96	6	3	0	55	140867
46	4	2	0	28	4014	97	10	5	0	55	148958
47	5	2	-1	29	4406	98	8	4	0	55	157456
48	4	2	0	29	4828	99	8	4	0	55	166353
49	4	2	0	29	5282	100	8	4	0	55	175400
50	7	4	1	30	5777						

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