Internat. J. Math. & Math. Sci.
VOL. 21 NO. 2 (1998) 341-346

ADAMS AND STEENROD OPERATORS IN DIHEDRAL HOMOLOGY

Y. GH. GOUDA
Department of Mathematics
Faculty of Science
South Valley University
Aswan, EGYPT

(Received January 30, 1996 and in revised form August 12, 1996)

ABSTRACT. In this article, we define the Adam's and Steenrod's operators in the dihedral homology

KEY WORDS AND PHRASES: Adam's and Steenrod's operators, dihedral homology.
1991 AMS SUBJECT CLASSIFICATION CODES: Primary 55N91, Secondary 55P91, 55Q91.

INTRODUCTION

The dihedral (co)homology of unital algebra with an involution, symmetry, bisymmetry and Weile has been studied by Looder [1], Krasauskas, Lapin and Solovev [2], Kolosov [3] and others 1987-89 In the present work we are concerned with Adam's and Steenrod's operators in the dihedral homology.

1. THE ADAM'S OPERATOR IN THE DIHEDRAL HOMOLOGY

We recall the Adam's operator ψ^k in the cyclic homology from [4] and [5]. Let A be a commutative, associative, and unital K-algebra with an involution $*$ ($* : A \longrightarrow A$ is an automorphism of degree zero, $*^2 = id$, $(a+b)^* = a^* + b^*$, $(ab)^* = b^* a^*, a, b \in A$), and K is a field with characteristic zero. Let $\lambda^k = \wedge^k(1_n - n)$ be the k^{th} exterior dimension representation of the Lie algebra $\mathfrak{gl}_n(k)$ and n is the direct sum of the one dimensional representation (n-argument). Following [6], the ring $R(\mathfrak{gl}_n(k))$ is isometric to the ring of polynomial $K[\lambda^1, ..., \lambda^n]$. Let $R(\mathfrak{gl}(k)) = \varprojlim R(\mathfrak{gl}_n(k))$. Consider, for an arbitrary representation ρ of an algebra $\mathfrak{gl}_n(k)$, the following sequence:

$$CC_\infty(A) \xrightarrow{S} \wedge^n(\mathfrak{gl}(k))_{\mathfrak{gl}(k)} \xrightarrow{\widehat{\rho}} \wedge^n(\mathfrak{gl}(k))_{\mathfrak{gl}(k)} \xrightarrow{\varphi}$$
$$\xrightarrow{\varphi} CC_n(M_\infty(A)) \xrightarrow{Tr} CC_\infty(A), \tag{1 1}$$

where $\wedge^\cdot(\mathfrak{gl}(k))_{\mathfrak{gl}(k)}$ is the coinvariant complex of Cherilley-Eilenberg Complex $\wedge(\mathfrak{gl}(k))$ (see [4]), $M_\infty(A) = \varinjlim M_n(A)$, $M_n(A)$ is the $(n \times n)$ matrix with coefficients in A. The composition maps in (1.1) are denoted by $\propto_n$ where $\propto = \varprojlim \propto_n$. The morphism S is given by:

$$S(a_1 \otimes a_2 \otimes ... \otimes a_n) = E_{12}a_1 \wedge E_{23}a_2 \wedge ... \wedge E_{n-1,n}a_{n-1} \wedge E_{n,1} \cdot a_n,$$

where E_{ij} is the matrix, whose only non zero elements are the identity element $1 \in k$. The map $\widehat{\rho}$ is given by:

$$\widehat{\rho}(X_1 a_1 \wedge ... \wedge X_n a_n) = \rho(x_1)a_1 \wedge ... \wedge \rho(x_n)a_n,\ x_i \in \mathfrak{gl}_n(k),$$
$$\varphi(Z_0 \wedge ... \wedge Z_n) = \sum_\sigma \operatorname{sgn}(\sigma)(-1)^n Z_0 \otimes Z_{\sigma(1)} \otimes ... \otimes Z_{\sigma(n)},\ Z_i \in \mathfrak{gl}_N(k),$$

$\rho : \mathfrak{gl}_n(k) \longrightarrow \mathfrak{gl}_N(k)$, and Tr is the trace map defined by:

$$Tr(X_1 a_1 \otimes ... \otimes X_n a_n) = tr(X_1 ... X_n) a_1 \otimes ... \otimes a_n.$$

We can easily check ([4]) that, $\propto(\rho + \tau) = \propto(\otimes)$, where ρ and τ are representations of $\mathfrak{gl}(k)$

From the above discussion we have the homomorphism $\alpha : R(\mathfrak{gl}(k)) \longrightarrow End(CC.(A))$. Clearly, for any $f \in K[\lambda^1, ..., \lambda^n, ...]$, the homomorphism $\alpha(f)$ coincides with the homomorphism α [5]. Suppose that Q_k, $k \geq 1$ is the Newton Polynomial, which is given by the symmetric function $\sum_{i=1}^{k} (u_i)^k$, such that $\sigma_r = \sum_{i_1<i_2<...<i_r}^{k} u_{i_1}...u_{i_r}$, $1 \leq r \leq k$. By acting with the morphism α on the Newton Polynomial, we get the Adams operators $\psi^k = \alpha(Q_k) = \alpha\big((-1)^k.k\,\lambda^k\big)$, since $(-1)^k.k\,\lambda^k$ is the linear part of K-Newton Polynomial. Consider the chain complex $(\mathcal{CH}.(A), b.')$ and the Connes-Tsygan bicomplex (see [1])

$$\begin{array}{ccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
b.\downarrow & & b'.\downarrow & & b.\downarrow & & b'.\downarrow & & b.\downarrow \\
\mathcal{CH}_2(A) & \xleftarrow{1-t.} & \mathcal{CH}_2(A) & \xleftarrow{N} & \mathcal{CH}_2(A) & \xleftarrow{1-t.} & \mathcal{CH}_2(A) & \xleftarrow{N} & \mathcal{CH}_2(A) \longleftarrow \dots \\
b.\downarrow & & b'.\downarrow & & b.\downarrow & & b'.\downarrow & & b.\downarrow \\
\mathcal{CH}_1(A) & \xleftarrow{1-t.} & \mathcal{CH}_1(A) & \xleftarrow{N} & \mathcal{CH}_1(A) & \xleftarrow{1-t.} & \mathcal{CH}_1(A) & \xleftarrow{N} & \mathcal{CH}_1(A) \longleftarrow \dots \\
b.\downarrow & & b'.\downarrow & & b.\downarrow & & b'.\downarrow & & b.\downarrow \\
\mathcal{CH}_0(A) & \xleftarrow{1-t.} & \mathcal{CH}_0(A) & \xleftarrow{N} & \mathcal{CH}_0(A) & \xleftarrow{1-t.} & \mathcal{CH}_0(A) & \xleftarrow{N} & \mathcal{CH}_0(A) \longleftarrow \dots ,
\end{array} \tag{1.1}$$

then, the subcomplex $(ker(1-t.), b.') \subset (\mathcal{CH}.(A), b.')$ has the same homology as the complex $(CC.(A), b.)$, that is,

$$\begin{aligned}
\mathcal{H}.(CC.(A)) &= \mathcal{H}.((\mathcal{CH}.(A), b.)/Im(1-t.)) = \mathcal{H}.((\mathcal{CH}.(A), b.)/Ker\,N.) \\
&= \mathcal{H}.(Im\,N, b.') = \mathcal{H}.(Ker(1-t.), b.'),
\end{aligned}$$

where $\mathcal{CH}_n(A) = A^{\otimes n+1} = A \otimes ... \otimes A (n+1 \text{ times})$, b_n, $b'_n : \mathcal{CH}_n(A) \longrightarrow \mathcal{CH}_{n-1}(A)$, such that $b'_n(a_0 \otimes ... \otimes a_n) = \sum_{i=0}^{n-1} (-1)^i (a_0 \otimes ... \otimes a_i a_{i+1} \otimes ... \otimes a_n)$, $b_n(a_0 \otimes ... \otimes a_n) = b'_n + (-1)^n (a_n a \otimes ... \otimes a_{n-1})$, $t_n : \mathcal{CH}_n(A) \longrightarrow \mathcal{CH}_n(A)$, such that $t_n(a_0 \otimes ... \otimes a_n) = (-1)^n (a_n \otimes a_0 \otimes ... \otimes a_{n-1})$ and $N_n = 1 + t_n^1 + ... + t_n^n$. Therefore, the complex $(Ker(1-t.), b.')$ is isomorphic to the complex $(CC.(A), b.)$. The isomorphism between them is given by the operator $N. : CC.(A) \longrightarrow (ker(1-t.), b.')$. Consequently, the action of the group $\mathbb{Z}/2$ on the complex $CC.(A)$, by means of the operator ${}^{\epsilon}r$, is equal to the action of $\mathbb{Z}/2$ on the complex $\big(\text{Ker}(1-t.), b'.\big)$, by means of the operator

$${}^{\epsilon}h : a_0 \otimes a_1 \otimes ... \otimes a_n \longrightarrow (-1)^{\frac{n(n+1)}{2}} \epsilon\, a_n^* \otimes a_{n-1}^* \otimes ... \otimes a_0^*,$$

where a^* is the image of element $a \in A$ under involution $* : A \longrightarrow A$, $\epsilon = \pm 1$. Since ${}^{\epsilon}h.t. = t.^{-1}\,{}^{\epsilon}h.$. Hence, $N.({}^{\epsilon}h.) = ({}^{\epsilon}h.)N.$. On the other hand, since ${}^{\epsilon}r. = t.{}^{\epsilon}h.$, then ${}^{\epsilon}h.N. = N.{}^{\epsilon}h. = (N.t.)\,{}^{\epsilon}h. = N.(t.{}^{\epsilon}h.) = N.{}^{\epsilon}r.$. So, the dihedral homology of A is given by the formula

$${}^{\epsilon}\mathcal{HD}.(A) = \mathcal{H}.(ker(1-t.)/(Im(1-{}^{\epsilon}h.) \cap ker(1-t.))).$$

Assume that the complex $CC.(A)$ is a subcomplex of $(\mathcal{CH}.(A), b.')$, then the direct calculation of homomorphism $\alpha\big((-1)^k k\,\lambda^k\big)$ gives the Adam's operator Ψ^k in additive algebraic K-theory (see [4]), that is, $\Psi(a_0 \otimes ... \otimes a_n) = \sum_I sgn(\sigma_I) a_{\sigma_I(0)} \otimes ... \otimes a_{\sigma_I(n)}$, where I is the division of the set $\{0, 1, 2, ..., n\}$ into non-empty intersected subsets, that is, $I = I_0 \cup ... \cup I_{k-1}$, and $\sigma_I \in \sum_{n+1}$ is the permutation of the set $\{0, 1, ..., n\}$, such that:

(i) If $i_1 \in I_{p_1}$, $i_2 \in I_{p_2}$, $P_1 < P_2$, then $\sigma_I(i_1) > \sigma_I(i_2)$,

(ii) For any P, $I_P = \{i_0, ..., i_q\}$, $(i_1 < i_2 < ... < i_q)$.

The permutation σ_I satisfies the following condition:

$$\sigma_I(i_q) = \sigma_I(i_{q-1}) + 1 = ... = \sigma_I(i_0) + q.$$

LEMMA 1.1. The following diagram is commutative:

$$\begin{array}{ccc} CC.(A) & \xrightarrow{\psi^k} & CC.(A) \\ {\scriptstyle \epsilon_r}\downarrow & & \downarrow{\scriptstyle \epsilon_r} \\ CC.(A) & \xrightarrow{\psi^k} & CC.(A) \end{array} \tag{1.2}$$

PROOF. Assume that the complex $CC.(A)$ is a subcomplex of the complex $(CH.(A), b.')$ and the element $a_0 \otimes ... \otimes a_n \in ker(1-t_n)$, then

$$\begin{aligned} \epsilon_h \psi^k(a_0 \otimes a_1 \otimes ... \otimes a_n) &= {}^\epsilon h \sum_I sgn(\sigma_I) a_{\sigma_I(0)} \otimes ... \otimes a_{\sigma_I(n)} \\ &= (-1)^{\frac{n(n+1)}{2}} \epsilon \sum_I sgn(\sigma_I) a^*_{\sigma_I(n)} \otimes ... \otimes a^*_{\sigma_I(0)}. \end{aligned} \tag{1.2}$$

On the other hand

$$\begin{aligned} \psi^k({}^\epsilon h)(a_0 \otimes ... \otimes a_n) &= (-1)^{\frac{n(n+1)}{2}} \epsilon \psi^k(a^*_n \otimes ... \otimes a^*_0) \\ &= (-1)^{\frac{n(n+1)}{2}} \epsilon \sum_J sgn(g_J) a^*_{g_J(n)} \otimes ... \otimes a^*_{g_J(0)}, \end{aligned} \tag{1.3}$$

where g_J is the permutation of the ordered set $\{n, n-1, ..., 0\}$ satisfies the conditions (i), (ii) and J is the division of the ordered set $\{n, n-1, ..., 0\}$. Note that, in general, the permutation g_J of the ordered set $\{0, 1, ..., n\}$, satisfies the following conditions:

i)' If $i_1 \in J_{p_1}$, $i_2 \in J_{p_2}$, $p_1 < p_2$, then $g_J(i_1) > g_J(i_2)$,

ii)" For any p, $g_J = \{i_1, ..., i_0\}$, $i_q > ... > i_0$, we have

$$g_J(i_0) = g_J(i_1) - 1 = ... = g_J(i_q) - q.$$

Note that the decreasing (by one) of the elements in the set $\{0, 1, ..., n\}$ met the increasing of elements (also by one) in the set $\{n, n-1, ..., 0\}$. Suppose that the arguments of the summation in (1 2) correspond to the permutation σ_I. The permutation g_J of the set $\{n, n-1, ..., 0\}$, where $g_J(i) = \sigma_I(i)$ will correspond to the division $J = I^*_{k-1} \cup ... \cup I^*_0$, where

$$I^*_i = \{P^i_{q_i}, ..., P^i_0\} \, (I = \{P^i_0, ..., P^i_{q_i}\}, P^i_0 < ... < P^i_{q_i}).$$

We can easily check, for any P and $I^*_p = \{i^p_{q_p}, ..., i^p_0\}$, $i^p_{q_p} < ... < i^p_0$, that $g_J(i^p_0) = g_J(i^p_1) - 1 = ... = g_J(i^p_{q_p}) - q_p$. If $i_1 \in I^*_{p_1}$, $i_2 \in I^*_{p_2}$, $p_1 < p_2$, then $g_J(i_1) > g_J(i_2)$. From the definition of σ_I and g_J we have ${}^\epsilon h\, \psi^k = \psi^k({}^\epsilon h)$ in $ker((1-t), b')$ and, hence ${}^\epsilon r\, \psi^k = \psi^k({}^\epsilon r)$ in $(CC.(A), b.)$. Clearly the inverse of the isomorphism $(CC.(A)) \longrightarrow ker(1-t.)$ is $\frac{1}{n} id : (ker(1-t.), b.') \longrightarrow (CC.(A)b)$. The operator ψ^k in $CC.(A)$ is given by $\frac{1}{n}\psi^k N$, where ψ^k is an operator in $(ker(1-t), b.')$. Since the operator ψ^k, on $CC.(A)$ commutes with the operator ${}^\epsilon r$, then we have the Adam's operator ${}^\epsilon\psi^k$ in the dihedral homology. Following [6] the multiplication in the cyclic homology of the algebra A is given as follows

$$\cup : \mathcal{H}C_p(A) \otimes \mathcal{H}C_q(A) \longrightarrow \mathcal{H}C_{p+q+1}(A),$$

such that

$$\cup : TotB(A) \otimes TotB(A) \longrightarrow TotB(A),$$

$$xuy = \begin{bmatrix} \rightarrow (x)T(\beta y), & r = 0 \\ \rightarrow 0 & , r \neq 0 \end{bmatrix} \in B(A)_{\ell+r, m+s+1}, x \in B(A)_{\ell,m} = A \otimes \overline{A}^{\otimes(m-\ell)},$$

$y \in B(A)_{r,s} = A \otimes \overline{A}^{\otimes(s-r)}$, where T is a product map [7], $Tot\,B(A)$ is the total complex of the bicomplex $\mathcal{B}(A)$, β is the Connes's operator. The group $\mathbb{Z}/2$ acts on the column of the bicomplex $\mathcal{B}(A)$ with the numbers $2\ell\,(n > 0)$ by means of the operator ${}^\epsilon r$, on the column with the numbers $(2\ell + 1)$ by means of the operator $(-1)^{\epsilon}r$, and on the complex $Tot^{\epsilon}B(A)\otimes Tot^{\delta}B(A)$ by means of ${}^{\epsilon}\hat{r}\otimes{}^{\delta}\hat{r}$, where ${}^{\epsilon}\hat{r}$ is the action of $\mathbb{Z}/2$ on $Tot^{\epsilon}B(A)$ induced by the action $\mathbb{Z}/2$ on ${}^{\epsilon}\mathcal{B}(A)$. Since the action of the group $\mathbb{Z}/2$ on the complex $Tot^{\epsilon}B(A)\otimes Tot^{\delta}B(A)$ commutes with the multiplication in the cyclic homology, then

$${}^{\epsilon}\hat{r}\otimes{}^{\delta}\hat{r}(a\otimes b) = {}^{E}\hat{r}(a)\otimes{}^{\delta}\hat{r}(b)\xrightarrow{\ \cup\ }{}^{\epsilon}\hat{r}(a)T\beta\big({}^{\delta}\hat{r}(b)\big),$$

$a\in Tot^{\epsilon}B(A)$, $b\in Tot^{\delta}B(A)$. On the other hand

$$\big(-\big({}^{\epsilon}\hat{r}(a)T\beta({}^{\delta}\hat{r}(b))\big)\big) = {}^{\epsilon}\hat{r}(a)T\beta\big(-{}^{\delta}\hat{r}(b)\big) = -{}^{\epsilon}\hat{r}(a)T\big({}^{\delta}\hat{r}(\beta(b)\big) = {}^{(\epsilon\delta)}\hat{r}(a\cup b).$$

Therefore ${}^{\epsilon}r(a)\cup{}^{\delta}r(b) = {}^{-(\epsilon\delta)}r(a\cup b)$. From the above we have the multiplication in the dihedral homology

$$\cup : {}^{\epsilon}\mathcal{HD}_p(A)\otimes{}^{\delta}\mathcal{HD}_q(A)\longrightarrow{}^{-(\epsilon\delta)}\mathcal{HD}_{p+q+1}(A).$$

It is well known that (see [1], [2]), the dihedral homology can be considered as the hyperhomology of the group $\mathbb{Z}/2$ with the coefficient in $Tot^{\epsilon}B(A)$, then

$$\begin{aligned}\mathbb{H}.(\mathbb{Z}/2,Tot^{\epsilon}B(A))\otimes\mathbb{H}.\big(\mathbb{Z}/2,Tot^{\delta}B(A)\big)&\longrightarrow\mathbb{H}.\big(\mathbb{Z}/2,Tot^{\epsilon}B(A)\otimes Tot^{\delta}B(A)\big)\\&\longrightarrow\mathbb{H}.\big(\mathbb{Z}/2,Tot^{-(\epsilon\delta)}B(A)\big).\end{aligned}$$

Consider the Adam's operator properties in the cyclic homology [4]. Since the Adam's operator ψ^k commutes with the action of the group $\mathbb{Z}/2$ and the multiplication $\cup$ in the cyclic homology anti-commutes with the action of group $\mathbb{Z}/2$, we get the following theorem.

THEOREM 1.2. Assume that A is a commutative K-algebra, where K is a field of characteristic zero. The Adam's operator ψ^k has the following properties:

1) ${}^{\epsilon}\psi^k\circ{}^{\epsilon}\psi^k = {}^{\epsilon}\psi^{k\ell}$,
2) ${}^{\epsilon}\psi^k(\alpha)\cup{}^{\delta}\psi^k(\beta) = {}^{-(\epsilon\delta)}\psi^k(\alpha\cup\beta)$, where $\alpha\in\mathcal{HD}.(A)$, $b\in\mathcal{HD}.(A)$.

2. THE STEENROD'S OPERATOR IN THE DIHEDRAL HOMOLOGY

In this part we define the Steenrod's operator in the dihedral homology. Let A be a commutative K-Hopf algebra, where K is a field with characteristic (not essential) zero. Let Ξ be the dihedral category and $K[\Xi]$ be an algebra associated with $[\Xi]$ over K (see [1], [2]). We can define on the $K[\Xi]$-module ${}^{\epsilon}A^D$, the structure of a co-commutative $K[\Xi]$-co-algebra by the formula

$${}^{\epsilon}A^D\xrightarrow{\ \nabla\ }{}^{\epsilon}(A\otimes A)\xrightarrow{\ f\ }{}^{\epsilon}A^D\otimes{}^{\epsilon}A^D,$$

where ∇ is the $K[\Xi]$ homomorphism, and f is given by

$$f((a_0\otimes b_0)\otimes(a_1\otimes b_1)\otimes\ldots\otimes(a_n\otimes b_n)) = (a_0\otimes a_1\otimes\ldots\otimes a_n)\otimes(b_0\otimes b_1\otimes\ldots\otimes b_n).$$

Suppose that $f\circ\nabla = {}^{\epsilon}\nabla^D$ gives the co-commutative co-multiplication in ${}^{\epsilon}A^D$. We show that ${}^{\epsilon}\nabla^D$ is a $K[\Xi]$-homomorphism Define on the algebra $K[\Xi]$ the co-multiplication

$$K[\Xi]\longrightarrow K[\Xi]\underset{k}{\otimes}K[\Xi];\quad\text{such that}\quad x\longrightarrow x\otimes x,\quad x\in K[\Xi].$$

Since ${}^{\epsilon}A^D\underset{k}{\otimes}{}^{\epsilon}A^D$ is $K[\Xi]\underset{k}{\otimes}K[\Xi]$ module, then by using the multiplication on ${}^{\epsilon}A^D\underset{k}{\otimes}{}^{\epsilon}A^D$, one can define the $K[\Xi]$-module structure and the $K[\Xi]$-module homomorphism f, since

$$\begin{aligned}f(x((a_0\otimes b_0)\otimes(a_1\otimes b_1)\otimes\ldots\otimes(a_n\otimes b_n))) &= x(a_0\otimes a_1\otimes\ldots\otimes a_n)\otimes x(b_0\otimes b_1\otimes\ldots\otimes b_n)\\&= x((a_0\otimes a_1\otimes\ldots\otimes a_n)\otimes(b_0\otimes b_1\otimes\ldots\otimes b_n))\\&= xf((a_0\otimes b_0)\otimes(a_1\otimes b_1)\otimes\ldots\otimes(a_n\otimes b_n)),\end{aligned}$$

$x\in K[\Xi]$. Hence the morphism ${}^{\epsilon}\nabla^D$ is a $K[\Xi]$-module homorphism.

The dihedral cohomology $Ext^n_{K[\Xi]}\big({}^{\epsilon}A^D;(K^D)^*\big)$ can be calculated by using the normalized bar construction $\beta(\mathfrak{L})$ (see [6]). Assume that $\mathfrak{L}$ and $\mathfrak{F}$ be the triples $({}^{\epsilon}A^D,K[\Xi],K^D)$, $(K[\Xi],K[\Xi],K^D)$, and $JK[\Xi]$ be the cokernel identity: $k \longrightarrow K[\Xi]$. The normalized bar construction $\beta(\mathfrak{L})$ is defined to be a k module $\beta(\mathfrak{L}) = {}^{\epsilon}A^D \otimes_{K[\Xi]} T(JK[\Theta]) \otimes_{K[\Xi]} K^D$, where $T(JK[\Xi])$ is the tensor algebra of $JK[\Xi]$. Clearly the K module $\beta(\mathfrak{L})$ is graded. The elements of the K-module $\beta(\mathfrak{L})$ can be written as follows: $a[g_1, g_2, ..., g_s]k \in \beta(\mathfrak{L})_s$, $a \in {}^{\epsilon}A$, $g_i \in k[\Xi]$ and $k \in K^D$. The differential $d : \beta(\mathfrak{L})_s \longrightarrow \beta(\mathfrak{L})_{s-1}$ and the argument $f : \beta(\mathfrak{L}) \longrightarrow {}^{\epsilon}A^D \otimes_{K[\Xi]} K^D$ can be written as follows

$$d[a[g_1|g_2\cdots|g_s]k] = ag_1[g_2|g_3|\cdots|g_s]k + \sum_{i=1}^{s-1}(-1)^i a[g_1|\cdots|g_{i-1}|g_ig_{i+1}|g_{i+2}|\cdots|g_s]k$$
$$+(-1)^s a[g_1|\cdots|g_{s-1}g_s]k, \quad \text{and} \quad f[g_1|\cdots|g_s]k = 0, \quad f(a[\,]k) = 0.$$

We can define also, for $\mathfrak{F}$,the maps d and f in the same manner. Note that for $\mathfrak{L}$, the differential d is a left $K[\Xi]$-module homomorphism, and $dS + Sd = 1 - \sigma f$, where the homomorphism σ

$$\sigma : K^D \longrightarrow \beta(\mathfrak{F}), \quad \text{and} \quad S : \beta(\mathfrak{F})_s \longrightarrow \beta(\mathfrak{F})_{s+1}$$

is given by the formulas

$$\sigma(k) = [\,]k \otimes [\,], \; S(g[g_1|\cdots|g_s]k) = [g|g_1|\cdots|g_s]k.$$

Clearly, that the differential d in the complex $\beta(\mathfrak{l}) = {}^{\epsilon}A^D \otimes_{K[\Xi]} \beta(\mathfrak{F})$ is equal to $1 \otimes_{K[\Xi]} d$. From [6], we have the following

$$Hom_{K[\Xi]}(\beta(\mathfrak{F}); ({}^{\epsilon}A^D)^*) = (\beta(\mathfrak{F}))^* = Hom_{K[\Xi]}(\beta({}^{\epsilon}A^D), K[\Xi], K[\Xi], (k^D)^*).$$

Then

$${}_{\epsilon}\mathcal{HD}^n(A) = Ext^n_{K[\Xi]}({}^{\epsilon}A^D; (K^D)^*) = \mathcal{H}^n(\beta(\mathfrak{L})^*).$$

Suppose the triples $\mathfrak{L}(({}^{\epsilon}A^D), k[\Xi], K^D)$ and $\widehat{\mathfrak{F}} = \left(\left({}^{\epsilon}\widehat{A}^D\right), \widehat{k}[\Xi], \widehat{K}^D\right)$ and consider the product $\perp : (\beta(\mathfrak{L} \otimes \widehat{\mathfrak{F}}) \longrightarrow \beta(\mathfrak{L}) \otimes \beta(\widehat{\mathfrak{F}})$. Define on $\beta(\mathfrak{L})$ the structure of co-associative co-algebra by means of co-multiplication $\tilde{\nabla} = \perp \beta({}^{\epsilon}\nabla^D, \nabla_{k[\Xi]}, \nabla_k D) : \beta(\mathfrak{L}) \longrightarrow \beta(\mathfrak{L}) \otimes \beta(\mathfrak{L})$and on the complex $\beta(\mathfrak{L})^*$ the following multiplication

$$\beta(\mathfrak{L})^* \otimes \beta(\mathfrak{L})^* \longrightarrow (\beta(\mathfrak{L}) \otimes \beta(\mathfrak{L}))^* \xrightarrow{(\tilde{\nabla})^*} \beta(\mathfrak{L})^*.$$

The following lemma can easily be proved by using the ordinary techniques of homological algebra (see [8]).

LEMMA 2.1. Let μ be an arbitrary subgroup of the symmetry group Σ_r, W is the $K[\mu]$-free resolution $K[\mu]$-module K that $W_0 = K[\mu]$ with the $K[n]$ generator e_0 and the module $W \otimes \beta(\mathfrak{L})$ is a graded module, since: $[W \otimes \beta(\mathfrak{L})]_s = \sum_{i+j=s} W_i \otimes \beta_j(\mathfrak{L})$, then there exist graded $K[n]$ complexes, with the following conditions of the homorphism $\triangle : W \otimes \beta(\mathfrak{L}) \longrightarrow \beta(\mathfrak{L})^{\otimes r}$:

1) $\triangle(W \otimes b) = 0$, $b \in \beta(\mathfrak{L})_0$ and $w \in W_i$, $i > 0$.

2) $\triangle(e_0 \otimes b) = \tilde{\nabla}^{\otimes r}(b)$, if $b \in \beta(\mathfrak{L})$, $\tilde{\nabla}^{\otimes r} : \beta(\mathfrak{L}) \longrightarrow \beta(\mathfrak{L})^{\otimes r}$.

3) For $\beta(\mathfrak{L})$ the map $\triangle$ is a left $K[\Xi]$-module homorphism, where $K[\Xi]$ acts on $W \otimes \beta(\mathfrak{L})$ by the relation $K(w \otimes b) = w \otimes kb$.

4) $\triangle(w_i \otimes \beta(\mathfrak{L})_s) = 0$, when $i > (r-1)_s$. Furthermore, there exists a $k[\mu]$-homotopy between any two homomorphisms $\triangle$ with the same properties. Now, define the $K[\mu]$-homomorphism Θ as follows: Θ:$W \otimes (\beta(\mathfrak{L})^*)^{\otimes r} \longrightarrow \beta(\mathfrak{L})^*$, since $\Theta(w \otimes x)(m) = \mathcal{B}(x)\triangle(w \otimes m)$, $w \in W$, $x \in (\beta(\mathfrak{L})^*)^{\otimes r}$, and $m \in \beta(\mathfrak{L})$, $\mathcal{B} : (\beta(\mathfrak{L})^*)^{\otimes r} \longrightarrow (\beta(\mathfrak{L})^{\otimes r})^*$ is a trivial homomorphism. Now we shall define the operator in $H(\beta(\mathfrak{L})^*)$. In the above lemma, let $\mu = Z/p$, $K = Z/p$. Consider the standard $K[Z/p]$-free resolution W. In this case W_i, $i \geq 0$, is a free $K[Z/p]$-module with the generator e_i. By considering the graded $W_i = W^{-i}$, which is a free $K[Z/p]$-module with the generator e^{-i}, let $x \in H^q(\beta(\mathfrak{L})^*)$, and define the following homomorphism: $R_i : H^q(\beta(\mathfrak{L})^*) \longrightarrow H^{pq-i}(\beta(\mathfrak{L})^*)$, since $R_i(x) = \Theta^*(e^{-i} \otimes x^p)$, $i \geq 0$ Now we can define the Steenrod operator P^i, by using the operator R_i, as follows:

1. If $p = 2$ then, $p^s(x) = R_{q-s}(x) \in H^{q+s}(\beta(\mathfrak{L})^*)$, where $R_i = 0$ if $i < 0$;

2. If $P > 2$, then

$$p^s(x) = (-1)^s\gamma(-q)R_{(q-2s)(p-1)}(x) \in H^{q+2s(p-1)}(\beta(\mathfrak{L})^*),$$
$$\mathcal{B}P^s(x) = (-1)^s\gamma(-q)R_{(q-2s)(p-1)-1}(x) \in H^{q+2s(p-1)+1}(\beta(\mathfrak{L})^*),$$

where $R_i = 0$ if $i < 0$, and if $q = 2j - \ell$, where $\ell = 0$ or 1, then $\gamma(-q) = (-1)^j(mI)^{\mathcal{L}}$, $m = \frac{p-1}{2}$.

Now we prove the main second theorem in this work.

THEOREM 2.2. Let A be a commutative K-Hopf algebra, where $K = Z/p$, then on the dihedral cohomology group ${}_\epsilon\mathcal{HD}^{\cdot}(A)$, we can define the following homomorphisms (Steenrod map):

a) $P^i : {}_\epsilon\mathcal{HD}^s(A) \longrightarrow {}_\epsilon\mathcal{HD}^{s+i}(A)$, if $p = 2$,

b) $P^i : {}_\epsilon\mathcal{HD}^s(A) \longrightarrow {}_\epsilon\mathcal{HD}^{s+2i(p-1)}(A)$, and $\mathcal{B}P^i : {}_\epsilon\mathcal{HD}^s(A) \longrightarrow {}_\epsilon\mathcal{HD}^{s+i+2i(p-1)}(A)$, if $p > 2$.

The operators P^i, βP^i have the following properties:

1) $P^i|_{{}_\epsilon\mathcal{HD}^s(A)} = 0$, if $p = 2$, $i > s$,
   $P^i|_{{}_\epsilon\mathcal{HD}^s(A)} = 0$, if $p > 2$, $2i > s$,
   $\mathcal{B}P^i|_{{}_\epsilon\mathcal{HD}^s(A)} = 0$, if $p > 2$, $2i \geq s$

2) $P^i(x) = x^p$, if $p = 2$ and $i = s$, or $p > 2$ and $2i = s$

3) $P_j = \Sigma P^i \otimes P^{j-i}$ and $\mathcal{B}P^j = \Sigma \mathcal{B}P^i \otimes P^{j-i} + P^i \otimes \mathcal{B}P^{j-i}$

4) The operators P^i and $\mathcal{B}P^i$ satisfy the following Adam's relations:

i) if $p \geq 2$ and $a < pb$, then

$$\mathcal{B}^\gamma P^a P^b \sum_i (-1)^{a+i}(a - pi, (p-1)b - a + i - 1).\mathcal{B}^\gamma P^{a+b-i}P^i,$$

where $\gamma = 0$ or 1 for $p = 2$, $\gamma = 1$ for $p > 2$, and for any two integers i and j let

$$(i,j) = \begin{cases} \dfrac{(i+j)!}{i!j!}, & \text{if } i \geq 0, j \geq 0, \\ 0 & \text{if } i < 0, j < 0, \end{cases}$$

ii) if $p > 2$, $a \leq Pb$, and $\gamma = 0$ or 1, then

$$\mathcal{B}^\gamma P^a P^b = (1-\gamma)\sum_i (-1)^{a+i}(a - pi, (p-1)(b - a + i - 1).\mathcal{B}P^{a+b-i}p^i - \sum_i (-1)^{a+i}$$
$$.(a - pi - 1, (p-1)b - a + i)\mathcal{B}^\gamma P^{a+b-i}\mathcal{B}P^i.$$

Note that the operators $\mathcal{B}^0 P^s$ and $\mathcal{B}^1 P^s$ are P^s and $\mathcal{B}P^s$, respectively.

PROOF. Suppose the triple $C = (E, \mathcal{A}, F)$ where $\mathcal{A}$ is a co-commutative Hopf algebra over $K = Z/p$, E and F are respectively the right and left co-commutative $\mathcal{A}$-co-algebra. From the above discussion and considering the triple $\mathfrak{L} = ({}^\epsilon A^D, K[\Xi], k^D)$, then $K[\Xi]$ is a co-commutative Hopf algebra over $K = Z/P$, ${}^\epsilon A^D$ and K^D are the left and right co-commutative $K[\Xi]$-co-algebra and hence $\mathcal{H}^{\cdot}(\mathcal{B}(\mathfrak{L})^*) = {}_\epsilon\mathcal{HD}^{\cdot}(A)$.

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