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POSITIVE SOLUTIONS OF AN ASYMPTOTICALLY PLANAR SYSTEM OF ELLIPTIC BOUNDARY VALUE PROBLEMS

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ABSTRACT. We will prove an existence result of positive solutions for an asymptotically planar system of two elliptic equations. It will be used as main tools for a Maximum Principle and a result on Bifurcation Theory.

KEY WORDS AND PHRASES: Positive solution, asymptotically planar, maximum principle, bifurcation point.

AMS SUBJECT CLASSIFICATION CODES: 35J25, 47H15.

1. INTRODUCTION

In this paper we will prove the existence of positive solutions for the elliptic system

$$-\Delta U = A(x)U + F(x,U) \text{ in } \Omega, \; U = 0 \text{ on } \partial\Omega \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $A(x) = \begin{pmatrix} a(x) & b(x) \\ c(x) & d(x) \end{pmatrix}$ whose entries are continuous in $\overline{\Omega}$, $U = \begin{pmatrix} u \\ v \end{pmatrix}$, $-\Delta U = \begin{pmatrix} -\Delta & 0 \\ 0 & -\Delta \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -\Delta u \\ -\Delta v \end{pmatrix}$ and $F(x,U) = \begin{pmatrix} f(x,u,v) \\ g(x,u,v) \end{pmatrix}$ with $f, g : \overline{\Omega} \times (\mathbb{R}^+)^2 \to \mathbb{R}$ locally lipschitzian continuous satisfying.

$$f(x,0,0) > 0 \text{ or } g(x,0,0) > 0 \text{ for all } x \in \overline{\Omega} \tag{2}$$

and there is a positive constant C so that

$$0 \leq f(x,u,v), g(x,u,v) \leq C \text{ for all } (x,u,v) \in \overline{\Omega} \times (IR^+)^2. \tag{3}$$

Condition (3) says that the function

$$\widehat{F}(x,U) = \begin{pmatrix} a(x)u + b(x)v + f(x,u,v) \\ c(x)u + d(x)v + g(x,u,v) \end{pmatrix}$$

is of asymptotically planar type. Since we are concerned with the existence of positive solutions we will suppose through this work that system (1) is cooperative, i.e., $b(x)$ and $c(x)$ are both nonnegative for all $x \in \overline{\Omega}$. This cooperativeness is imposed in order we may use a *Maximum Principle (MP in short)* In particular we will deal with the one due to the author of this paper in collaboration with M A S Souto [1]. Using this (*MP*) and a result on *Bifurcation Theory* we prove the following:

THEOREM 1. *If* $a(x) < \lambda_1$, $d(x) < \lambda_1$ *and if either*

$$(i) \qquad |a|_\infty \leq |d|_\infty \;\; and \;\; 1 < \frac{2\lambda_1}{2|d|_\infty + |b|_\infty + |c|_\infty}$$

or

$$(ii) \qquad |d|_\infty \leq |a|_\infty \;\; and \;\; 1 < \frac{2\lambda_1}{2|a|_\infty + |b|_\infty + |c|_\infty}$$

then problem (1) *possesses a positive (classical) solution*

Here $|\cdot|_\infty$ denotes the usual sup norm, that is, $|u|_\infty = \sup_{x\in\overline{\Omega}} |u(x)|$ and λ_1 is the first eigenvalue of $-\Delta$ in Ω under *Dirichlet* boundary condition.

To tackle this theorem we proceed as follows: Since f and g are both defined only for $u, v \geq 0$ we ought to consider the extensions of f and g, respectiveley

$$f_1(x,u,v) = f(x,|u|,|v|) \;\; \text{and} \;\; g_1(x,u,v) = g(x,|u|,|v|)$$

now defined for all $(x,u,v) \in \overline{\Omega} \times \mathbb{R}^2$ We now carry on by setting

$$L = \begin{pmatrix} -\Delta - a(x) & 0 \\ 0 & -\Delta - d(x) \end{pmatrix}, \; \overline{F}(x,U) = \begin{pmatrix} f_1(x,u,v) \\ g_1(x,u,v) \end{pmatrix}, \; B(x) = \begin{pmatrix} 0 & b(x) \\ c(x) & 0 \end{pmatrix}.$$

Fixing these notations we are going to pay attention to the following nonlinear eigenvalue problem

$$LU = \lambda\big[B(x)U + \overline{F}(x,U)\big] \;\text{ in } \Omega, \; U = 0 \;\text{ on } \partial\Omega \tag{4}$$

where $\lambda > 0$ is a real parameter and it will be proved the existence of a continuum $\Sigma \subset \mathbb{R}^+ \times \big[C(\overline{\Omega})\big]^2$ of solutions (λ, U) of (4) that begins at $(0,0)$ and extends beyond the line $\{1\} \times \big[C(\overline{\Omega})\big]^2$ arising a solution of (1) which in view of the (*MP*), should be positive.

As we will show after proving Theorem 1 the motivation in studying problem (1) came of the scalar one

$$-\Delta u = f(x,u) \;\text{ in } \Omega, \; u = 0 \;\text{ on } \partial\Omega, \tag{5}$$

where f has a sublinear behavior.

2. PRELIMINARY RESULTS

In order to establish the (*MP*) we begin by fixing some notations. Let $\mathbf{X}$ be a Banach space ordered by the positive cone $\mathbf{K} \subset \mathbf{X}$ and $\widehat{L} : \mathbf{X} \to \mathbf{X}$ a linear operator. By a (*MP*) to problem

$$U = \widehat{L}U + F, U \in \mathbf{X}, \tag{6}$$

we mean the statement $F \geq 0$ (i.e. $F \in \mathbf{K}$) imply $U \geq 0$ whenever that U is a solution of (6).

PROPOSITION 2 (*Maximum Principle*). *Let $\widehat{L} : \mathbf{X} \to \mathbf{X}$ be a positive linear compact operator (positive means $\widehat{L}(\mathbf{K}) \subset K$). Then* (6) *satisfies the* (*MP*) *if the condition below holds true*

$$\big\{U \in \mathbf{X}, t \in [0,1], U = t\widehat{L}U\big\} \Rightarrow U = 0. \tag{7}$$

Now we shall focus our attention on the problem

$$LU = B(x)U + \overline{F}(x,U) \;\text{ in } \Omega, \; U = 0 \;\text{ on } \partial\Omega, \tag{8}$$

to prove the following:

THEOREM 3. *If $a(x) < \lambda_1$, $d(x) < \lambda_1$ and if either*

$$(i) \qquad |a|_\infty \leq |d|_\infty \;\; and \;\; 1 < \frac{2\lambda_1}{2|d|_\infty + |b|_\infty + |c|_\infty}$$

or

$$(ii) \qquad |d|_\infty \leq |a|_\infty \text{ and } 1 < \frac{2\lambda_1}{2|a|_\infty + |b|_\infty + |c|_\infty}$$

then every solution of (8) *is positive and so is a solution of* (1).

PROOF. We first observe that the extension $\overline{F}(x,U)$ is also nonnegative. Second we notice that the operator $L = \begin{pmatrix} -\Delta - a(x) & 0 \\ 0 & -\Delta - d(x) \end{pmatrix}$ has an inverse $L = \begin{pmatrix} (-\Delta - a(x))^{-1} & 0 \\ 0 & (-\Delta - d(x))^{-1} \end{pmatrix} : [C(\overline{\Omega})]^2 \to [C(\overline{\Omega})]^2$ which is compact and positive in view of $a(x), d(x) < \lambda_1$ in $\overline{\Omega}$ So we will analyze uniqueness for the problem

$$U = tL^{-1}B(x)U,\ U \in \left[C(\overline{\Omega})\right]^2,\ t \in [0,1]$$

that is equivalent to

$$\begin{cases} -\Delta u - a(x)u = tb(x)v & \text{in } \Omega \\ -\Delta v - d(x)v = tc(x)u & \text{in } \Omega \\ u = v = 0 & \text{on } \partial\Omega \end{cases} \tag{9}$$

By multiplying both sides of the first equation in (9) by u and both sides of the second one by v and integrating by parts we obtain

$$\int |\nabla u|^2 = \int a(x)u^2 + t\int b(x)uv$$

and

$$\int |\nabla v|^2 = \int c(x)uv + \int d(x)v^2$$

Since a, b, c and d belong to $C(\overline{\Omega})$ one gets, thanks to both Holder's and Pioncare's inequalities,

$$\int |\nabla u|^2 \leq \frac{|a|_\infty}{\lambda_1}\int |\nabla u|^2 + \frac{|b|_\infty}{2\lambda_1}\left(\int [|\nabla u|^2 + |\nabla u|^2]\right)$$

and

$$\int |\nabla v|^2 \leq \frac{|c|_\infty}{2\lambda_1}\left(\int [|\nabla u|^2 + |\nabla v|^2]\right) + \frac{|d|_\infty}{\lambda_1}\int |\nabla v|^2.$$

Summing up these two inequalities and assuming that $|a|_\infty \leq |d|_\infty$ one has

$$\int |\nabla u|^2 + \int |\nabla v|^2 \leq \frac{1}{\lambda_1}\left[\frac{2|d|_\infty + |b|_\infty + |c|_\infty}{2}\right]\left[\int (|\nabla u|^2 + |\nabla v|^2)\right].$$

Since $1 < \frac{2\lambda_1}{2|d|_\infty + |b|_\infty + |c|_\infty}$ we conclude that $U = 0$. We arrive at the same conclusion by assuming assumption (ii). Thus system (8) enjoys the (*MP*) and in view of $\overline{F}(x,U) \geq 0$ we have $U \geq 0$ and so it is a solution of (1). □

We now enunciate a proposition, due to Rabinowitz [5], which is another tool in proving Theorem 1.

PROPOSITION 4. *Let X be a Banach space and suppose that* $T : \mathbb{R}^+ \times \mathbf{X} \to \mathbf{X}$ *is a continuous map. Then the nonlinear eigenvalue problem* $u = T(\lambda, u)$ *possesses an unbounded continuum of solutions meeting* $(0,0) \in \mathbb{R} \times \mathbf{X}$, *if in addition, we suppose* $T(0,u) = 0$ *for all* $u \in \mathbf{X}$.

3. MAIN RESULTS AND REMARKS

We start this section proving Theorem 1.

PROOF OF THEOREM 1. Set $\mathbf{X} = \left[C(\overline{\Omega})\right]^2$ endowed with the usual norm $|U|_\infty = |u|_\infty + |v|_\infty$. Hence $\mathbf{X}$ is a Banach space and, as we said before, $L^{-1} : \mathbf{X} \to \mathbf{X}$ is linear, compact and positive. So problem (4) is equivalent to the following functional equation in $\mathbb{R}^+ \times \mathbf{X}$:

$$U = \lambda\left[L^{-1}A(x)U + L^{-1}\tilde{F}(x,U)\right],\ \lambda \geq 0,\ U \in \mathbf{X}, \tag{10}$$

where $\tilde{F}(x,U)$ is the Nemytskii operator associated with the function $\overline{F}$, i.e., for each $U \in \mathbf{X}$ one has

$$\tilde{F}(\,\cdot\,,U(\,\cdot\,))(x) = \overline{F}(x,U(x)).$$

Since $L^{-1}A$ and $L^{-1}\tilde{F}$ are compact operators we are able to conclude the existence of an unbounded continuum Σ of solutions of (10) beginning at $(0,0) \in \mathbb{R}^+ \times \mathbf{X}$. If $(\lambda,0) \in \Sigma$ then $\lambda = 0$ because $f(x,0,0) > 0$ or $g(x,0,0) > 0$ Plainly $(0,U) \in \Sigma$ implies $U = 0$. Thus Σ meets $\{0\} \times \left[C(\overline{\Omega})\right]^2$ and $\mathbb{R}^+ \times \{0\}$ only at $(0,0)$. Note that bootstraping these solutions, that at first sight belong only to $\left[C(\overline{\Omega})\right]^2$, we obtain classical solutions.

It is worthy to say that hitherto we cannot affirm that Σ contains positive solutions. In spite of this we can say that a piece (or perhaps pieces) of Σ contains only positive solutions. Indeed, if $\lambda \leq 1$ we may prove, reasoning as in the proof of Theorem 3, that every solution U of problem (10) is positive. It rests to show that in fact Σ reaches $\lambda = 1$

Since Σ is unbounded it may be unbounded with respect to λ, or with respect to U or with respect to both λ and U. If Σ is unbounded in λ then it crosses the line $\{1\} \times \mathbf{X}$ and so we find a solution U of the problem (4) and in view of assumptions (i) and (ii) of Theorem 3 is positive and so is a solution of (1). We now suppose that if $(\lambda,U) \in \Sigma$ then $\lambda \leq 1$. Hence there is a sequence $(\lambda_n, U_n) \in \Sigma$ with $\lambda_n \leq 1$ and $|U_n|_\infty \to \infty$ Thus

$$LU_n = \lambda\left[B(x)U_n + \overline{F}(x,U_n)\right] \text{ in } \Omega,\ U_n = 0 \text{ on } \partial\Omega.$$

Setting $W_n = \frac{U_n}{|U_n|_\infty}$ we obtain

$$LW_n = \lambda_n\left[B(x)W_n + \frac{\overline{F}(x,U_n)}{|U_n|_\infty}\right] \text{ in } \Omega,\ W_n = 0 \text{ on } \partial\Omega.$$

Passing to a subsequence if necessary we obtain $\lambda_n \to \lambda_0 \in [0,1]$, $W_n \to W$ in $\left[C(\overline{\Omega})\right]^2$ and $LW_n \to \lambda_0 B(x)W$ As $L : D(L) \to \left[C(\overline{\Omega})\right]^2$, where $D(L) = \left\{U \in \left[C(\overline{\Omega})\right]^2;\ LU \in \left[C(\overline{\Omega})\right]^2 \text{ and } U = 0 \text{ on } \partial\Omega\right\}$, is closed one has that $W \in D(L)$ and

$$LW = \lambda_0 A(x)W \text{ in } \Omega,\ W = 0 \text{ on } \partial\Omega.$$

Because $W_n \to W$ in $\left[C(\overline{\Omega})\right]^2$ and $|W_n|_\infty = 1$ then $|W|_\infty = 1$, i.e., W is a nontrivial solution of the above problem. But, in view of (*MP*) and $\lambda_0 \leq 1$, $W \equiv 0$ which is absurd. Thus Σ crosses $\{1\} \times \mathbf{X}$ and, by Theorem 3, such solution is positive and the proof of Theorem 1 is over. □

REMARK 1. The proof of Theorem 1 rests heavily on the existence of a (*MP*) like the one contained in [1] We must observe that this (*MP*) is valid for a more general elliptic operator. Indeed, if we consider uniformly elliptic operators in the divergence form

$$L_k = -D_j\left(a_{ij}^k D_i u\right) + D_j\left(a_j^k u\right),\ k = 1,2$$

(the symbols of summation are implicit in the expressions) where coefficients are regular enough, $a_{ij}^k = a_{ij}^k$ and setting $\lambda_1(L_k)$ as being the first eigenvalue of $(L_k, H_0^1(\Omega))$, the system

$$\mathcal{L}U = A(x)U + F(x) \text{ in } \Omega,\ U = 0 \text{ on } \partial\Omega$$

where $\mathcal{L} = \begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix}$, enjoys the (*MP*) if

$$(B(x)\xi,\xi) < \lambda_1(L_k)\left(\xi_1^2 + \xi_2^2\right)$$

for all $\xi = (\xi_1, \xi_2)$ and $x \in \overline{\Omega}$ Here $(.,.)$ is the usual inner product in $\mathbb{R}^2$ and $B(x) = \begin{pmatrix} 0 & b(x) \\ c(x) & 0 \end{pmatrix}$
Note that the above condition provides the uniqueness required by the *(MP)* in [1]. So Theorems 1 and 3 remain valid, with slight modifications, if system (1) is considered with $-\Delta$ substituted by the nonselfadjoint operators L_1 and L_2.

REMARK 2. At the outset of our motivations in studying problem (1) we had considered the following

$$Lu := -\sum_{i,j=1}^{N} a_{ij}(x)D_{ij}u + \sum_{i,j=1}^{N} a_i(x)D_i u = f(x,u) \text{ in } \Omega,\ u = 0 \text{ on } \partial\Omega \tag{11}$$

where L is a second order uniformly elliptic operator in Ω with real smooth coefficients satisfying $a_{ij} = a_{ji}$ in $\overline{\Omega}$, for all $1 \le i,\ j \le N$, and $f : \overline{\Omega} \times \mathbb{R}^+ \to \mathbb{R}^+$ is a sublinear nonlinearity. It is to say, setting

$$a_0(x) = \lim_{t\to 0^+} \frac{f(x,t)}{t},\ a_\infty(x) = \limsup_{t\to\infty} \frac{f(x,t)}{t} \tag{12}$$

one must have

$$\lambda_1(a_0) < 1 < \lambda_1(a_\infty) \tag{13}$$

where $\lambda_1(a_i)$, $i = 0, \infty$, is the first eigenvalue of the linear eigenvalue problem

$$Lu = \lambda a_i(x)u \text{ in } \Omega,\ u = 0 \text{ on } \partial\Omega.$$

Condition (13) says that we are working with a sublinear problem, i.e, in case, for instance, a_0 and a_∞ are constants the nonlinearity f begins, above the straight line $\lambda_1 t$ and at the end it remains below the same line.

In Brezis-Oswald [2] the authors consider $L = -\Delta$ and use Variational Methods by exploring the selfadjointness of $-\Delta$ and f is not necessarily a smooth function. In fact $a_0(x)$ and $a_\infty(x)$ may take values $+\infty$ and $-\infty$, respectively, so we address the reader to Section 3 of [2] for the precise meaning of (13).

In de Figueiredo [3] problem (11) is studied under condition (13) where L is a selfadjoint operator more general than $-\Delta$ but f is a C^α-function, $0 < \alpha < 1$, and $f(x,t) + Kt$ is nondecreasing in t for some $K \ge 0$. In this case the sub and supersolution method is used.

If L is not necessarily selfadjoint problem (11) was studied by Costa-Gonçalves [4] under condition (13), still using the sub and supersolution technique. In the works quoted above the authors always show existence of a positive solution as well as give sufficient condition for uniqueness.

This scalar problem arises a very natural question: How can we formulate a sublinear problem like before when we take a system into account?

We think that the best motivation towards a more general situation is to consider the biharmonic problem because it brings up for attention a very simple system and from it we would deal a more general sublinear problem. More precisely we first analyze the simplest biharmonic problem

$$\Delta^2 u = mu + g(u) \text{ in } \Omega,\ u = \Delta u = 0 \text{ on } \partial\Omega, \tag{14}$$

that is, the biharmonic equation under the so called Navier boundary conditions. Here m is a positive constant. Setting $v = -\Delta u$, $A = \begin{pmatrix} 0 & 1 \\ m & 0 \end{pmatrix}$, $G(U) = \begin{pmatrix} 0 \\ g(u) \end{pmatrix}$ we get the system

$$-\Delta U = AU + G(U) \text{ in } \Omega,\ U = 0 \text{ on } \partial\Omega. \tag{15}$$

Taking g a bounded function then $f(u) = mu + g(u)$ would be sublinear if it begins at $t = 0$ zero above $\lambda_1^2 t$ and remains below $\lambda_1^2 t$ for t large enough. Note that this is the counterpart of condition (13) when we are dealing with Δ^2 Observe that λ_1^2 is the first eigenvalue of Δ^2 in Ω under Navier boundary conditions and the situation described above occurs, for instance, if $g(0) > 0$ and $m < \lambda_1^2$.

REMARK 3. Now we are going to analyze the condition given in Theorem 1 for the system (15) In this case one has that $\frac{1+m}{2} < \lambda_1$ is a sufficient condition in order system (15) enjoys the (*MP*).

Next we will show that this condition leads to a sublinear problem related to (14). Let us suppose that $\frac{1+m}{2} < \lambda_1$

a) If $m \neq 1$ one has $(m-1)^2 > 0$ which implies $\frac{(1+m)^2}{4} > m$ and since $\lambda_1 > \frac{1+m}{2}$ we get $\lambda_1^2 > \frac{(1+m)^2}{4} > m$ and so we have a sublinear problem.

b) If $m = 1$ then $\lambda_1 > \frac{1+m}{2} = 1$ and hence $\lambda_1^2 > \lambda_1 > 1 = m$. In this case we still have a sublinear problem.

Therefore we believe that conditions (i) and (ii) are two kinds of sublinearity conditions when we deal with a system of two equations.

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