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SOME PROPERTIES OF PREREFLEXIVE SUBSPACES OF OPERATORS

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ABSTRACT. In the paper, we define a notion of prereflexivity for subspaces, give several equivalent conditions of this notion and prove that if $\mathcal{S} \subseteq L(H)$ is prereflexive, then every σ-weakly closed subspace of $\mathcal{S}$ is prereflexive if and only if $\mathcal{S}$ has the property WP(see definition 2.11). By our result, we construct a reflexive operator A such that $A \oplus 0$ is not prereflexive.

KEY WORDS AND PHRASES: Prereflexive subspace, reflexive operator.
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1. INTRODUCTION

The concept of reflexivity for algebras of operators was introduced by Halmos [1]. There is a natural generalization which was first formulated by Loginov and Sul'man [2]. Arveson [3] introduced the concept of prereflexivity for algebras but nothing corresponding to this has been studied in the generalized version. The concept of prereflexivity has already proved its worth. In this paper, we define a notion of prereflexivity for subspaces of operators which extends the concept of prereflexivity for algebras. In Section 2, we give several equivalent conditions of prereflexivity for subspaces, prove that if $\mathcal{S}$ is a σ-weakly closed subspace, then $\mathcal{S}$ has the property WP if and only if $\mathcal{S}$ is hereditarily prereflexive in the sense that every σ-weakly closed subspace of $\mathcal{S}$ is prereflexive. In Section 3, using the results in Section 2, we construct a prereflexive but not reflexive operator and prove that there exists a reflexive operator A such that $A \oplus 0$ is not prereflexive.

Throughout the paper, let H denote a complex separable Hilbert space and let $L(H)$ denote the algebra of all bounded linear operators on H. We write $T(H)$ for the ideal of trace class operators in $L(H)$, F for the finite rank operators in $T(H)$ and F_k for the subset of F consisting of operators of rank k or less. The trace norm is denoted by $\|\cdot\|_1$. If $\mathcal{S} \subseteq L(H)$, we denote $\mathcal{S}_\perp$ for its preannihilator, i.e., $\mathcal{S}_\perp \equiv \{t \in T(H) : tr(at) = 0 \text{ for all } a \in \mathcal{S}\}$; dually, the notation $\mathcal{M}^\perp$ indicates the annihilator of a subset $\mathcal{M}$ of $T(H)$, that is $\mathcal{M}^\perp \equiv \{a \in L(H) : tr(at) = 0 \text{ for all } a \in \mathcal{M}\}$. For any $A \in L(H)$, the symbol $A^{(n)}$ denotes $\underbrace{A \oplus \ldots \oplus A}_{n\ times}$. If $\mathcal{S}$ is a subset of $L(H)$, $\mathcal{S}^{(n)}$ denotes $\{A^{(n)} : A \in \mathcal{S}\}$. For any x, y in H, let $x \otimes y$ denote the rank-1 operator $u \to (u,x)y$. Let $\mathcal{L}$ be a collection of (closed linear) subspaces of H, $alg\mathcal{L}$ denotes the set of all operators acting on H that leave every member of $\mathcal{L}$ invariant. Dually, if φ is a set of operators acting on H, $lat\varphi$ denotes the collection of subspaces of H which are left invariant by every member of φ

2. SOME RESULTS OF PREREFLEXIVE SUBSPACES

In [3,4], Arveson introduced the following concept of prereflexivity for algebras.

DEFINITION 2.1. A σ-weakly closed algebra $\mathcal{A} \subseteq L(H)$ is called prereflexive if $\mathcal{A} \cap \mathcal{A}^* = alglat\mathcal{A} \cap (alglat\mathcal{A})^*$.

DEFINITION 2.2. A σ-weakly closed subspace of $L(H)$ is called n-prereflexive if whenever $T \in L(H^{(n)})$ satisfies the condition that $Tx \in [\mathcal{S}^{(n)}x]$ and $T^*x \in [\mathcal{S}^{(n)}x]$ for all x in H then T

is in $\mathcal{S}^{(n)}$.(Here $[\,\cdot\,]$ denotes norm closed linear span.)

When reference to n is omitted, it is understood to be 1.

REMARKS. Since $L(H)$ is n-prereflexive, to prove that $\mathcal{S}$ is n-prereflexive we need only to prove that whenever $T \in L(H)$ satisfies $T^{(n)}x \in [\mathcal{S}^{(n)}x]$ and $T^{(n)*}x \in [\mathcal{S}^{(n)}x]$ for all x in $H^{(n)}$ then T is in $\mathcal{S}$.

By the definition 2.2, we easily prove that if U is a unitary operator in $L(H)$ then $U\mathcal{S}U^*$ is prereflexive if and only if $\mathcal{S}$ is prereflexive.

PROPOSITION 2.3. A unital σ-weakly closed algebra $\mathcal{A}$ is prereflexive as a subspace if and only if it is prereflexive as an algebra (i.e. $\mathcal{A} \cap \mathcal{A}^* = (alglat\mathcal{A})^* \cap alglat\mathcal{A}$).

PROOF. Suppose that $\mathcal{A}$ is prereflexive as a subspace of operators. Let $T \in (alglat\mathcal{A})^* \cap alglat\mathcal{A}$. Then we have that for any $M \in lat\mathcal{A}, TM \subseteq M, T^*M \subseteq M$. For any $x \in H$, $[\mathcal{A}x] \in lat\mathcal{A}$ and $I \in \mathcal{A}$, we have that $T^*x \in [\mathcal{A}x]$ and $Tx \in [\mathcal{A}x]$. By prereflexivity of $\mathcal{A}$ as a subspace, we have that $T \in \mathcal{A}$ and $T^* \in \mathcal{A}$, thus $\mathcal{A} \cap \mathcal{A}^* \supseteq (alglat\mathcal{A})^* \cap alglat\mathcal{A}$. The reverse inclusion always holds, hence $\mathcal{A}$ is prereflexive as an algebra.

Conversely, let $Tx \in [\mathcal{A}x]$ and $T^*x \in [\mathcal{A}x]$ for all $x \in H$. Then $TM \subseteq M$, $T^*M \subseteq M, \forall M \in lat\mathcal{A}$. Since $\mathcal{A}$ is prereflexive as an algebra, we have that $T \in \mathcal{A}$. Hence $\mathcal{A}$ is prereflexive as a subspace. Q.E.D.

If φ is an arbitrary subset of $L(H)$, then we use $preref(\varphi)$ to denote the closure of $span\{S, T : S \in \varphi, T \in L(H), Tx \in [\varphi x]$ and $T^*x \in [\varphi x]$ for all $x \in H\}$ in σ-weak operator topology. It follows that $preref(\varphi)$ is the smallest prereflexive subspace containing φ, and φ is prereflexive if and only if $\varphi = preref(\varphi)$.

PROPOSITION 2.4. If $\mathcal{S}$ is a σ-weakly closed subspace of $L(H)$, then $\mathcal{S}$ is prereflexive if and only if $preref(\mathcal{S}) \cap (preref(\mathcal{S}))^* = ref(\mathcal{S}) \cap (ref(\mathcal{S}))^* = \mathcal{S} \cap \mathcal{S}^*$.

PROOF. The necessity is trivial, so we have only to prove the sufficiency.

If $T \in L(H), Tx \in [\mathcal{S}x]$ and $T^*x \in [\mathcal{S}x]$, so $T \in preref(\mathcal{S}) \cap (preref(\mathcal{S}))^* = \mathcal{S} \cap \mathcal{S}^* \subseteq \mathcal{S}$. Hence $\mathcal{S}$ is prereflexive. Q.E.D.

By the previous proposition, we get that $\mathcal{S}$ is prereflexive if and only if $\mathcal{S}^*$ is prereflexive; and if $\mathcal{S}$ is a unital algebra, Proposition 2.4 is the analogy of the definition of prereflexivity for unital algebras that Arveson gives.

THEOREM 2.5. If $\mathcal{S}$ is a σ-weakly closed subspace of $L(H)$, then $\mathcal{S}$ is n-prereflexive if and only if

$$\mathcal{S}_\perp \subseteq \overline{span\{(\mathcal{S}_\perp \cup \mathcal{S}_\perp{}^*) \cap F_n\}}^{\|\cdot\|_1}.$$

PROOF. If rank $f \le n$, we have $x_1, ..., x_n, y_1, ..., y_n$ in H such that $f = x_1 \otimes y_1 + ... + x_n \otimes y_n$. Let $T \in L(H)$, then $tr(Tf) = \sum_{i=1}^{n}(Ty_i, x_i) = (T^{(n)}\widetilde{y}, \widetilde{x})$ where $\widetilde{x} = x_1 \oplus ... \oplus x_n, \widetilde{y} = y_1 \otimes ... \otimes y_n, \widetilde{x}$ and $\widetilde{y}$ in $H^{(n)}$. Hence $f \in \mathcal{S}_\perp$ if and only if $(S^{(n)}\widetilde{y}, \widetilde{x}) = 0$ for all $S \in \mathcal{S}$ if and only if $\widetilde{x} \in [\mathcal{S}^{(n)}\widetilde{y}]^\perp$. So $tr(Tf) = 0, tr(T^*f) = 0 = \overline{tr(Tf^*)}$ for all f in $\mathcal{S}_\perp$ with rank $f \le n$ if and only if $T^{(n)}\widetilde{y} \in [\mathcal{S}^{(n)}\widetilde{y}]$ and $T^{(n)*}\widetilde{y} \in [\mathcal{S}^{(n)}\widetilde{y}]$, for all $\widetilde{y}$ in $H^{(n)}$.

If $\mathcal{S}$ is n-prereflexive, the above paragraph shows

$$\overline{span\{(\mathcal{S}_\perp \cup \mathcal{S}_\perp{}^*) \cap F_n\}}^{\perp} \subseteq \mathcal{S}.$$

Hence

$$\mathcal{S}_\perp \subseteq \overline{span\{(\mathcal{S}_\perp \cup \mathcal{S}_\perp{}^*) \cap F_n\}}^{\|\cdot\|_1}.$$

Conversely, if $\mathcal{S}_\perp \subseteq \overline{span\{(\mathcal{S}_\perp \cup \mathcal{S}_\perp{}^*) \cap F_n\}}^{\|\cdot\|_1}$, let $T \in L(H)$ such that for any $\widetilde{y} \in H^{(n)}, T^{(n)}\widetilde{y} \in [\mathcal{S}^{(n)}\widetilde{y}]$, $T^{(n)*}\widetilde{y} \in [\mathcal{S}^{(n)}\widetilde{y}]$. Then $tr(Tf) = 0, tr(Tf^*) = 0$, for any $f \in \mathcal{S}_\perp$ with rank $f \le n$, so

$$T \in (\overline{span\{(\mathcal{S}_\perp \cup \mathcal{S}_\perp{}^*) \cap F_n\}}^{\|\ \|_1})^\perp \subseteq \mathcal{S}.$$

Hence $\mathcal{S}$ is prereflexive. Q.E.D.

By Theorem 2.5, we have that if $\mathcal{S}$ is self-adjoint, then $\mathcal{S}$ is reflexive if and only if $\mathcal{S}$ is prereflexive.

COROLLARY 2.6. If a subspace $\mathcal{S}$ of $L(H)$ is n-prereflexive, then it is m-prereflexive for $m \geq n$.

PROPOSITION 2.7. For $i, j = 1, ..., n$, let $\mathcal{S}_{ij}$ be a σ-weakly closed subspace of $L(H)$ and let $\mathcal{S}$ be the subspace of $L(H^{(n)})$ defined by

$$\mathcal{S} = \{(t_{ij})_{n\times n} : t_{ij} \in \mathcal{S}_{ij}\}.$$

Then $\mathcal{S}$ is prereflexive if and only if $\overline{span\{(\mathcal{S}_{ij\perp} \cup \mathcal{S}_{ji\perp}{}^*) \cap F_1\}}^{\|\cdot\|_1} \supseteq \mathcal{S}_{ij\perp}$.

PROOF. For $\mathcal{S}_\perp = \{(a_{ij})_{n\times n} : a_{ij} \in \mathcal{S}_{ji\perp}\}$, by Theorem 2.5, we have that $\mathcal{S}$ is prereflexive if and only if

$$\overline{span\{(\mathcal{S}_{ij\perp} \cup \mathcal{S}_{ji\perp}{}^*) \cap F_1\}}^{\|\cdot\|_1} \supseteq \mathcal{S}_{ij\perp}. \text{ Q.E.D.}$$

COROLLARY 2.8. Let $\mathcal{S}_{ij}$ $(1 \leq i \leq j \leq n)$ be σ-weakly closed subspace of $L(H)$, define

$$\mathcal{S} = \left\{ \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{nn} \end{pmatrix} \mid a_{ij} \in \mathcal{S}_{ij}, 1 \leq i \leq j \leq n \right\}.$$

Then $\mathcal{S}$ is prereflexive if and only if every $\mathcal{S}_{ii}$ is prereflexive.

PROPOSITION 2.9. Let $\mathcal{S} = \mathcal{S}_1 \oplus ... \oplus \mathcal{S}_n$, where $\mathcal{S}_i$ is a σ-weakly closed subspace of $L(H_i)$. Then $\mathcal{S}$ is a prereflexive subspace of $L(H_1 \oplus ... \oplus H_n)$ if and only if every $\mathcal{S}_i$ is prereflexive.

The proof is easy, we leave the proof to the reader.

PROPOSITION 2.10. Let $\mathcal{S}$ be a σ-weakly closed subspace of $L(H)$, define below the subalgebra of $L(H \oplus H)$

$$\mathcal{A} = \left\{ \begin{pmatrix} \lambda I & s \\ 0 & \lambda I \end{pmatrix} \mid \lambda \in \mathbf{C}, s \in \mathcal{S} \right\}.$$

Then $\mathcal{A}$ is prereflexive if and only if $\mathcal{S}_\perp \cap F_1 \neq 0$.

PROOF. Suppose that $\mathcal{A}$ is prereflexive. If $\mathcal{S}_\perp \cap F_1 = 0$, we have that for all $z \in H, z \neq 0, [\mathcal{S}z] = H$. For any $\eta = \binom{x}{y} \in H^{(2)}$, if y = 0, let $b_n = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$; if $y \neq 0$, take $a_n \in \mathcal{S}$ such that $\lim_{n\to\infty} a_n y = x$, let $b_n = \begin{pmatrix} 0 & a_n \\ 0 & 0 \end{pmatrix}$. In either case, we have that $\lim_{n\to\infty} b_n\eta = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}\eta$. Since $\mathcal{A}$ is prereflexive, we have that $\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$ belongs to $\mathcal{A}$. This is a contradiction, hence $\mathcal{S}_\perp \cap F_1 \neq 0$.

Conversely, by Corollary 2.8, we have that

$$\tilde{\mathcal{A}} = \left\{ \begin{pmatrix} \lambda I & s \\ 0 & \mu I \end{pmatrix} \mid \lambda, \mu \in \mathbf{C}, s \in \mathcal{S} \right\}$$

is prereflexive, so $preref(\mathcal{A}) \subseteq \tilde{\mathcal{A}}$. In the following, we prove that $\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \notin preref(\mathcal{A})$. By $\mathcal{S}_\perp \cap F_1 \neq 0$, we get that there exist x and y in H satisfying that $\|x\| = \|y\| = 1, x \otimes y \in \mathcal{S}_\perp$, hence $\eta = \binom{x}{-y} \otimes \binom{x}{y} \in \mathcal{A}_\perp$. Since $tr(\eta \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}) \neq 0$, we have that $\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \notin preref(\mathcal{A})$. Hence $\mathcal{A}$ is prereflexive. Q.E.D.

In [5], we prove that if $\mathcal{S}$ is a σ-weakly closed subspace of $L(H)$, and we let

$$\mathcal{A} = \left\{ \begin{pmatrix} \lambda I & 0 & \dots & 0 & s \\ 0 & \lambda I & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \lambda I & 0 \\ 0 & 0 & \dots & 0 & \lambda I \end{pmatrix}_{n\times n} \mid \lambda \in \mathbf{C}, s \in \mathcal{S} \right\}$$

where $n \geq 3$, then $\mathcal{A}$ is prereflexive.

By Propositions 2.7, 2.10 and Proposition 3.10 [6], we know that the reflexivity is very different to the prereflexivity. Let $\mathcal{S}$ be a prereflexive subspace of $L(H)$. Then $\mathcal{S}$ is said to be *hereditarily prereflexive* if every σ-weakly closed subspace of $\mathcal{S}$ is prereflexive. In the following we discuss hereditary prereflexivity.

DEFINITION 2.11. Let $\mathcal{S}$ be a σ-weakly closed subspace of $L(H)$. We say that $\mathcal{S}$ has the property WP if it statisfies

$$(\mathcal{S}_\perp + F_1) \cup (\mathcal{S}_\perp + \overline{span\{(\mathcal{S}_\perp \cup \mathcal{S}_\perp{}^*) \cap F_1\}}^{\|\cdot\|_1}) = T(H).$$

REMARK. The property WP is a property which is weaker than the property P.

THEOREM 2.12. Let $\mathcal{S}$ be a prereflexive subspace of $L(H)$. Then $\mathcal{S}$ is hereditarily prereflexive if and only if $\mathcal{S}$ has the property WP.

PROOF. Suppose that $\mathcal{S}$ has the property WP. Let $\mathcal{V}$ be any σ-weakly closed subspace of $\mathcal{S}$. For any t in $\mathcal{V}_\perp \subseteq T(H)$, we consider below the two cases:

(i) If $t \in \mathcal{S}_\perp + F_1$, then t=f + g with $f \in \mathcal{S}_\perp$ and $g \in F_1, g = t - f \in \mathcal{V}_\perp \cap F_1$. Since $\mathcal{S}$ is prereflexive, we have $f \in \mathcal{S}_\perp \subseteq \overline{span\{(\mathcal{S}_\perp \cup \mathcal{S}_\perp{}^*) \cap F_1\}}^{\|\cdot\|_1} \subseteq \overline{span\{(\mathcal{V}_\perp \cup \mathcal{V}_\perp{}^*) \cap F_1\}}^{\|\cdot\|_1}$. Hence $t \in \overline{span\{(\mathcal{V}_\perp \cup \mathcal{V}_\perp{}^*) \cap F_1\}}^{\|\cdot\|_1}$.

(ii) If $t \in \mathcal{S}_\perp + \overline{span\{(\mathcal{S}_\perp \cup \mathcal{S}_\perp{}^*) \cap F_1\}}^{\|\cdot\|_1}$, for $\mathcal{S}_\perp \subseteq \overline{span\{(\mathcal{S}_\perp \cup \mathcal{S}_\perp{}^*) \cap F_1\}}^{\|\cdot\|_1} \subseteq \overline{span\{(\mathcal{V}_\perp \cup \mathcal{V}_\perp{}^*) \cap F_1\}}^{\|\cdot\|_1}$, we have $t \in \overline{span\{(\mathcal{V}_\perp \cup \mathcal{V}_\perp{}^*) \cap F_1\}}^{\|\cdot\|_1}$.

By the above two cases, we have that $\mathcal{V}_\perp \subseteq \overline{span\{(\mathcal{V}_\perp \cup \mathcal{V}_\perp{}^*) \cap F_1\}}^{\|\cdot\|_1}$. By Theorem 2.5, we have that $\mathcal{V}$ is prereflexive.

Conversely, suppose that

$$(\mathcal{S}_\perp + F_1) \cup (\mathcal{S}_\perp + \overline{span\{(\mathcal{S}_\perp \cup \mathcal{S}_\perp{}^*) \cap F_1\}}^{\|\cdot\|_1}) \neq T(H).$$

Let $t \notin (\mathcal{S}_\perp + F_1) \cup (\mathcal{S}_\perp + \overline{span\{(\mathcal{S}_\perp \cup \mathcal{S}_\perp{}^*) \cap F_1\}}^{\|\cdot\|_1})$ but $t \in T(H)$ and define $\mathcal{V} = (\mathbf{C}t + \mathcal{S}_\perp)^\perp$, we have that $\mathcal{V}$ is a σ-weakly closed subspace of $\mathcal{S}$. In the following we prove that $\mathcal{V}$ is not prereflexive. Since $\mathcal{V}_\perp \cap F_1 = \mathcal{S}_\perp \cap F_1$, we have

$$(\mathcal{V}_\perp \cup \mathcal{V}_\perp{}^*) \cap F_1 = (\mathcal{S}_\perp \cup \mathcal{S}_\perp{}^*) \cap F_1.$$

Suppose $\mathcal{V}$ is prereflexive. We have

$$\mathcal{V} \supseteq \{(\mathcal{S}_\perp \cup \mathcal{S}_\perp{}^*) \cap F_1\}^\perp = \{(\mathcal{V}_\perp \cup \mathcal{V}_\perp{}^*) \cap F_1\}^\perp,$$

then $\mathcal{V}_\perp = \mathbf{C}t + \mathcal{S}_\perp \subseteq \overline{span\{(\mathcal{S}_\perp \cup \mathcal{S}_\perp{}^*) \cap F_1\}}^{\|\cdot\|_1}$. It is impossible since $t \notin (\mathcal{S}_\perp + F_1) \cup (\mathcal{S}_\perp + \overline{span\{(\mathcal{S}_\perp \cup \mathcal{S}_\perp{}^*) \cap F_1\}}^{\|\cdot\|_1})$. Q.E.D.

PROPOSITION 2.13. Let $\mathcal{S}$ be a weakly closed subspace of $L(H)$ such that

$$(\mathcal{S}_\perp + F_k) \cup \overline{span\{(\mathcal{S}_\perp \cup \mathcal{S}_\perp{}^*) \cap F_{2k+1}\}}^{\|\cdot\|_1} = T(H).$$

Then $\mathcal{S}$ is (2k+1)-prereflexive.

PROOF. Since $\mathcal{S}$ is weakly closed, it follows that $\overline{\mathcal{S}_\perp \cap F}^{\|\cdot\|_1} = \mathcal{S}_\perp$. By Theorem 2.5 we only need to prove that $\overline{span\{(\mathcal{S}_\perp \cup \mathcal{S}_\perp{}^*) \cap F_{2k+1}\}}^{\|\cdot\|_1} \supseteq \mathcal{S}_\perp$. Since $\mathcal{S}_\perp \cap F = \bigcup_{i=1}^{\infty} (\mathcal{S}_\perp \cap F_i)$, it suffices to prove for all $l > 2k + 1$,

$$\mathcal{S}_\perp \cap F_l \subseteq \overline{span\{(\mathcal{S}_\perp \cup \mathcal{S}_\perp{}^*) \cap F_{2k+1}\}}^{\|\cdot\|_1}. \tag{2.1}$$

If we can show

$$\overline{span\{(\mathcal{S}_\perp \cup \mathcal{S}_\perp{}^*) \cap F_{l-1}\}}^{\|\cdot\|_1} = \overline{span\{(\mathcal{S}_\perp \cup \mathcal{S}_\perp{}^*) \cap F_l\}}^{\|\cdot\|_1}$$

with $l > 2k+1$, we have that (2.1) is true. Let $t \in (\mathcal{S}_\perp \cup \mathcal{S}_\perp{}^*) \cap F_l$ with $l > 2k+1$, we may assume that $t \in \mathcal{S}_\perp \cap F_l$(if $t \notin \mathcal{S}_\perp \cap F_l$ we may consider t^*), write t= f+g with $f \in F_{k+1}$ and $g \in F_{l-k-1}$. By hypothesis, we have

$$f, g \in (\mathcal{S}_\perp + F_k) \cup \overline{span\{(\mathcal{S}_\perp \cup \mathcal{S}_\perp{}^*) \cap F_{2k+1}\}}^{\|\ \|_1}.$$

If $f \in \mathcal{S}_\perp + F_k$, we have that there exists an h in F_k such that $f - h \in \mathcal{S}_\perp, t = f - h + g + h$. Since $f - h \in \mathcal{S}_\perp \cap F_{2k+1},\ g + h \in F_{l-1} \cap \mathcal{S}_\perp$, it follows that

$$t \in \overline{span\{(\mathcal{S}_\perp \cup \mathcal{S}_\perp{}^*) \cap F_{l-1}\}}^{\|\cdot\|_1}.$$

Similarly, if $g \in \mathcal{S}_\perp + F_k$, we may prove that

$$t \in \overline{span\{(\mathcal{S}_\perp \cup \mathcal{S}_\perp{}^*) \cap F_{l-1}\}}^{\|\cdot\|_1}.$$

If $f \notin \mathcal{S}_\perp + F_k$ and $g \notin \mathcal{S}_\perp + F_k$, we have that $f, g \in \overline{span\{(\mathcal{S}_\perp \cup \mathcal{S}_\perp{}^*) \cap F_{2k+1}\}}^{\|\cdot\|_1}$. Hence $t = f + g \in \overline{span\{(\mathcal{S}_\perp \cup \mathcal{S}_\perp{}^*) \cap F_{2k+1}\}}^{\|\cdot\|_1} \subseteq \overline{span\{(\mathcal{S}_\perp \cup \mathcal{S}_\perp{}^*) \cap F_{l-1}\}}^{\|\cdot\|_1}$. Q.E.D.

PROPOSITION 2.14. Let $\mathcal{S}$ be a weakly closed subspace of $L(H)$ satisfying that given $x_1, ..., x_n \in H$, there exists $x \in H$ such that $\|Tx_i\| \leq \|Tx\|$, for all $T \in \mathcal{S}$. Then every weakly closed subspace of $\mathcal{S}$ is prereflexive.

The proof is easy, we omit it.

3. AN APPLICATION.

If $A \in L(H)$, let $\omega(A)$ denote the closure in the weak operator topology of $L(H)$ of the set of polynomials in A and I, let $\omega_0(A)$ denote the weakly closed principal ideal generated by A. An operator A is called prereflexive if $\omega(A)$ is prereflexive. In [7], Larson and Wogen construct a reflexive operator A such that $A \oplus 0$ is not reflexive. In the section, as an application of the results in Section 2, we prove that there exists a reflexive operator A such that $A \oplus 0$ is not prereflexive. By the idea in [8], we first construct a prereflexive but not reflexive operator.

Let H be a separable Hilbert space of dimension j and let $K = \bigoplus_{n=1}^{\infty} H$. Consider the Hilbert space $K \oplus H$. If $1 \leq k < \infty$, let P_k be the orthogonal projection of $K \oplus H$ onto the k^{th} summand of H in K and let P_∞ be the projection of $K \oplus H$ onto $0 \oplus H$. For any $T \in L(K \oplus H), T$ admits a matrix representation $T = (T_{ij})_{1 \leq i,j \leq \infty}$, with $T_{ij} \in L(H)$.

If $\mathcal{A} \subseteq L(K \oplus H)$ let $\mathcal{A}_{ij} = P_i \mathcal{A} P_j$, we may choose to view $\mathcal{A}_{ij}$ either as a subset of $L(K \oplus H)$ or as a subset of $L(H)$. For any $\varphi \subseteq L(H)$ let $[\varphi]_{ij} = \{S \in L(K \oplus H) : S_{ij} \in \varphi \text{ and } S_{kl} = 0 \text{ if } (k,l) \neq (i,j)\}$. Let $\tilde{\mathcal{A}} = \{A - A_{1,\infty} : A \in \mathcal{A}\}$. Let φ be a weakly closed subspace of $L(H)$ such that $\varphi^{(2)}$ is prereflexive but not reflexive. By Proposition 3 [8], we may construct an operator T such that

$$\omega(T^{(2)}) = \omega(T^{(2)})\tilde{} \dotplus [\varphi^{(2)}]_{1,\infty}.$$

By Lemma 6 [8], we have $\omega(T^{(2)})\tilde{}$ is reflexive. Since $\varphi^{(2)}$ is not reflexive, it follows $\omega(T^{(2)})$ is not reflexive. In the following we prove that $\omega(T^{(2)})$ is prereflexive. Since $preref(\omega(T^{(2)}) \subseteq \omega(T^{(2)})\tilde{} \dotplus [preref(\varphi^{(2)})]_{1,\infty}$. we have that if $A \in L(K^{(2)} \oplus H^{(2)})$ such that for all x in $K^{(2)} \oplus H^{(2)}, Ax \in [\omega(T^{(2)})x], A^*x \in [\omega(T^{(2)})x]$, then $A = A_1 \dotplus A_2$, where $A_1 \in \omega(T^{(2)})\tilde{}$, and $A_2 = \begin{pmatrix} 0 & A_{1\infty} \\ 0 & 0 \end{pmatrix}$ satisfying that for any $y \in H^{(2)}$, $A_{1\infty} y \in [\varphi^{(2)} y], A_{1\infty}^* y \in [\varphi^{(2)} y]$. Since $\varphi^{(2)}$ is prereflexive, we have $A_{1\infty} \in \varphi^{(2)}$, so $A \in \omega(T^{(2)})\tilde{} \dotplus [\varphi^{(2)}]_{1,\infty}$. Hence $\omega(T^{(2)})$ is prereflexive.

PROPOSITION 3.1. Suppose that H and $\tilde{H}$ are Hilbert spaces with $\dim \tilde{H} \geq 1$. Let $A \in L(H)$ and let $0 \in L(\tilde{H})$.

(1) $A \oplus 0$ is prereflexive if and only if $\omega_0(A)$ is prereflexive.

(2) If A is prereflexive, then $A \oplus 0$ is not prereflexive if and only if $I \notin \omega_0(A)$ but $I \in preref(\omega_0(A))$.

PROOF. (1) Let $B \in L(H \oplus \tilde{H})$ such that for any $x \oplus y \in H \oplus \tilde{H}$

$$B(x \oplus y) \in [\omega(A \oplus 0)(x \oplus y)], \tag{3.1}$$

$$B^*(x \oplus y) \in [\omega(A \oplus 0)(x \oplus y)], \tag{3.2}$$

we have that $B \in preref(\omega(A \oplus 0))$. For $preref(\omega(A)) \oplus \mathbf{C}I$ is prereflexive and contains $\omega(A \oplus 0)$, it follows that $B = B_1 \oplus \lambda I$, where $B_1 \in preref(\omega(A))$. It suffices to prove that $B_1 - \lambda I \in \omega_0(A)$, since $\omega(A \oplus 0) = \omega_0(A \oplus 0) + \mathbf{C}(I \oplus I) = \omega_0(A) \oplus 0 + \mathbf{C}(I \oplus I)$. For a fixed nonzero vector y in K and for any x in H, by (3.1), we have a sequence of polynomials $\{p_n\}$ such that $\lim_{n \to \infty} p_n(A \oplus 0)(x \oplus y) = (B_1 \oplus \lambda I)(x \oplus y) = B_1 x \oplus \lambda y$. Since $p_n(A \oplus 0) = p_n(A) \oplus p_n(0)I$, thus

$$p_n(0) \to \lambda, p_n(A)x \to B_1 x.$$

Let $q_n = p_n - p_n(0)$, then $q_n(0) = 0, q_n(A)x \to (B_1 - \lambda I)x$, that is $(B_1 - \lambda I)x \in [\omega_0(A)x]$. By (3.2), we may prove that $(B_1^* - \bar{\lambda} I)x \in [\omega_0(A)x]$. Since $\omega_0(A)$ is prereflexive, we have $B_1 - \lambda I \in \omega_0(A)$.

Conversely, suppose that $\omega(A \oplus 0)$ is prereflexive. Let $T \in preref(\omega_0(A))$. For

$$\begin{aligned} T \oplus 0 \in & preref(\omega_0(A)) \oplus 0 = preref(\omega_0(A \oplus 0)) \subseteq preref(\omega(A \oplus 0)) \\ = & \omega(A \oplus 0) = \omega_0(A \oplus 0) + \mathbf{C}(I \oplus I) = \omega_0(A) \oplus 0 + \mathbf{C}(I \oplus I), \end{aligned}$$

it follows that $T \in \omega_0(A)$. Hence $\omega_0(A)$ is prereflexive.

(2) Suppose that A is prereflexive. For

$$\omega_0(A) \subseteq preref(\omega_0(A)) \subseteq preref(\omega(A)) = \omega(A) = \omega_0(A) + \mathbf{C}I. \tag{3.3}$$

By (1), we have that $A \oplus 0$ is not prereflexive if and only if $\omega_0(A)$ is not prereflexive. By (3.3), it follows that $\omega_0(A)$ is not prereflexive if and only if $I \notin \omega_0(A)$ but $I \in preref(\omega_0(A))$. Q.E.D.

By Proposition 2.1 and Theorem 3.7 [7] as well as the the above proposition we have the following results.

COROLLARY 3.2. If A is reflexive, then $A \oplus 0$ is reflexive if and only if $A \oplus 0$ is prereflexive.

COROLLARY 3.3. There exists a reflexive operator A such that $A \oplus 0$ is not prereflexive.

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