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ON DIRICHLET CONVOLUTION METHOD

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ABSTRACT. In this paper we have proved limitation theorem for $(D, h(n))$ summability methods and have shown that it is best possible.

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1. INTRODUCTION

In his studies on the prime number theorem, Ingham [1] defined a novel summability method called (I) This was generalized by Segal [2] and he defined the notion of $(D, h(n))$ summability, where $h : N \to R$ denotes a function with $h(1) = 1$ We define the "Dirichlet inverse" $h^*(n)$ of $h(n)$ by $\sum_{d|n} h(d)h^*(n/d) = \begin{cases} 1, & n = 1 \\ 0, & n > 2 \end{cases}$. A series $\sum a_n$ is said to be $(D, h(n))$ summable to L if and only if

$$n \overset{\lim}{\to} \infty \frac{1}{n} \sum_{v=1}^{n} v \sum_{d|v} a_d h(v/d) = L. \tag{1 1}$$

Given a series $\sum a_n$ and a specific $h(n)$, define the function

$$D(t) = \frac{1}{t} \sum_{n<t} n \sum_{d|n} a_d h(n/d). \tag{1.2}$$

Since $D([t]) = \frac{t}{[t]} D(t)$, it clearly makes no difference to the existence or value of the limit (1 2) whether $t \to \infty$ is through real values or integers. Ingham's method corresponds to the case $h(n) = \frac{1}{n}$

Segal [3] proved the limitation theorem for (I) summability. If $\sum a_n$ is (I) summable, then $\sum_{n<x} a_n = o(\log x)$ and has shown in the following theorem that his result is best possible

THEOREM A [4] Let $\in (x)$ be any positive function decreasing to 0 monotonically but arbitrarily slowly as $x \to \infty$. Then there exists a series $\sum a_n$ which is (I) summable and such that

$$\sum_{n<x} a_n \neq 0(\in (x) \log x) \quad \text{as} \quad x \to \infty.$$

Sukla [5] has shown an analogous limitation theorem for $(D, h(n))$ summability.

THEOREM B. If $\sum a_n$ is $(D, h(n))$ summable then $\sum_{n<x} a_n = O(\log x)$ if

(i) $$H^*(r) = \sum_{n<r} h^*(n) = O(1)$$

and

(ii) $$\sum_{v=1}^{n} |h^*(v)| = O(\log n).$$

It is remarked in that paper that the condition (ii) cannot be dropped However if we replace (i) by a slightly stronger condition then we get the result to be true without assuming (ii) In section 4 we show that our revised version of Theorem A is best possible.

2. MAIN RESULTS

THEOREM 1. If $\sum a_n$ is $(D, h(n))$ summable then

$$\sum_{n\le x} a_n = O\left(\sum_{n<x} |h^*(n)|\right) \tag{2 1}$$

if

$$\sum_{n\le r} h^*(n) = O\big((\log r)^{-1-\epsilon}\big) \quad \text{as} \quad r\to\infty. \tag{2 2}$$

We will show that (3 1) is a best possible result

THEOREM 2. Let $\epsilon\,(x)$ be any positive function decreasing to 0 monotonically but arbitrarily slowly as $x\to\infty$. Then there exists a series $\sum a_n$ which is $(D, h(n))$ summable and (3 2) holds and

$$\sum_{1\le d\le \frac{r}{[r^{1/2}]+1}} |h^*(d)| \Big/ \sum_{n\le r} |h^*(n)| \tag{2 3}$$

does not tend to zero as $r\to\infty$ holds and such that

$$\sum_{n\le x} a_n \ne o\left(\epsilon\,(x)\sum_{n\le x}|h^*(n)|\right) \quad \text{as} \quad x\to\infty.$$

PROOF OF THEOREM 1. For $m\ge 0$, let

$$K(m) = \begin{cases} mD(m) & \text{if } m\ge 1 \\ 0 & \text{if } m=0\end{cases}$$

then by (1 1) and (1.2) it follows that

$$K(m) = O(m), \quad \text{as} \quad n\to\infty, \quad \text{and} \tag{2 4}$$

$$\sum_{n\le r} a_n = \sum_{d\le r}\frac{K(d)}{d}\, d\left(\frac{H^*(\frac{r}{d})}{d} - \frac{H^*(\frac{r}{d+1})}{d+1}\right).$$

By (2 4) it is enough to show that

$$\sum_{d\le r} d\left|\frac{H^*(\frac{r}{d})}{d} - \frac{H^*(\frac{r}{d+1})}{d+1}\right| = O\left(\sum_{n\le r}|h^*(n)|\right). \tag{2 5}$$

The left hand side of (2.5) is maximized by

$$\sum_{d\le r}\left|H^*\left(\frac{r}{d}\right) - H^*\left(\frac{r}{d+1}\right)\right| + \sum_{d\le r}\frac{|H^*(\frac{r}{d+1})|}{d+1}, \tag{2.6}$$

Now

$$\sum_{d\le r}\left|H^*\left(\frac{r}{d}\right)\right| - H^*\left(\frac{r}{d+1}\right) = \sum_{d\le r}\left|\sum_{\frac{r}{d+1}\le v\le \frac{r}{d}} h^*(v)\right| \le \sum_{1\le v\le r}|h^*(v)|$$

and

$$\sum_{d\le r}\left|\frac{H^*(\frac{r}{d+1})}{d+1}\right| = O\left(\sum_{d\le r-2}\left(\log\frac{r}{d+1}\right)^{-1-\epsilon}\frac{1}{d+1}\right) = O(1)$$

since $H^*(x) = O$ for $x<1$

PROOF OF THEOREM 2. Define b_n by

$$b_n = \sum_{d|n} h^*\left(\frac{n}{d}\right)\left(\frac{dD(d) - (d-1)D(d-1)}{d}\right), \quad \text{where} \quad D(t) \to 0 \quad \text{as} \quad t \to \infty \tag{2 7}$$

then

$$\frac{1}{t}\sum_{n<t} n \sum_{d|n} b_d h\left(\frac{n}{d}\right) = \frac{1}{t}\sum_{n<t} n \sum_{d|n} h\left(\frac{n}{d}\right) \sum_{r|d} h^*\left(\frac{r}{d}\right)\left(\frac{rD(r) - (r-1)D(r-1)}{r}\right) = D(t). \tag{2 8}$$

Since $D(t) \to 0$, $\sum b_n$ is $(D, h(n))$summable to 0.

$$\begin{aligned}
\sum_{n\le r} b_n &= \sum_{n<r}\sum_{d|n} h^*\left(\frac{n}{d}\right)\left(\frac{dD(d) - (d-1)D(d-1)}{d}\right) \\
&= \sum_{d\le r}\left(\frac{dD(d) - (d-1)D(d-1)}{d}\right)\sum_{m\le \frac{r}{d}} h^*(m) \\
&= \sum_{d\le r} D(d) - D(d-1)H^*\left(\frac{r}{d}\right) + \sum_{d\le r}\frac{D(d-1)}{d} H^*\left(\frac{r}{d}\right) = \sum_1 + \sum_2 .
\end{aligned}$$

Now

$$\sum_1 = \sum_{d\le r} D(d)\left[H^*\left(\frac{r}{d}\right) - H^*\left(\frac{r}{d+1}\right)\right].$$

Since $H^*(x) = 0$ for $x < 1$

$$\sum_2 = O\left(\sum_{d\le r}\frac{1}{d}\left|H^*\left(\frac{r}{d}\right)\right|\right) = O(1) \quad \text{as} \quad r \to \infty \quad by\,(2.2).$$

We have now

$$\sum_1 = \sum_{n\le r} b_n + O(1). \tag{2 9}$$

Suppose the theorem does not hold then

$$\sum_{n\le r} b_n = o\left(\epsilon(r)\sum_{n\le r}|h^*(n)|\right).$$

So (2 9) becomes

$$\sum_1 = o\left(\epsilon(r)\sum_{n\le r}|h^*(n)|\right). \tag{2 10}$$

Since $D(d) \to 0$ as $n \to \infty$, let

$$\alpha_{r,d} = \frac{1}{\epsilon(r)\sum_{n\le r}|h^*(n)|}\left[H^*\left(\frac{r}{d}\right) - H^*\left(\frac{r}{d+1}\right)\right].$$

It is well known that in order for $\alpha_{r,d}$ to transform all sequences tending to 0 into sequences tending to 0,

$$\frac{1}{\epsilon(r)\sum_{n\le r}|h^*(n)|}\sum_{d\le r}\left|H^*\left(\frac{r}{d}\right) - H^*\left(\frac{r}{d+1}\right)\right| < c$$

must hold for all r where c is independent of r

$$\sum_{d\le r}\left|H^*\left(\frac{r}{d}\right) - H^*\left(\frac{r}{d+1}\right)\right| = \sum_{d\le r}\left|\sum_{\frac{r}{d+1}<m\le\frac{r}{d}} h^*(m)\right| \ge \sum_{r^{1/2}<d\le r}\left|\sum_{\frac{r}{d+1}<m\le\frac{r}{d}} h^*(m)\right|.$$

Since in this last sum $\frac{r}{d} - \frac{r}{d+1} < 1$ the inner sum contains at most one term, and so

$$\frac{1}{\epsilon(r)\sum_{n\le r}|h^*(n)|}\sum_{d\le r}\left|H^*\left(\frac{r}{d}\right) - H^*\left(\frac{r}{d+1}\right)\right| \ge \frac{1}{\epsilon(r)}\left(\frac{\sum_{1<d\le\frac{r}{[r^{1/2}]+1}}|h^*(d)|}{\sum_{n\le r}|h^*(n)|}\right)$$

tends to infinity as $r \to \infty$ since by (2.3) the expression in the bracket does not tend to zero as $r \to \infty$ This completes the proof of Theorem 2

Agnew [6] showed directly that, for $r > 0$ the Cesáro and Riesz transforms $C_r(n)$, $R_r(n)$ respectively of a given series $\sum a_n$ are equiconvergent i.e $C_r(n)$, $R_r(n)$ exist for each n and

$$\lim_{r \to \infty}(C_r(n) - R_r(n)) = 0.$$

These concepts are applied to arithmetic summation methods (I) and $(D, h(n))$ for particular values of $h(n)$ by Jukes [7] He has found different conditions under which the equiconvergence of $\frac{6}{\pi^2}(I)$ and $\left(D, \frac{\mu^2(n)}{n}\right)$ have been established. The $\left(D, \frac{\mu^2(n)}{n}\right)$ and $\frac{6}{\pi^2}(I)$ transform are given by

$$b_{nk} = \frac{k}{n}\sum_{r\le\frac{n}{k}}\mu^2(r), \quad C_{nk} = \frac{6}{\pi^2}\frac{k}{n}\left[\frac{n}{k}\right]$$

respectively Let $M_2 = \lim_n \sup\sum k\left|\triangle\left(\frac{b_{nk}-c_{nk}}{k}\right)\right|$

$$A_2 = \lim_n \sup\left|\sum_{k=\infty}^{n}\frac{ka_k}{(n+1)}\right|.$$

THEOREM C [7] Tauherian constants M_2 do not exist for comparisons of conservative matrices with non-conservative matrices

THEOREM D [7] The $\left(D, \frac{\mu^2(n)}{n}\right)$ and $6/\pi^2(I)$ transform are not equiconvergent whenever $A_2 < \infty$

We have proved (see Kuttner and Sukla [8]) that

THEOREM E. The $(D, h(n))$ is conservative if and only if $\sum_{n=1}^{\infty}|h(n)| < \infty$ It is to note that if part of the above theorem was proved earlier by Jukes [9] See S. L. Segal, *Math. Reviews* 86e 11093 (May 1986, p 1864)

THEOREM 3. The $(D, h(n))$ and (I) are not equiconvergent whenever $A_2 < \infty$ and $\sum|h(n)| < \infty$

PROOF. By Theorem C since (I) is not conservative and $(D, h(n))$ is conservative for $\sum|h(n))| < \infty$ whenever $A_2 < \infty (D, h(n))$ and (I) are not equiconvergent

From Theorem E also we get that the following theorem of Jukes as corollaries

COROLLARY 1. The methods $\left(D, \frac{\mu(n)}{n}\right)$ and $\left(D, \frac{\lambda(n)}{n}\right)$ are not conservative

PROOF. Since $\sum_{n=1}^{\infty}\frac{\mu(n)}{n}$ and $\sum_{n=1}^{\infty}\frac{\lambda(n)}{n}$ are not absolutely convergent So by Theorem 3 the result follows.

COROLLARY 2. $(D, \mu^2(n)/n)$ and $(D, \in \lambda(n)/\pi^2(n))$ transforms are not equiconvergent whenever $A_2 < \infty$.

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