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REGULARIZATION OF DISTRIBUTIONS

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ABSTRACT. We give a simple necessary and sufficient condition for the existence of distributional regularizations. Our results apply to functions and distributions defined in the complement of a point, in one or several variables. We also consider functions defined in the complement of a hypersurface. We apply these results to the existence of distributional boundary values of harmonic and analytic functions.

KEY WORDS AND PHRASES: Generalized function, harmonic function, analytic function.
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1. INTRODUCTION

The distributional regularization of functions with non-integrable singularities is an important and mostly well-understood topic. In its simplest form, the problem is to assign a distribution $\tilde{f} \in \mathcal{D}'(\mathbf{R})$ to a function f, defined and locally integrable on $\mathbf{R} \setminus \{x_0\}$ in such a way that $\tilde{f}$ coincides with f on $\mathbf{R} \setminus \{x_0\}$.

Many textbooks on the theory of distributions have sections dealing with this regularization problem, where one usually finds conditions that guarantee the existence of a distributional regularization and where one can find examples of functions that do not admit regularizations. As far as we know, however, there is no place in the literature where one can find a simple necessary and sufficient condition for the existence of a regularization.

The purpose of this article is to give such a criterion for the existence of regularizations. Actually our result applies to a somewhat more general situation, namely, if $f \in \mathcal{D}'(\mathbf{R} \setminus \{x_0\})$, when is there an extension $\tilde{f} \in \mathcal{D}'(\mathbf{R})$ such that $\tilde{f}|_{\mathbf{R}\setminus\{x_0\}} = f$?

The answer is obtained by using the recently developed theory of distributional asymptotic expansions as presented by Estrada–Kanwal [1, 2]. It is well known that if f is locally integrable in $\mathbf{R} \setminus \{x_0\}$ and if x_0 is an algebraic singularity of f in the sense that $f(x) = O(|x - x_0|^{-\alpha})$, as $x \to x_0$, for some $\alpha \in \mathbf{R}$, then f admits a regularization in $\mathcal{D}'(\mathbf{R})$. It is also well known that the converse does not hold. However, as we show, the existence of regularizations is equivalent to the order relation $f(x) = O(|x - x_0|^{-\alpha})$ *not in the ordinary but in the distributional or average sense.*

In the second section we define the order relations $f(x) = O(|x - x_0|^{-\alpha})$ in the distributional or average sense. By using the ideas of the theory of distributional asymptotic expansions we show

how order relations can be defined by one of two equivalent procedures, namely, by considering the parametric behavior or by considering the behavior of primitives. In the third section we show how the results can be extended to the regularization of generalized functions of several variables with a singularity at the origin. Singularities located on a hypersurface of $\mathbf{R}^n$ are considered in section 4. In the last section we apply these results to the existence of distributional boundary values of harmonic and analytic functions.

2. CHARACTERIZATION OF REGULARIZABLE DISTRIBUTIONS

In this section we study the behavior as $x \to 0^+$ of distributions defined on $x > 0$ and this allows us to characterize those distributions that admit a regularization in $\mathcal{D}'(\mathbf{R})$. We start by clarifying the notation.

Let (a, b) be an open interval, finite, semi-infinite or infinite. The space $\mathcal{D}(a, b)$ consists of those smooth functions ϕ with support, $\operatorname{supp}\phi = \overline{\{x \in (a,b) : \phi(x) \neq 0\}}$ compact, the closure being taken in (a, b). This space carries the standard Schwartz topology, and $\mathcal{D}'(a, b)$ is its dual, the space of standard distributions on (a, b).

The space $\mathcal{E}(a, b)$ is the space of all smooth functions on (a, b), without restriction of its support. This space is equipped with the topology of uniform convergence of all derivatives on compacts of (a, b). Since the inclusion of $\mathcal{D}(a, b)$ into $\mathcal{E}(a, b)$ is continuous and has dense image, it follows that the dual space $\mathcal{E}'(a, b)$ can be identified with a subspace of $\mathcal{D}'(a, b)$, namely, the subspace of distributions with compact support in (a, b).

If (a, b) is finite we can define the space $\mathcal{E}[a, b]$, formed by the smooth functions on $[a, b]$ (at the endpoints we ask for the existence of the one-sided derivatives). The topology of $\mathcal{E}[a, b]$ is that of the uniform convergence of all derivatives on $[a, b]$. The dual space $\mathcal{E}'[a, b]$ can be identified with the closed subspace of $\mathcal{D}'(\mathbf{R})$ formed by those distributions f with $\operatorname{supp} f \subseteq [a, b]$.

Finally, the space $\mathcal{S}(a, b)$ is formed by those smooth functions ϕ for which the seminorms

$$\|\phi\|_{k,j} = \sup\{\rho_k(x)|\phi^{(j)}(x)| : a < x < b\}, \tag{2.1}$$

where $\rho_k(x)$ are any continuous functions defined in (a, b) that satisfy $\rho_k(x) > 0$, $a < x < b$ and $\rho_k(x) = |x-a|^{-k} + O(|x-a|^{\infty})$, $x \to a$, if $|a| < \infty$ or $\rho_k(x) = |x|^k + O(|x|^{-\infty})$, as $x \to a$ if $|a| = \infty$, and similarly $\rho_k(x) = |x-b|^{-k} + O(|x-b|^{\infty})$, $x \to b$, if $|b| < \infty$ or $\rho_k(x) = |x|^k + O(|x|^{-\infty})$, as $x \to b$ if $|b| = \infty$. If $(a, b) = \mathbf{R}$ we obtain the space $\mathcal{S}'(\mathbf{R})$ of tempered distributions, see Estrada–Kanwal [2], while if (a, b) is a finite interval we obtain the space of regularizable distributions on (a, b), introduced by Orton [3] with a somewhat different notation. Indeed, if (a, b) is a finite interval then the inclusion of $\mathcal{S}(a, b)$ into $\mathcal{E}[a, b]$ is continuous and thus, by duality, we obtain a canonical projection $\pi : \mathcal{E}'[a, b] \to \mathcal{S}'(a, b)$ which is onto and whose kernel consists of the distributions with support contained on the two point set $\{a, b\}$. Hence, $\mathcal{S}(a, b)$ is the subspace of $\mathcal{D}(a, b)$ formed by those distributions that admit extensions to $\mathcal{D}'(\mathbf{R})$.

It is also convenient to consider the space $\mathcal{D}_{ij}(a, b)$ formed by those smooth functions defined in (a, b) that show the behavior of the space i at $x = a$ and that of the space j at $x = b$, where $i = 1$ corresponds to $\mathcal{D}(a, b)$, $i = 2$ corresponds to $\mathcal{E}(a, b)$, $i = 3$ to $\mathcal{S}(a, b)$ and $i = 4$ to $\mathcal{E}[a, b]$. When $i = 4$ it is better to use the notation $\mathcal{D}_{4j}[a, b)$. These spaces were introduced by Estrada–Kanwal [4] and play an important role in the study of integral equations in spaces of distributions.

The regularization problem we want to consider is the following. If f is a distribution of the space $\mathcal{D}'(0, \infty)$, when is there an extension $\tilde{f} \in \mathcal{D}'(\mathbf{R})$ such that $\tilde{f}|_{(0,\infty)} = f$? Observe that if such extensions exist then we can find an extension $\tilde{f}$ with support in $[0, \infty)$, that is, $\tilde{f} \in \mathcal{D}'_{41}[0, \infty)$. Alternatively, we would like to know when the distribution $f \in \mathcal{D}'(0, \infty)$ belongs to $\mathcal{D}'_{31}(0, \infty)$. Our characterization uses the distributional version of the Landau order relations.

DEFINITION. Let $f \in \mathcal{D}(0,\infty)$. Let $\alpha \in \mathbf{R}\setminus\{-1,-2,-3,\ldots\}$. We say that f is big O of x^α as $x \to 0^+$ in the average or distributional sense if there exists $n \in \mathbf{N}$ such that every primitive of order n of f, F, is an ordinary function near $x=0$ and satisfies

$$F(x) = p(x) + O(x^{\alpha+n}), \quad \text{as } x \to 0^+, \tag{2.2}$$

for a suitable polynomial p of order $n-1$ that depends on F. In that case we write

$$f(x) = O(x^\alpha) \qquad (C), \quad \text{as } x \to 0^+. \tag{2.3}$$

The notation $f(x) = o(x^\alpha)$ (C), as $x \to 0^+$ is defined similarly. The letter (C) refers to Cesàro. We suppose $\alpha \neq -1,-2,-3,\ldots$ to avoid the consideration of the primitives of $x^{-1}, x^{-2}, x^{-3}, \ldots$.

Observe that in (2.2) we have $\dfrac{d^n p}{dx^n} = 0$ and thus $F-p$ is a primitive of order n of f. Therefore, the definition can be restated by saying that there exists $n \in \mathbf{N}$ and a primitive of order n, F, such that $F(x) = O(x^{\alpha+n})$, $x \to 0^+$.

EXAMPLE. Consider the function $f(x) = x^{-2}e^{1/x}\sin e^{1/x}$. Then $f(x)$ is not $O(x^\alpha)$ as $x \to 0^+$ for any α in the ordinary sense because $|f(x)|$ grows very fast as $x \to 0^+$. However, since $F(x) = \cos e^{1/x}$ is a primitive of f, we obtain that *on the average* $f(x)$ is $O(x^{-1-\varepsilon})$, as $x \to 0^+$, for $0 < \varepsilon < 1$. ∎

This notation is related to the idea of distributional point values and limits introduced by Łojasiewicz [5]. Indeed, the distribution $f(x)$ has the limit L, as $x \to 0^+$ if and only if

$$\lim_{\varepsilon\to 0^+} f(\varepsilon x) = L \tag{2.4}$$

in $\mathcal{D}'(0,\infty)$, that is, if and only if

$$\lim_{\varepsilon\to 0^+} \langle f(\varepsilon x), \phi(x)\rangle = L\int_{-\infty}^{\infty} \phi(x)\,dx, \qquad \phi \in \mathcal{D}(0,\infty). \tag{2.5}$$

However, as Łojasiewicz shows, $\lim\limits_{x\to 0^+} f(x) = L$ distributionally if and only if there exists a primitive of order n of f, F, such that

$$\lim_{x\to 0^+} \frac{n!F(x)}{x^n} = L, \tag{2.6}$$

namely, if and only if

$$f(x) = L + o(1) \qquad (C), \quad \text{as } x \to 0^+. \tag{2.7}$$

Actually, the parametric behavior of $f(\varepsilon x)$, as $\varepsilon \to 0^+$, and the behavior of $f(x)$ as $x \to 0^+$ in the (C) sense are also related for the order relations. As we shall show, the statements $f(\varepsilon x) = O(\varepsilon^\alpha)$, $\varepsilon \to 0^+$ and $f(x) = O(x^\alpha)$, (C) as $x \to 0^+$, are equivalent. In order to show this result we need some preliminary results.

LEMMA 1. Let A be a function defined in $(0,\infty)$. Suppose that for each compact $K \subseteq (0,\infty)$ we have

$$A(\rho\varepsilon) = A(\varepsilon) + O(\varepsilon^\alpha), \quad \text{as } \varepsilon \to 0^+, \tag{2.8}$$

uniformly on $\rho \in K$. If $\alpha < 0$ then

$$A(\varepsilon) = O(\varepsilon^\alpha), \quad \text{as } \varepsilon \to 0^+, \tag{2.9}$$

while if $\alpha > 0$, then there exists a constant a such that

$$A(\varepsilon) = a + O(\varepsilon^\alpha), \quad \text{as } \varepsilon \to 0^+. \tag{2.10}$$

PROOF. Taking $K = [\frac{1}{2}, 2]$, we can find a constant M and $\varepsilon_0 > 0$ such that

$$|A(\rho\varepsilon) - A(\varepsilon)| \le M\varepsilon^\alpha, \qquad 0 < \varepsilon \le \varepsilon_0,\ \frac{1}{2} \le \rho \le 2. \tag{2.11}$$

Suppose first that $\alpha < 0$. Let $\rho \in [\frac{1}{2^n}, \frac{1}{2^{n-1}}]$. Then if $0 < \varepsilon \le \varepsilon_0$,

$$\begin{aligned} |A(\rho\varepsilon) - A(\varepsilon)| &\le \sum_{j=0}^{n-2} |A(2^j\rho\varepsilon) - A(2^{j+1}\rho\varepsilon)| + |A(2^{n-1}\rho\varepsilon) - A(\varepsilon)| \\ &\le \sum_{j=0}^{n-1} M(2^j\rho\varepsilon)^\alpha \\ &\le \frac{M}{1-2^\alpha}\rho^\alpha\varepsilon^\alpha. \end{aligned}$$

Thus, if $\varepsilon < \varepsilon_0$,

$$\begin{aligned} |A(\varepsilon)| &\le |A(\varepsilon) - A(\varepsilon_0)| + |A(\varepsilon_0)| \\ &\le \frac{M}{1-2^\alpha}\varepsilon^\alpha + |A(\varepsilon_0)|, \end{aligned}$$

and consequently

$$A(\varepsilon) = O(\varepsilon^\alpha), \quad \text{as } \varepsilon \to 0^+.$$

When $\alpha > 0$ we proceed as follows. Take $\rho \in [\frac{1}{2^n}, \frac{1}{2^{n-1}}]$. Then if $0 < \varepsilon \le \varepsilon_0$,

$$|A(\rho\varepsilon) - A(\varepsilon)| \le \sum_{j=0}^{n-1} M(2^j\rho\varepsilon)^\alpha \le \frac{2^\alpha M}{2^\alpha - 1}\varepsilon^\alpha. \tag{2.12}$$

Thus,

$$\overline{\lim_{\varepsilon,\delta\to 0}} |A(\varepsilon) - A(\delta)| \le \overline{\lim_{\varepsilon,\delta\to 0}} \frac{2^\alpha M}{2^\alpha - 1}(\max\{\varepsilon, \delta\})^\alpha = 0,$$

and it follows that the limit

$$a = \lim_{\varepsilon\to 0} A(\varepsilon) \tag{2.13}$$

exists. Taking the limit as $\rho \to 0$ in (2.12) we obtain

$$A(\varepsilon) = a + O(\varepsilon^\alpha),$$

as required. ∎

Observe that we just need to require (2.8) to hold uniformly for some compact K with non-empty interior.

The following lemma is easy to prove, but will be useful.

LEMMA 2. Let Λ be a topological space, let $\lambda_0 \in \Lambda$ and let $\Phi: \Lambda \setminus \{\lambda_0\} \to (0, \infty)$. Let $\{f_\lambda\}_{\lambda\in\Lambda\setminus\{\lambda_0\}}$ be a family of distributions of $\mathcal{D}'(a, b)$ and let $\{F_\lambda\}_{\lambda\in\Lambda\setminus\{\lambda_0\}}$ be a family of first order primitives: $F'_\lambda = f_\lambda$. Suppose

$$f_\lambda(x) = O(\Phi(\lambda)), \quad \text{as } \lambda \to \lambda_0, \tag{2.14}$$

distributionally. Then there exists a function $A: \Lambda \setminus \{\lambda_0\} \to \mathbf{R}$ such that

$$F_\lambda(x) = A(\lambda) + O(\Phi(\lambda)), \quad \text{as } \lambda \to \lambda_0, \tag{2.15}$$

distributionally. ∎

We can now prove the following

THEOREM 1. Let $f \in \mathcal{D}'(0, \infty)$ and $\alpha \in \mathbf{R} \setminus \{-1, -2, -3, \dots\}$. Then

$$f(\varepsilon x) = O(\varepsilon^\alpha), \quad \text{as } \varepsilon \to 0^+, \tag{2.16}$$

distributionally if and only if

$$f(x) = O(x^\alpha), \qquad (C), \quad \text{as } x \to 0^+. \tag{2.17}$$

PROOF. Suppose first that (2.17) holds. Then there exists $n \in \mathbf{N}$ and a primitive of order n of f, F, which is continuous on $(0,1]$ and satisfies

$$|F(x)| \le Mx^{\alpha+n} \quad , \qquad 0 < x \le 1, \tag{2.18}$$

for some constant M. We may suppose $\alpha + n > 0$.

Let $\phi \in \mathcal{D}(0,\infty)$. Let $c > 0$ be such that $\operatorname{supp}\phi \subseteq (0,c]$. Then if $\varepsilon < c^{-1}$,

$$\begin{aligned} |\langle F(\varepsilon x), \phi(x)\rangle| &\le \int_0^c |F(\varepsilon x)|\,|\phi(x)|\,dx \\ &\le M\varepsilon^{\alpha+n}\int_0^c x^{\alpha+n}|\phi(x)|\,dx, \end{aligned}$$

so that

$$F(\varepsilon x) = O(\varepsilon^{\alpha+n}), \quad \text{as } \varepsilon \to 0^+ \tag{2.19}$$

in $\mathcal{D}'(0,\infty)$.

Hence,

$$\langle f(\varepsilon x), \phi(x)\rangle = (-1)^n \varepsilon^{-n}\langle F(\varepsilon x), \phi^{(n)}(x)\rangle = O(\varepsilon^\alpha),$$

and (2.16) follows.

Conversely, suppose (2.16) holds. Then there exists $n \in \mathbf{N}$ and a primitive $G(x,\varepsilon)$ of order n of $f(\varepsilon x)$ with respect to x, $\dfrac{\partial^n G(x,\varepsilon)}{\partial x^n} = f(\varepsilon x)$, such that

$$G(x,\varepsilon) = O(\varepsilon^\alpha), \tag{2.20}$$

uniformly for $\frac{1}{2} \le x \le 1$. But according to the Lemma 2, if F is a primitive of order n of f, then there exist functions $a_0(\varepsilon), a_1(\varepsilon), \ldots, a_{n-1}(\varepsilon)$ such that

$$G(x,\varepsilon) = \varepsilon^{-n}F(\varepsilon x) + \sum_{j=0}^{n-1} a_j(\varepsilon)x^j. \tag{2.21}$$

Hence

$$F(\varepsilon x) + \sum_{j=0}^{n-1} \varepsilon^n a_j(\varepsilon)x^j = O(\varepsilon^{\alpha+n}), \tag{2.22}$$

uniformly on $x \in [\frac{1}{2}, 2]$. Let $A_j(\varepsilon) = a_j(\varepsilon)\varepsilon^{n-j}$. Then replacing εx by $\rho\varepsilon x$ and grouping in two different ways in (2.22) we obtain:

$$F(\rho\varepsilon x) + \sum_{j=0}^{n-1} A_j(\rho\varepsilon)\rho^j\varepsilon^j x^j = O(\varepsilon^{\alpha+n}), \tag{2.23a}$$

$$F(\rho\varepsilon x) + \sum_{j=0}^{n-1} A_j(\varepsilon)\rho^j\varepsilon^j x^j = O(\varepsilon^{\alpha+n}), \tag{2.23b}$$

thus

$$\sum_{j=0}^{n-1}(A_j(\rho\varepsilon) - A_j(\varepsilon))\rho^j\varepsilon^j x^j = O(\varepsilon^{\alpha+n}). \tag{2.24}$$

Taking n different values of x and solving the corresponding linear system arising from (2.24), shows that each term of the sum is $O(\varepsilon^{\alpha+n})$. Hence

$$A_j(\rho\varepsilon) = A_j(\varepsilon) + O(\varepsilon^{\alpha+n-j}), \tag{2.25}$$

for $0 \le j \le n-1$. This holds uniformly for $\rho \in [\frac{1}{2}, 2]$. Using Lemma 1, we can find constants $a_0, \ldots, a_{n-1}$ such that

$$A_j(\varepsilon) = a_j + O(\varepsilon^{\alpha+n-j}), \quad \varepsilon \to 0^+, \tag{2.26}$$

where we take $a_j = 0$ if $\alpha + n - j < 0$. Therefore,

$$F(\varepsilon x) + \sum_{j=0}^{n-1} a_j \varepsilon^j x^j = O(\varepsilon^{\alpha+n}), \quad \varepsilon \to 0^+. \tag{2.27}$$

Taking $x = 1$ and replacing ε by x we thus obtain

$$F(x) + \sum_{j=0}^{n-1} a_j x^j = O(x^{\alpha+n}), \quad x \to 0^+, \tag{2.28}$$

and (2.17) follows. ■

Summarizing, the *distributional* relation $f(x) = O(x^\alpha)$, $x \to 0^+$, admits two equivalent interpretations, one in terms of primitives and one in terms of parametric behavior.

We can now give our characterization of regularizable distributions. As it is well known, if f is locally integrable in $(0, \infty)$ and has an algebraic singularity at $x = 0$ in the sense that $f(x) = O(x^\alpha)$, as $x \to 0^+$ for some $\alpha < 0$, then f admits an extension to $\mathcal{D}'(\mathbf{R})$. However, there are examples of locally integrable functions, that admit extensions to $\mathcal{D}'(\mathbf{R})$, but that do not have algebraic singularities at $x = 0$. Nonetheless, we have

THEOREM 2. Let $f \in \mathcal{D}'(0, \infty)$. Then f admits an extension to $\mathcal{D}'(\mathbf{R})$ if and only if there exists $\alpha \in \mathbf{R} \setminus \{-1, -2, -3, \ldots\}$ such that $f(x) = O(x^\alpha)$, as $x \to 0^+$, distributionally, in the sense either of the two equivalent conditions (2.16) or (2.17) is satisfied.

PROOF. Suppose f admits an extension $\tilde{f}$ to $\mathcal{D}'(\mathbf{R})$. Then these exists $n \in \mathbf{N}$ and a primitive of order n of $\tilde{f}, F$, which is continuous on $[-1, 1]$. But then $F(x) = O(1)$ as $x \to 0^+$ and since F is a primitive of f on $(0, \infty)$, it follows that if $0 < \delta < 1$, then

$$f(x) = O(x^{-n-\delta}), \qquad (C), \quad \text{as } x \to 0^+. \tag{2.29}$$

Conversely, if $f(x) = O(x^\alpha) \quad (C)$, as $x \to 0^+$, some $\alpha \in \mathbf{R} \setminus \{-1, -2, -3, \ldots\}$ then there exists $n \in \mathbf{N}$ and a primitive of f on $(0, \infty)$, F, which is continuous on $(0, \infty)$ and satisfies $F(x) = O(x^{\alpha+n})$, as $x \to 0^+$. Then F admits an extension to $\mathcal{D}'(\mathbf{R})$, say $\tilde{F}$. It follows that $\tilde{f} = \tilde{F}^{(n)}$ is an extension of f to $\mathcal{D}'(\mathbf{R})$. ■

3. REGULARIZATION IN SEVERAL DIMENSIONS

Our next aim is to consider the regularization of generalized functions of several variables. We shall apply the theory of topological tensor products and thus we start with the vector valued distributions.

Let E be a Banach space with norm $\|\ \|$. Then we can consider the spaces $\mathcal{D}'_{ij}((a, b), E)$ of vector valued distributions. The space $\mathcal{D}'_{ij}((a, b), E)$ can be considered as the space of continuous linear operators from $\mathcal{D}_{ij}(a, b)$ to E or, alternatively, as the topological tensor product $\mathcal{D}'_{ij}(a, b) \hat{\otimes} E$, completion of the algebraic tensor product $\mathcal{D}'(a, b) \otimes E$ equipped with the π-topology: see Treves [6].

The analysis of the regularization of the distributions with values in a Banach space is completely analogous to the analysis of the previous section. Indeed, if $\mathbf{f} \in \mathcal{D}'((a, b), E)$ then $\mathbf{f}(x) = O(x^\alpha) \qquad (C)$, as $x \to 0^+$, means that there exists $n \in \mathbf{N}$ and a primitive of order n of $\mathbf{f}$,$\mathbf{F}$, such that $\|\mathbf{F}(x)\| = O(x^{\alpha+n})$, as $x \to 0^+$. Actually, it is not hard to see that Lemmas 1, 2 and Theorems 1 and 2 remain valid in this case.

Summarizing:

THEOREM 3. Let E be a Banach space with norm $\| \ \|$. Let $\mathbf{f} \in \mathcal{D}'((0,\infty), E)$. Then the following are equivalent:

1. $\mathbf{f}$ admits an extension to $\mathcal{D}'(\mathbf{R}, E)$.
2. $\mathbf{f}$ is algebraically bounded in the (C) sense at $x = 0$,

$$\mathbf{f}(x) = O(x^\alpha) \qquad (C), \quad \text{as } x \to 0^+, \tag{3.1}$$

for some $\alpha \in \mathbf{R} \setminus \{-1, -2, -3, \ldots\}$.

3. $\mathbf{f}$ is algebraically bounded parametrically at $x = 0$,

$$\mathbf{f}(\varepsilon x) = O(\varepsilon^\alpha) \quad \text{as } \varepsilon \to 0^+, \tag{3.2}$$

for some $\alpha \in \mathbf{R} \setminus \{-1, -2, -3, \ldots\}$ in the sense that if $\phi \in \mathcal{D}(0, \infty)$ then

$$\|\langle \mathbf{f}(\varepsilon x), \phi(x)\rangle\| = O(\varepsilon^\alpha), \tag{3.3}$$

as $\varepsilon \to 0^+$. ∎

Let us now consider the regularization of distributions of the space $\mathcal{D}'(\mathbf{R}^n \setminus \{0\})$. Functions and distributions on $\mathbf{R}^n \setminus \{0\}$ and on $\mathbf{R}^n$, of course, can be described by using polar coordinates

$$\mathbf{x} = r\omega, \qquad r = |\mathbf{x}|, \quad \omega \in S, \tag{3.4}$$

where $S = \{\omega \in \mathbf{R}^n : |\omega| = 1\}$ is the unit sphere. A test function $\phi(\mathbf{x})$ of the space $\mathcal{D}(\mathbf{R}^n \setminus \{0\})$ can be considered as a test function $\tilde{\phi}(r, \omega) = \phi(r, \omega)$ of the space $\mathcal{D}(0, \infty) \hat{\otimes} \mathcal{D}(S) \cong \mathcal{D}((0, \infty) \times S)$. Similarly, a distribution $f \in \mathcal{D}'(\mathbf{R}^n \setminus \{0\})$ corresponds to a distribution $\tilde{f}(r, \omega)$ of $(\mathcal{D}(0, \infty) \hat{\otimes} \mathcal{D}(S))' \cong \mathcal{D}'((0, \infty) \times S)$ given by

$$\langle f(\mathbf{x}), \phi(\mathbf{x})\rangle = \langle r^{n-1} \tilde{f}(r, \omega), \phi(r, \omega)\rangle. \tag{3.5}$$

The factor r^{n-1} is the Jacobian of the transformation (3.4):

$$d\mathbf{x} = r^{n-1}\, dr\, d\sigma(\omega), \tag{3.6}$$

where σ is the surface measure on S.

The situation in $\mathbf{R}^n$ is somewhat more complicated. The application $\phi \mapsto \tilde{\phi}$ allows us to identify $\mathcal{D}(\mathbf{R}^n)$ with a closed proper subspace of $\mathcal{D}_{41}[0, \infty) \hat{\otimes} \mathcal{D}(S)$. Therefore, each $g \in (\mathcal{D}_{41}[0, \infty) \hat{\otimes} \mathcal{D}(S))'$ admits a restriction $\pi(g) \in \mathcal{D}'(\mathbf{R}^n)$. Conversely, by the Hahn–Banach theorem, each $f \in \mathcal{D}'(\mathbf{R}^n)$ admits several extensions $\tilde{f} \in (\mathcal{D}_{41}[0, \infty) \hat{\otimes} \mathcal{D}(S))'$.

Therefore, a distribution $f \in \mathcal{D}'(\mathbf{R}^n \setminus \{0\})$ admits an extension in $\mathcal{D}'(\mathbf{R})$ if and only if the corresponding distribution $\tilde{f} \in \mathcal{D}'(0, \infty) \hat{\otimes} \mathcal{D}'(S)$ admits an extension in the space $\mathcal{D}'_{41}[0, \infty) \hat{\otimes} \mathcal{D}'(S)$.

Theorem 3 is not directly applicable to distributions $\tilde{f} \in \mathcal{D}'(0, \infty) \hat{\otimes} \mathcal{D}'(S) = \mathcal{D}'((0, \infty), \mathcal{D}'(S))$ because $\mathcal{D}'(S)$ is not a Banach space. However, $\mathcal{D}'(S)$ is the inductive limit of the Banach spaces $(C^k(S))'$ as $k \to \infty$. Hence, the relation $\tilde{f}(\varepsilon r, \omega) = O(\varepsilon^\alpha)$ in $\mathcal{D}'((0, \infty), \mathcal{D}'(S))$ implies the existence of $k \in \mathbf{N}$ such that the relation holds in $\mathcal{D}'((0, \infty), (C^k(S))')$ and, consequently, the existence of an extension in the space $\mathcal{D}'_{41}([0, \infty), (C^k(S))')$, and the latter is a subspace of $\mathcal{D}'_{41}([0, \infty), \mathcal{D}'(S))$. Therefore, we have

LEMMA 3. Let $\tilde{f} \in \mathcal{D}'((0, \infty), \mathcal{D}'(S))$. Then the following are equivalent:

1. $\tilde{f}$ admits an extension to $\mathcal{D}'_{41}([0, \infty), \mathcal{D}'(S))$.
2. There exists $\alpha \in \mathbf{R} \setminus \{-1, -2, -3, \ldots\}$ such that

$$\tilde{f}(r, \omega) = O(r^\alpha) \qquad (C), \quad \text{as } r \to 0^+. \tag{3.7}$$

3. There exists $\alpha \in \mathbf{R} \setminus \{-1, -2, -3, \ldots\}$ such that

$$\tilde{f}(\varepsilon r, \omega) = O(\varepsilon^{\alpha}) \quad \text{as } \varepsilon \to 0^{+}, \tag{3.8}$$

distributionally, in the sense that if $\phi \in \mathcal{D}(0, \infty)$ then

$$\langle \tilde{f}(\varepsilon r, \omega), \phi(r) \rangle = O(\varepsilon^{\alpha})$$

in $\mathcal{D}'(S)$. ■

The lemma immediately gives

THEOREM 4. Let $f \in \mathcal{D}'(\mathbf{R}^n \setminus \{\mathbf{0}\})$. Then the following are equivalent:

1. f admits an extension to $\mathcal{D}'(\mathbf{R}^n)$.
2. There exists $\alpha \in \mathbf{R} \setminus \{n-2, n-3, \ldots\}$ such that

$$f(r\omega) = O(r^{\alpha}) \qquad (C), \quad \text{as } x \to 0^{+}, \tag{3.8}$$

in the sense that there exists $p \in \mathbf{N}$ and a function F that satisfies $\tilde{F}(r, \omega) = O(r^{\alpha+p})$ as $r \to 0^{+}$ and $\dfrac{\partial^p \tilde{F}}{\partial r^p} = \tilde{f}(r, \omega)$.

3. There exists $\alpha \in \mathbf{R} \setminus \{n-2, n-3, \ldots\}$ such that

$$f(\varepsilon \mathbf{x}) = O(\varepsilon^{\alpha}) \quad \text{as } \varepsilon \to 0^{+} \tag{3.9}$$

distributionally, in the space $\mathcal{D}'(\mathbf{R}^n \setminus \{\mathbf{0}\})$. ■

4. SINGULAR HYPERSURFACES

We now consider the regularization of distributions defined in the complement of a hypersurface.

Let us start with some remarks on the theorem 3 on regularization of vector valued distributions. The result applies if E is a Banach space and, as we observed, it can be extended to the case when E is an inductive limit of Banach spaces. However, it does not apply to other topological vector spaces, as the next example shows.

EXAMPLE. Let $E = C(\mathbf{R})$ be the Fréchet space of continuous functions on $\mathbf{R}$, with the topology of uniform convergence on compacts. Let $f \in \mathcal{D}'((0, \infty), E)$ be defined as

$$f(x, y) = \sum_{n=1}^{\infty} x^{-n} \phi(y - n), \qquad x > 0, \ y \in \mathbf{R}, \tag{4.1}$$

where $\phi \in C(\mathbf{R})$ is a continuous function with compact support in $[0, 1]$. Then f admits a regularization,

$$\tilde{f}(x, y) = \sum_{n=0}^{\infty} \mathcal{P}f(x^{-n} H(x)) \phi(y - n), \quad (x, y) \in \mathbf{R}^2 \tag{4.2}$$

in the space $\mathcal{D}'(\mathbf{R}, E)$. But there is no $\alpha \in \mathbf{R}$ such that $f(\varepsilon x, y) = O(\varepsilon^{\alpha})$ as $\varepsilon \to 0^{+}$. ■

Actually, the same example shows that there are distributions in $\mathcal{D}'((0, \infty) \times \mathbf{R})$ that admit extensions to $\mathcal{D}'(\mathbf{R} \times \mathbf{R})$ but do not satisfy $f(\varepsilon x, y) = O(\varepsilon^{\alpha})$, as $\varepsilon \to 0^{+}$, for any α. The appropriate criterion takes the following form.

THEOREM 5. Let $\mathcal{D}'((0, \infty) \times \mathbf{R}^{n-1})$. Then f admits an extension to $\mathcal{D}'(\mathbf{R}^n)$ if and only if for each compact $K \subseteq \mathbf{R}^{n-1}$ there exists $\alpha_K \in \mathbf{R}$ such that

$$\langle f(\varepsilon x, \mathbf{y}), \phi(x, \mathbf{y}) \rangle = O(\varepsilon^{\alpha_K}), \quad \text{as } \varepsilon \to 0^{+}, \tag{4.3}$$

for each $\phi \in \mathcal{D}'((0, \infty) \times \mathbf{R}^{n-1})$ with $\operatorname{supp} \phi \subseteq (0, \infty) \times K$.

PROOF. Let $\mathcal{D}_K = \{\phi \in \mathcal{D}(\mathbf{R}^{n-1}) : \operatorname{supp} \phi \subseteq K\}$, for K compact. We endow $\mathcal{D}_K$ with the topology of the uniform convergence on $\mathbf{R}^{n-1}$. Then $\mathcal{D}_K$ is a Fréchet space and consequently the theorem 3 applies to $\mathcal{D}'_K$. Hence (4.3) is equivalent to the existence of a regularization in $\mathcal{D}'(\mathbf{R})\hat{\otimes}\mathcal{D}'_K$ for each K and since $\mathcal{D}'(\mathbf{R}^{n-1})$ is the inductive limit of the $\mathcal{D}'_K$ as $K \nearrow$, it follows immediately that the existence of a regularization in $\mathcal{D}'(\mathbf{R}^n) \cong \mathcal{D}'(\mathbf{R}) \hat{\otimes} \mathcal{D}'(\mathbf{R}^{n-1})$ is equivalent to the existence of regularizations in $\mathcal{D}'(\mathbf{R}) \hat{\otimes} \mathcal{D}'_K$ for each K compact. ∎

Observe that (4.3) is equivalent to the order relation

$$f(x, \mathbf{y}) = O(x^{\alpha_K}) \quad (C), \quad \text{as } x \to 0^+ \tag{4.4}$$

in the space $\mathcal{D}'((0,\infty), \mathcal{D}'_K)$ for each compact $K \subseteq \mathbf{R}^{n-1}$.

The above analysis applies to distributions defined in $\mathbf{R}^n \setminus \Sigma$, where Σ is a hyperplane. However, as we now show, the same results hold if Σ is a smooth regular hypersurface.

Let R be a region of $\mathbf{R}^n$. Let Σ be a hypersurface of $\mathbf{R}^n$, contained in R, that divides it into two subregions R_+ and R_-. Suppose Σ is smooth and regular, in the sense that at each point of Σ there is a well-defined normal vector. This means that locally Σ is given by an equation of the form

$$u(\mathbf{x}) = 0, \tag{4.5}$$

for some smooth function whose gradient $\mathbf{D}u$ does not vanish near Σ. Let R_0 be the subregion of R where u is defined and where its gradient does not vanish.

It follows that if $|\varepsilon|$ is small enough, the equation

$$u(\mathbf{x}) = \varepsilon \tag{4.6}$$

defines a smooth regular hypersurface Σ_ε. We may suppose that $\Sigma_\varepsilon \subseteq R_+$ if $\varepsilon > 0$ and $\Sigma_\varepsilon \subseteq R_-$ if $\varepsilon < 0$.

Observe that if $(v_1, \ldots, v_{n-1})$ is a local coordinate system in Σ, then we can use $(v_1, \ldots, v_{n-1}, u)$ as a coordinate system in R_0 and use $(v_1, \ldots, v_{n-1})$ as a local coordinate system on Σ_ε if $|\varepsilon|$ is small. Therefore, any test function in $\mathcal{D}(\Sigma_\varepsilon \cap R_0)$ can be considered, by using this local coordinate system, as a test function in $\mathcal{D}(\Sigma \cap R_0)$ and, by duality, the same is true of distributions on $\mathcal{D}'(\Sigma_\varepsilon \cap R_0)$.

Let f be a distribution of the space $\mathcal{D}'(R \setminus \Sigma)$. We would like to find a necessary and sufficient condition for the existence of an extension on $\mathcal{D}'(R)$.

If $\phi \in \mathcal{D}'(\Sigma)$ has $\operatorname{supp}\phi \subseteq \Sigma \cap R_0$, then we can define the distribution of one variable $f_{u,\phi}$ on $\mathcal{D}'((-\varepsilon_0, 0) \cup (0, \varepsilon_0))$ for ε_0 small by

$$\langle f_{u,\phi}(t), \psi(t)\rangle = \langle f(\mathbf{x}(v_1, \ldots, v_{n-1}, u)), J\phi(v_1, \ldots, v_{n-1})\psi(u)\rangle, \tag{4.7}$$

where J is the Jacobian of the transformation $\mathbf{x} = \mathbf{x}(v_1, \ldots, v_{n-1}, u)$. Then f admits an extension to $\mathcal{D}'(R_0)$ if and only if $f_{u,\phi}$ admits an extension to $\mathcal{D}'((-\varepsilon_0, \varepsilon_0))$ for each such ϕ. This is equivalent to the extension of f considered as an element of the space $\mathcal{D}'((-\varepsilon_0, 0) \cup (0, \varepsilon_0), \mathcal{D}'(\Sigma \cap R_0))$. Thus, we obtain

THEOREM 6. Let R be a region of $\mathbf{R}^n$ and Σ be a smooth regular hypersurface that divides it into two parts R_+ and R_-. Let f be a distribution defined in $R \setminus \Sigma = R_+ \cup R_-$. Then f admits an extension to $\mathcal{D}'(R)$ if and only if for each compact subset K of Σ there exists $\alpha_K \in \mathbf{R} \setminus \{-1, -2, -3, \ldots\}$ such that

$$f_{u,\phi}(\varepsilon t) = O(\varepsilon^{\alpha_K}), \quad \text{as } \varepsilon \to 0, \tag{4.8}$$

distributionally, in $\mathcal{D}'((-\varepsilon_0, 0) \cup (0, \varepsilon_0))$ for each $\phi \in \mathcal{D}(\Sigma)$ with $\operatorname{supp}\phi \subseteq K$, where u is a smooth function that represents Σ in a neighborhood of K. Condition (4.8), in turn, is equivalent to the condition

$$f_{u,\phi}(t) = O(|t|^{\alpha_K}) \quad (C), \quad \text{as } t \to 0, \tag{4.9}$$

for all such ϕ. ■

Two remarks are in order. First, the hypersurface Σ might be compact, in which case we can take $\alpha_K = \alpha$, a fixed constant. Second, observe that we did not assume that f has a restriction to Σ_ε for $0 < |\varepsilon| < \varepsilon_0$: that is why we used the auxiliary function ϕ. However, when f admits a restriction g_t to the hypersurface Σ_t, $0 < t < |\varepsilon_0|$, then g_t defines a distribution of $\mathcal{D}'((-\varepsilon_0, 0) \cup (0, \varepsilon_0), \mathcal{D}'(\Sigma \cap R_0))$ and it thus follows that the existence of the regularization is equivalent to the condition $g_t = O(|t|^{\alpha_K})$ distributionally as $t \to 0$.

5. DISTRIBUTIONAL BOUNDARY VALUES

Let R be a region of $\mathbf{R}^n$ and let Σ is a smooth regular hypersurface that divides it into two subregions R_+ and R_-. Let u be a smooth function with never vanishing gradient on the subregion R_0 such that the equation $u = 0$ represents $\Sigma \cap R_0$. Let $f \in \mathcal{D}'(R \setminus \Sigma)$. We say that f has distributional boundary values $f_+ \in \mathcal{D}'(\Sigma)$ from the positive side if for each $\phi \in \mathcal{D}'(\Sigma)$ and for each u that represents Σ in a neighborhood of $\operatorname{supp} \phi$ the limit

$$\langle f_+, \phi \rangle = \lim_{t \to 0^+} f_{u,\phi}(t) \tag{5.1}$$

exists in the *distributional* sense. The distributional boundary value f_- from the negative side is defined similarly as the limit of $f_{u,\phi}(t)$ as $t \to 0^-$.

As a simple corollary of the results of the previous section, it follows that if the distributional boundary values f_+ and f_- exist then f admits an extension to $\mathcal{D}'(R)$. The converse is plainly false as the function $f(x, y) = \dfrac{1}{x}$ considered as a distribution on $\mathcal{D}'(\mathbf{R}^2 \setminus \mathbf{R})$ shows. The existence of distributional boundary values is a much stronger condition than the existence of an extension.

It is then quite interesting that in some case the mere existence of an extension implies the existence of boundary values.

Indeed, let $B = \{(x_1, \ldots, x_n) \in \mathbf{R}^n : x_1^2 + \cdots + x_n^2 < 1\}$ be the unit ball of $\mathbf{R}^n$ and let $S = \{(x_1, \ldots, x_n) \in \mathbf{R}^n : x_1^2 + \cdots + x_n^2 = 1\}$ be its boundary. Let U be defined and harmonic in B. Then it is known, see Bremermann [7], Estrada–Kanwal [8], that the distributional boundary values

$$u(\omega) = \lim_{r \to 1^-} U(r\omega), \qquad \omega \in S, \tag{5.2}$$

defined as

$$\langle u(\omega), \phi(\omega) \rangle = \lim_{r \to 1^-} \langle U(r\omega), \phi(\omega) \rangle \tag{5.3}$$

for $\phi \in \mathcal{D}(S)$ exist if and only if there exist M and β such that

$$|U(\mathbf{x})| \le \frac{M}{(1 - |\mathbf{x}|)^\beta}, \qquad |\mathbf{x}| < 1. \tag{5.4}$$

As we pointed out, the existence of the distributional limit (5.2), or equivalently, the bound (5.4) imply the existence of extensions of U to $\mathcal{D}'(\mathbf{R}^n)$. Actually, we can take extensions that satisfy $\tilde{U}|_{\mathbf{R}^n \setminus B} = 0$. Interestingly, the converse also holds.

THEOREM 7. Let U be a harmonic function in $B = \{(x_1, \ldots, x_n) \in \mathbf{R}^n : x_1^2 + \cdots + x_n^2 < 1\}$. Then the following are equivalent:

1. U has distributional boundary values at $S = \partial B$.
2. There exists $\beta \in \mathbf{R}$ such that

$$U(r\omega) = O((1 - r)^{-\beta}), \quad \text{as } r \to 1^-, \tag{5.5}$$

uniformly on $\omega \in S$.

3. U admits an extension to $\mathcal{D}'(\mathbf{R}^n)$.
4. There exists $\beta \in \mathbf{R}$ such that for each $\phi \in \mathcal{D}(S)$,

$$\langle U(r\omega), \phi(\omega)\rangle = O((1-r)^{-\beta}), \quad (C), \quad \text{as } r \to 1^-. \tag{5.6}$$

PROOF. As we explained, (1) and (2) are equivalent and from the results of the previous section so are (3) and (4). Also, (1) implies (3) in general, whether U is harmonic or not. Thus, it remains to show that (3) implies (1). So suppose (3) holds.

Let V be an extension of U to $\mathcal{D}'(\mathbf{R}^n)$. We may suppose $V(\mathbf{x}) = 0$, if $|\mathbf{x}| > 1$. Then V is harmonic in $\mathbf{R}^n \setminus S$, so $f = \Delta v$ has support on S. Hence there are distributions $f_0, \dots, f_k \in \mathcal{D}'(S)$ such that

$$f = f_0\,\delta(S) + f_1 d_n\,\delta(S) + \cdots + f_k d_n^k\,\delta(S), \tag{5.7}$$

where the multilayer distributions $f_j d_n^j\,\delta(S)$ are the distributions of $\mathcal{D}'(\mathbf{R}^n)$ defined in Estrada–Kanwal [2] as

$$\langle f_j d_n^j\,\delta(S), \phi\rangle = (-1)^j \left\langle f_j(\omega), \frac{d^j\phi(\omega)}{dn^j}\right\rangle \tag{5.8}$$

where $\phi \in \mathcal{D}(\mathbf{R}^n)$, d/dn is the derivative in the normal direction to S and where the evaluation on the right is on $\mathcal{D}'(S) \times \mathcal{D}(S)$.

Let

$$K(\mathbf{x}) = C_n|\mathbf{x}|^{2-n}, \quad n > 2, \tag{5.9a}$$

$$K(\mathbf{x}) = \frac{1}{2\pi}\log|\mathbf{x}|, \quad n = 2, \tag{5.9b}$$

where $C_n = \dfrac{1}{(2-n)\omega_n}$ and ω_n is the area of S, be the fundamental solution of the Laplace equation:

$$\Delta K(\mathbf{x}) = \delta(\mathbf{x}). \tag{5.10}$$

Let

$$V_1 = K * f = \sum_{j=0}^{k}(K * f_j d_n^j\,\delta(S)). \tag{5.11}$$

Then V_1 is harmonic in $\mathbf{R}^n \setminus S$ and if $|\mathbf{x}| \neq 1$ then

$$V_1(\mathbf{x}) = \sum_{j=0}^{k}(-1)^j \left\langle f_j(\omega), \frac{d^j K(\omega - \mathbf{x})}{dn_\omega^j}\right\rangle. \tag{5.12}$$

But it is easy to see that the function $\left\langle f_j(\omega), \dfrac{d^j K(\omega - \mathbf{x})}{dn_\omega^j}\right\rangle$ satisfies an estimate of the form

$$\left|\left\langle f_j(\omega), \frac{d^j K(\omega - \mathbf{x})}{dn_\omega^j}\right\rangle\right| \le M_j(1 - |\mathbf{x}|)^{-\beta_j}, \quad |\mathbf{x}| < 1, \tag{5.13}$$

for some M_j, β_j. Thus V_1 has distributional boundary limits

$$V_1(\omega) = \lim_{r \to 1^-} V_1(r\omega). \tag{5.14}$$

But $\Delta(V_1 - V) = 0$ and it follows that

$$V = V_0 + V_1, \tag{5.15}$$

where V_0 is harmonic in $\mathbf{R}^n$. Since V_0 also has (ordinary!) boundary values at S it follows that so does V. But $U(\mathbf{x}) = V(\mathbf{x})$ if $|\mathbf{x}| < 1$ and the result follows. ∎

A similar analysis gives the following result on the distributional boundary values of analytic functions.

THEOREM 8. Let Ω be a region of $\mathbf{C}$ and let γ be a smooth regular curve in Ω that divides Ω into two subregions Ω_+ and Ω_-. Let F be analytic in $\Omega \setminus \gamma$. Then F admits an extension to $\mathcal{D}'(\Omega)$ if and only if F has distributional boundary values F_+ and F_- in $\mathcal{D}'(\gamma)$. ∎

The jump of F across γ is $[F] = F_+ - F_-$, a quantity that can be defined even if F_+ and F_- do not exist as the distributional limit $\lim_{t\to 0^+}(F_{u,\phi}(t) - F_{u,\phi}(-t))$. Using the ideas of the proof of Theorem 7 we can show that the existence of the jump $[F] \in \mathcal{D}'(\gamma)$ of a sectionally analytic function F implies the existence of an extension to $\mathcal{D}'(\Omega)$ and consequently, the existence of the boundary values F_+ and F_-.

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